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ON THE OPTIMAL CONTROL FOR A SEMILINEAR EQUATION WITH COST DEPENDING ON THE FREE BOUNDARY

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Dedicated to Hiroshi Matano with friendship and admiration in occasion of his 60th birthday

Abstract. We study an optimal control problem for a semilinear elliptic boundary value problem giving rise to a free boundary. Here, the free boundary is generated due to the fact that the nonlinear term of the state equation is not differentiable. The new aspect considered in this paper, with respect to other control problems involving free boundaries, is that here the cost functional explicitly depends on the location of the free boundary. The main difficulty is to show the continuous dependence (in measure) of the free boundary with respect to the control function. The crucial tool to get such continuous dependence is to know how behaves the state solution near the free boundary, as in previous works by L.A. Caffarelli and D. Phillips among other authors. Here we improved previous results in the literature thanks to a suitable application of the Fleming-Rishel formula.

1. **Introduction.**

We consider the optimal control problem

$$\min_{u\in U_{\rm ad}} J(u)$$

where

$$J(u)=\int_\Omega \chi_{S(y(x;u))\cap B}(x)\,dx+\int_\Omega \frac{1}{G(y(x;u))}\,dx,$$

with χ_A the characteristic function of a subset $A\subset\Omega$ ($\chi_A(x)=1$ if $x\in A$ and $\chi_A(x)=0$ if $x\in\Omega-A$), G a given real continuous increasing function such that $G(0)>0$, and for a suitable set $U_{\rm ad}$ of admissible controls which will be specified later. The state function $y(x;u)$ is the (unique) solution of the boundary value problem

$$\begin{cases} -Ly(x)+f(y(x))=u(x)\chi_\omega & \text{in } \Omega,\\ y=0 & \text{on } \partial\Omega,\end{cases} \tag{1}$$

where L is an elliptic linear operator of the form

$$Ly=\sum_{i,j=1}^N a_{ij}(x)\frac{\partial^2 y}{\partial x_i\partial x_j}+\sum_{i=1}^N b_i(x)\frac{\partial y}{\partial x_i} \tag{2}$$

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with $a_{ij} \in C^1(\overline{\Omega})$, $b_i \in L^\infty(\Omega)$ such that there exist $\Lambda, \lambda > 0$ for which

$$\lambda|\xi|^2 \leq \sum a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \forall \xi \in \mathbb{R}^N. \tag{3}$$

The set $\Omega \subset \mathbb{R}^N$ is a bounded domain, ω (an open set) and B (a compact set) are two subset contained in Ω. The reaction nonlinearity is given by

$$f(t) = |t|^{q-1}t, \text{ for some } q \in (0,1) \tag{4}$$

which is a crucial fact in our study. It is well known that for any given control, for instance $u \in L^\infty(\omega)$, there exists a unique weak solution $y \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for any $p \in [1, +\infty)$ (see, e.g., the presentation made in Díaz [16] and its references).

Given a general function $y : \Omega \to \mathbb{R}$, we define $N(y) = \{x \in \Omega : y(x) = 0\}$ (the null set of y), $S(y) = \{x \in \Omega : y(x) \neq 0\}$, and if $y(x)$ is the solution of (1) we set $\mathcal{F} = \partial S(u) \cap \Omega$, the free boundary of y.

Before introducing the set of admissible controls U_{ad} we need to present an important growth condition property which we shall impose over all the possible controls. First of all we ask the support of the control functions to satisfy an *internal uniform sphere condition*,

$$\begin{cases} \text{there is a } R_0 > 0 \text{ such that for each } x_0 \in \partial S(u), \\ \text{there exists a } x_1 \in S(u) \text{ with } dis(x_0, x_1) = R_0 \text{ and } B_{R_0}(x_1) \subset S(u). \end{cases} \tag{5}$$

We shall use also the notation

$$S_u(R) = \{x \in S(u) : dis(x, \partial S(u)) = R\},$$

for any $0 < R \leq R_0$.

As we shall show later (see Example 1), if the control $u(x)$ is permitted to be arbitrarily flat near the boundary of its support then the continuous dependence of the free boundary of the state solution may fail. This is the reason why we shall suppose that any admissible control satisfies the following additional *no flat condition*: there exist $R, C > 0$ and $\gamma = \frac{2q}{1-q}$ such that for any $x_1 \in S_u(R)$

$$u(x) \geq C(R - |x - x_1|)^\gamma \quad \text{if } x \in B_R(x_1). \tag{6}$$

Note that the above condition implicitly implies some kind of constraint on the weak derivatives of u near the boundary of its support $\partial S(u)$. Indeed, if we take for instance $n = 1$, $x_0 \in \partial S(u)$ and $x_m \in B_R(x_1)$ (i.e. $|x_m - x_1| < R$) with $|x_0 - x_1| = R$, then (since $u^{1/\gamma}(x_0) = 0$) (6) implies that

$$\frac{u^{1/\gamma}(x_m) - u^{1/\gamma}(x_0)}{|x_m - x_0|} \geq C.$$

Passing to the limit, when $x_m \to x_0$, we get that necessarily $\frac{d}{dx}(u^{1/\gamma}(x_0)) \geq C$. This explains why we shall use, in what follows, some requirements on the derivatives of the controls in order to be able to ensure that condition (6) remains true for a control u which is the limit (in some suitable sense) of a sequence of admissible controls u_n satisfying each of them the associated property (6).

Given M, M^*, R_0, C_0 (C_0 will be made more explicit later in Theorem 2.2), the set of admissible controls we shall consider in this paper is defined by

$$U_{\text{ad}} = \{u \in H^1(\omega) \cap L^\infty(\omega) \,|\, 0 \leq u(x) \leq M, \, ||u||_{H^1(\omega)} \leq M^* \text{and} \\ u \text{ satisfies (5) and (6) for some } 0 < R^* \leq R \leq R_0 \text{ and } C > C_0\}.$$

Theorem 1.1. *Under the above assumptions there exists at least one minimum of J in U_{ad}.*

We have to point out that the first term of the functional J is non trivial. In fact, due to the presence of the non differentiable nonlinearity, it is well known that a dead core can be formed, and so the intersection $S(y) \cap B$ is not always equal to B (according to the properties of the control u). The existence of a non-empty null set of y in Ω, and so of the associated free boundary $\mathcal{F}$, is discussed in the monograph of Díaz [16] where the reader can find many references dealing also with the existence and uniqueness of the state solution $y(x)$.

The problem under consideration is relevant in many different applied contexts: catalysis in a porous medium (see, e.g., Aris [5] and the homogenization results of Díaz [17] and Conca, Díaz, Liñán and Timofte [17]), desalination plants (see, for instance, Bleninger-Jirka [9] and Díaz, Sánchez, N. Sánchez, Veneros and Zarzo [20]), other environmental discharge problems (see, e.g. [8] and [22]), etc. In many of those problems $f(y) = \lambda |y|^{q-1} y$ where $q \in [0, 1)$ is the so called *order of the chemical reaction*. Since the order of the chemical reaction is less than one, it is well known that some free boundary is formed corresponding to the boundary of the support of $y(\cdot : u)$ (which we denote by $S(y(u(\cdot)))$.

There is a very intensive literature on optimal control for nonlinear elliptic equations. Many authors considered also the special case in which the state solution give rise to a free boundary, but in most of the cases the cost functional does not involve explicitly the location of the free boundary (see, for instance, [7], [35], [34] and their references). Moreover, although the problem can be considered quite close to the shape optimization (see, e.g. the monographs [31] and [23]) here the control is not directly the domain given by the support of the state solution but the heterogeneous source introduced at the state equation over the small subdomain ω.

Before giving the proof of our main theorem we have to introduce a central tool in our study: the continuous dependence of the free boundary with respect to the control. Let us mention that the continuous dependence with respect to other types of data admits several different meanings. For instance, in Chen, Matano and Mimura [13] the continuos dependence is proved with respect to the initial datum for some one dimensional homogeneous parabolic equation with strong absorption. Here the presence of the control function introduces a different type of heterogeneity and, besides that, the multidimensional formulation of the problem requires to understand the continuous dependence property in a weaker sense: we shall state it in terms of the measure of the symmetric difference of the null set of the associate state solutions, in the same style than works with a different motivation as the ones by Caffarelli [11], [12], Brezzi and Caffarelli [10], Phillips [28], Rodrigues [33] and Nochetto [27], [26], among other authors.

This paper improves a preliminary presentation of this type of results (see the communication by the authors in the electronic proceedings Díaz, Mingazzini and Ramos [18]). Some numerical experiences were also presented in the above mentioned reference but its detailed version will be the object of a separated future publication.

2. **On the continuous dependence of the free boundary.** The aim of this part is to prove the continuous dependence of the support of the solution of (1) with respect to the data u. Let us recall first some properties of the solutions of (1). According to the theory of semilinear elliptic equations, whenever u is

bounded and non negative, the solution y belongs also to $L^\infty(\Omega)$, and due to the comparison principle it is also a non negative function. Concerning some additional regularity, we recall that in this case the solution belongs at least to $C^{1,\alpha}(\overline{\Omega})$ for some $0 \le \alpha < 1$. For details we refer, for instance, to [16].

A curious property, studied in different papers, is the so called "non-diffusion" property of the support (see [16], [3], [2] and [4]), which under suitable hypothesis on u guarantees that the support of the solution coincides with that of the datum u. So, in these special cases, it is clear that we can control exactly the support of our solution just by considering the support of the data.

But here we are not interested in this very special case. Our aim is to control the state function under the non flat condition (6) which, as we shall show, ensures that the support of the solution is strictly larger than the one of the control.

The strict propagation of the support was studied initially in Álvarez-Díaz [3] and Álvarez [2] for the case $L = \Delta$ and later generalized in Álvarez-Díaz [4] to the case in which the second order linear operator L is replaced by the quasilinear operator Δ_p. We present here the extension for the general operator L (see Theorem 2.2).

Our main idea in order to control the behavior of $\mathcal{F}(y(u(\cdot)))$ relies on the use of some non-degeneracy property of the solution near its free boundary in a way very close to the one followed by Álvarez and Díaz in [3]. To be more precise we shall prove the following result:

Theorem 2.1. *. Let $u, u_n \in U_{ad}$, with $u_n \to u$ strongly in $L^2(\omega)$ and weakly star in $L^\infty(\omega)$, and let y_n and y be the solutions of the associated problems (1). Then there exist a subsequence (still labeled as y_n), $\varepsilon_0 > 0$ and $h_\infty : \mathbb{R}_+ \to \mathbb{R}_+$ continuous with $h_\infty(0) = 0$, such that for all $\varepsilon < \varepsilon_0$ and for any element of this subsequence*

$$|\{x : 0 < y_n(x) < \varepsilon\}| \le h_\infty(\varepsilon), \tag{7}$$

where $|A|$ stands for the Lebesgue measure of A in $\mathbb{R}^N$.

To prove this result we shall need to divide the analysis on two different subsets, $S(u)$ and $\Omega \setminus S(u)$. On $S(u)$ we shall build a family of subsolutions showing the strict diffusion of the support of y (see Theorem 2.2 below) and the non-degeneracy property (7) for y on this part of the domain. We introduce the quantity

$$S = ess \sup_{\Omega} \left[\sum_i a_{ii}(x) - \sum_{i,j} a_{ij}(x) \frac{x_i x_j}{r^2} + \sum_i b_i(x) x_i \right], \tag{8}$$

with $r = |x|$, which we know to be finite because the coefficients of L are bounded.

Theorem 2.2. *Let $u \in L^1_{loc}(\omega)$, $u \ge 0$, $x_0 \in \partial S(u) \cap \omega$ and $y \ge 0$ such that*

$$-L(y) + y^q \ge u \quad in\ \omega. \tag{9}$$

Let $1 \le \delta \le 1 + \lambda(\beta q + 1)/S$, with $\beta = \gamma/q$ and λ given by (3). Then there exist $C, K_1, K_2, K_3 > 0$ such that if $\epsilon > 0$, $x_1 \in \omega$ satisfy $\delta\epsilon > |x_1 - x_0| \ge ((\delta + 1)/2)\epsilon$, $B_\epsilon(x_1) \subset \omega$ and

$$u(x) \ge C_0 |x - x_0|^{\beta q} \quad a.e.\ x \in B_\epsilon(x_1), \tag{10}$$

then

$$y(x) \ge \begin{cases} K_1 \epsilon^\beta - K_2 |x - x_1|^\beta & if\ 0 \le |x - x_1| \le \epsilon, \\ K_3(\delta\epsilon - |x - x_1|)^\beta & if\ \epsilon \le |x - x_1| \le \delta\epsilon. \end{cases}$$

In particular, $y > 0$ in $B_{(\delta\epsilon - |x_1 - x_0|)}(x_0)$.

Proof. As in Álvarez-Díaz [3], [4] (and Álvarez [2]) we shall built a subsolution with the desired growth near the free boundary of y. Suppose $\underline{y}$ is a radially symmetric function defined on $B_{\delta\epsilon}(x_1)$, i.e. $\underline{y}(x) = \eta(|x - x_1|)$, then the linear operator (2) can be written as

$$L(\eta(r)) = \eta''(r) \sum_{i,j} a_{ij}(x) \frac{x_i x_j}{r^2} + + \frac{\eta'(r)}{r} \left[\sum_i a_{ii}(x) - \sum_{i,j} a_{ij}(x) \frac{x_i x_j}{r^2} + \sum_i b_i(x) x_i \right],$$

with $r \in (0, \delta\epsilon)$. Using assumption (3) and property (8), we deduce that if $\eta', \eta'' \le 0$ then

$$-L\underline{y} \le -\Lambda \eta''(r) - \frac{S}{r} \eta'(r).$$

Moreover, if $\eta' \le 0, \eta'' \ge 0$, then

$$-L\underline{y} \le -\lambda \eta''(r) - \frac{S}{r} \eta'(r).$$

Now we set the function η to be

$$\eta(r) = \begin{cases} \eta_1(r) = K_1 \epsilon^\beta - K_2 r^\beta & 0 \le r \le \epsilon, \\ \eta_2(r) = K_3(\delta - r)^\beta & \epsilon \le r \le \delta\epsilon, \end{cases}$$

and we put $\mathcal{L}(\eta) = -L\underline{y} + \underline{y}^q$. We want to show that for suitable constants

$$\mathcal{L}(\eta(r)) \le C r^{\beta q} \quad \text{in } D'(0, \delta\epsilon).$$

$$\mathcal{L}(\eta_2) \le -\lambda\beta(\beta-1)K_3(\delta\epsilon - r)^{\beta-2} + S\beta K_3 \tfrac{(\delta\epsilon - r)^{\beta-1}}{r} + K_3^q (\delta\epsilon)^{\beta q} \le$$

$$\le \beta K_3 (\delta\epsilon - r)^{\beta q} \left(-\lambda(\beta - 1) + S \tfrac{(\delta\epsilon - r)}{r} + K_3^{q-1} \beta^{-1} \right).$$

Now $\frac{(\delta\epsilon - r)}{r} \le \delta - 1$ when $\epsilon \le r \le \delta\epsilon$; so if we choose K_3 as

$$K_3 = \left(\frac{\beta^{-1}}{\lambda(\beta q + 1) - S(\delta - 1)} \right)^{\frac{1}{1-q}}$$

we obtain that $-L\eta_2 + \eta_2^q \le 0$.
On the other hand

$$\mathcal{L}(\eta_1) \le \beta(\beta-1)\Lambda K_2 r^{\beta-2} + \beta S K_2 r^{\beta-2} + [K_1 \epsilon^\beta - K_2 r^\beta]^q = \qquad (r = \epsilon t)$$

$$= \left[\beta(\beta-1)\Lambda K_2 t^{\beta q} + \beta S K_2 t^{\beta q} + (K_1 - K_2 t^\beta)^q \right] \epsilon^{q\beta} \le$$

$$\le K_4 \epsilon^{\beta q}$$

where $K_4 = \beta[(\beta - 1)\Lambda + S K_2] + K_1$.
We observe that if $x \in B_\epsilon(x_1)$

$$|x - x_0| \ge dis(x_0, B_\epsilon(x_1)) \ge \frac{\delta - 1}{2} \epsilon,$$

therefore if we take

$$C = \frac{K_4 2^{\beta q}}{(\delta - 1)^{\beta q}}$$

we obtain that

$$u(x) \geq C|x - x_0|^{\beta q} \geq \frac{K_4 2^{\beta q}}{(\delta - 1)^{\beta q}} \left(\frac{\delta - 1}{2} \epsilon \right)^{\beta q} \geq \mathcal{L}(\eta_1).$$

Finally on $\partial B_{\delta\epsilon}$ we have $\underline{y} \leq y$, and as

$$-L\underline{y} + \underline{y}^q \leq g(x)$$

we conclude that $\underline{y}$ is actually a subsolution and so $\underline{y} \leq y$ holds on $B_{\delta\epsilon}$. □

Remark 1. A careful study of the influence of a possible convection velocity $\mathbf{v}(x)$ in the formation of the free boundary for the stationary problem (but with $y = 1$ at $\partial\Omega$ and without any control, $u \equiv 0$) was carried out in Pinsky [29] and [30]. His results, when particularized to a ball, show the important difference between inward and outward pointing convection vector fields.

The non-degeneracy property (7) near the free boundary $\mathcal{F}(y(u(\cdot)))$ is now a trivial consequence (see the argument in Corollary 2 of [3] or Lemma 2.2 of [2]):

Corollary 1. *Let y be the solution of (1), and assume that u satisfies (6) for suitable $R, C > 0$. Then for any compact $K \subset \Omega$ there exist $\epsilon_0, k > 0$ such that*

$$|\{x \in K \cap S(u) \,:\, 0 < y(x) < \epsilon^\beta\}| \leq k\epsilon \tag{11}$$

for any $\epsilon < \epsilon_0$, with $\beta = \frac{2}{1-q}$.

Now we pass to analyze the behavior of the solution on $\Omega \setminus S(u)$.

Theorem 2.3. *Let D be an open subset of Ω, $y \in W^{1,p}(D)$, for some $p \geq 1$, $y \geq 0$, such that $y(x)$ satisfies $-Ly + y^q = 0$, with $q \in (0,1)$, in a weak sense on D. Then there exist ε_0 and $h : \mathbb{R}_+ \to \mathbb{R}_+$ continuous, with $h(0) = 0$ such that for all $\varepsilon < \varepsilon_0$*

$$|\{x \in D : 0 < y(x) < \varepsilon\}| \leq h(\varepsilon). \tag{12}$$

Proof. By the Fleming-Rishel-Federer formula (see, e.g., Rakotoson [32] Proposition 6.2.2) we know that, if we define the function of distribution of y by

$$m_y(t) := |\{x \in D : t < y(x)\}|,$$

and if we define

$$m_{o,y}(t) := |\{x \in D : t < y(x), \nabla y(x) = 0\}|,$$

then the function

$$m_{1,y}(t) := m_y(t) - m_{o,y}(t) \tag{13}$$

is absolutely continuous on $\mathbb{R}$. But, thanks to the assumptions on the coefficients of L, and since $q \in (0,1)$, we know (by the Agmon-Douglis-Nirenberg regularity result) that $y \in W^{2,p}_{loc}(D)$ and so, by Lemma A.4 of Kinderlehrer-Stampacchia [24], if the subset $\{x \in D : \nabla y(x) = 0\}$ has a positive measure then $Ly = 0$ a.e. on this set. Thus, since $Ly = y^q$ a.e. on D, we deduce that necessarily $\{x \in D : \nabla y(x) = 0\} \subset \{x \in D : y(x) = 0\}$. In other words, $m_{o,y}(t) := |\{x \in D : y(x) < t, \nabla y(x) = 0\}| = 0$ for any $t \geq 0$. Thus, (13) implies that $m_y(t)$ is absolutely continuous on $[0, +\infty)$ and so

$$m_{1,y}(t + \varepsilon) - m_{1,y}(t) = \int_t^{t+\varepsilon} \left(\int_{\{u=s, \nabla y(x) \neq 0\}} \frac{d\mathcal{L}^{N-1}}{|\nabla y(x)|} \right) ds, \text{ for any } t \in [0, +\infty).$$

Finally, it suffices to notice that

$$|\{x \in D : 0 < y(x) < \varepsilon\}| = m_{1,y}(\varepsilon) - m_{1,y}(0)$$

and to take

$$h(\varepsilon) := \int_0^{\varepsilon} \left(\int_{\{u=s, \nabla y(x) \neq 0\}} \frac{d\mathcal{L}^{N-1}}{|\nabla y(x)|} \right) ds.$$

□

Remark 2. Notice that the conclusion (12) is ensured merely on the subset where the control vanishes.

Remark 3. The above theorem extends (in different senses) many previous results in the literature. For instance, in the special case $L = \Delta$, a stronger property was obtained firstly in Caffarelli [11] for the obstacle problem ($q = 0$) and then in Phillips [28] and Alt-Phillips [6] for $0 < q < 1$: it was shown there that the property holds with $h(\varepsilon) = \varepsilon^{2/(1-q)}$. Notice that this is equivalent to say that the function of distribution of y, $m_y(t)$, is Hölder continuous near $t = 0$. Our result is weaker in this sense (although it is enough for our purposes) but it is more general since it applies to the general operators L under the assumptions indicated above. We point out that the deep local study made in Phillips [28] and Alt-Phillips [6] leads to many other qualitative information on y, but it requires sharp properties on the elliptic operator which, although well-known for the Laplacian operator, are not satisfied by the operator L under the generality assumed here.

Remark 4. The case of $q = 0$ requires a different treatment since it leads to a variational inequality of the type of the obstacle problem. In this case the nonlinearity must be understood in the sense of multivalued maximal monotone operators. Although Theorem 2.3 can be extended to this case (see, e.g., the extension of some results on the location of the free boundary, from the semilinear case to the multivalued case, in [16]), there is an extensive literature on the continuous dependence of the free boundary (the boundary of the coincidence set) with respect to the source term (see, e.g., Caffarelli [11], Rodrigues [33], Garroni and Vivaldi [21], Challal, Lyaghfouri and Rodrigues[14] and their references).

Theorem 2.4. *Let $u_n \to u$ in $L^2(\Omega)$ and weakly star in $L^\infty(\omega), u_n \geq 0$, and let y_n and y be the solutions of the associated problems (2). Then there exists a subsequence (still labeled as y_n) such that $y_n \to y$ in $W^{2,p}(\Omega)$ for any $p \in [1, +\infty)$, y is a co-area regular function in $N(u) \cap S(y)$ (in the sense of [1]) and, in particular, there exist ε_0 and $\bar{h} : \mathbb{R}_+ \to \mathbb{R}_+$ continuous, with $\bar{h}(0) = 0$ such that for all $\varepsilon < \varepsilon_0$ and for any n of this subsequence*

$$|\{x \in N(u_n) : 0 < y_n(x) < \varepsilon\}| \leq \bar{h}(\varepsilon).$$

Proof. Since $u_n(x)\chi_\omega - f(y_n(x))$ are uniformly bounded, by the Agmon-Douglas-Nirenberg regularity result we know that $y_n \to y$ in $W^{2,p}(\Omega)$ (the inverse of the operator L, with zero Dirichlet boundary conditions, is a compact operator from $L^p(\Omega)$ into $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$). Moreover, from the monotonicity of the nonlinear term y^q and Sobolev inequalities we know that $y_n \to y$ in $L^\infty(\Omega)$ and so y is the solution corresponding to the limit control u. As in Theorem 2.3, y_n and y are co-area regular functions in $N(u_n) \cap S(y_n)$ and $N(u) \cap S(y)$ respectively (since $|\{x \in N(u) \cap S(y) : y(x) = t \text{ and } \nabla y(x) = 0\}| = 0$). Then we know (see [1] and the presentation made in [32]) that *length of* $\{y_n(x) = t\}$ →*length of* $\{y(x) = t\}$, once $t \in (0, \varepsilon_0)$, and since $|\nabla y_n(x)| \to |\nabla y(x)|$ uniformly in any compact subset of Ω and $\int_0^\varepsilon (\int_{\{u=s, \nabla y(x) \neq 0\}} \frac{d\mathcal{L}^{N-1}}{|\nabla y(x)|}) ds < \infty$ for any $\varepsilon < \varepsilon_0$, by the dominated convergence theorem we deduce that $\exists\, \bar{h} : \mathbb{R}_+ \to \mathbb{R}_+$ continuous, with $\bar{h}(0) = 0$, such that

$$h_n(\varepsilon) = \int_0^\varepsilon \left(\int_{\{u_n = s, \nabla y_n(x) \neq 0\}} \frac{d\mathcal{L}^{N-1}}{|\nabla y_n(x)|} \right) ds \leq \bar{h}(\varepsilon)$$

for all $\varepsilon < \varepsilon_0$ and for any n of this subsequence. □

We want to give now a simple example showing how the non-degeneracy condition (6) is, in some sense, optimal if we want to be able to have the continuous dependence of the support with respect to the data. In other words, we cannot expect it without a condition of type (6).

Example 1. Let us consider the one dimensional case

$$\begin{cases} -\varphi''(r) + \varphi^q(r) = u(r) & r \in (-2, 2), \\ \varphi(-2) = \varphi(2) = 0. \end{cases}$$

We set

$$\varphi_\epsilon(r) = \begin{cases} 0 & r \in (1, 2), \\ e^{-\frac{1}{1-r}} & r \in (1-\epsilon, 1), \\ C_2 - C_1 r^2 & r \in (0, 1-\epsilon), \end{cases}$$

and define $\varphi_\epsilon(r)$ by reflection on the interval $(-2, 0)$. The constants C_1 and C_2 have to be chosen so as to make $\varphi_\epsilon \in C^1(-2, 2)$, which means

$$C_1 = e^{-\frac{1}{\epsilon}} \frac{1}{2\epsilon^2(1-\epsilon)}, \quad C_2 = e^{-\frac{1}{\epsilon}} + e^{-\frac{1}{\epsilon}} \frac{1-\epsilon}{2\epsilon^2}.$$

We want to check now that these functions satisfy

$$-\varphi_\epsilon'' + \varphi_\epsilon^q \geq 0. \tag{14}$$

On the interval $[0, 1-\epsilon)$ the functions are concave and positive and the result follows. On $(1-\epsilon, 1)$ the behavior is the same of $e^{-\frac{1}{1-r}}$. In this case

$$-\varphi_\epsilon''(r) + \varphi_\epsilon^q(r) = e^{-\frac{1}{1-r}} \left[\frac{2}{(1-r)^3} - \frac{1}{(1-r)^4} \right] + e^{-\frac{q}{1-r}} \geq 0,$$

for $1 - r < 1 - r_0$ for some $r_0 > 0$. So if we take $\epsilon < r_0$ we obtain (14) on the whole interval. Now it easy to check that $u_\epsilon := -\varphi_\epsilon'' + \varphi_\epsilon^q \to 0$ uniformly as $\epsilon \downarrow 0$, and that $\varphi_\epsilon \to 0$. Nevertheless $S(\varphi_\epsilon) = (-1, 1)$, for any $\epsilon < r_0$, and so there is not continuous dependence of the free boundary. We point out that condition (11) is not satisfied by the family φ_ϵ.

Let us show now how we can prove that the support depends continuously (in measure) on u by using this kind of non-degeneracy property.

Lemma 2.5. *Let $\{y_n\}$ converge in $L^\infty(\Omega)$ to y. Suppose that the following non-degeneracy property holds uniformly for all $n \in \mathbb{N}$: there exist $\epsilon_0 > 0$ and $h : \mathbb{R}_+ \to \mathbb{R}_+$ with $lim_{t \to 0} h(t) = 0$ and*

$$|\{x \in \Omega \,:\, |y_n(x)| < \epsilon\}| \leq h(\epsilon) \quad \forall \epsilon < \epsilon_0. \tag{15}$$

Then $|N(y_n) \div N(y)| \to 0$, where $\div$ stands for the symmetric difference of two sets, i.e.

$$N(y_n) \div N(y) = (N(y_n) \setminus N(y)) \cup (N(y) \setminus N(y_n)).$$

Proof. Let us consider the case of $N(y) \setminus N(y_n) = N(y) \cap S(y_n)$, and let $\epsilon < \epsilon_0$. For n sufficiently large we know that $|y_n(x) - y(x)| < \epsilon$, for a.e. $x \in \Omega$, and hence $|y_n(x)| < \epsilon$ a.e on $N(y)$. But due to the non-degeneracy property (15) we have that $|\{|y_n < \epsilon|\}| \leq h(\epsilon)$, and so we conclude that

$$|N(y) \cap S(y_n)| \leq |\{|y_n| < \epsilon\}| < h(\epsilon) \quad \forall n \geq n(\epsilon).$$

Hence, letting $\epsilon \to 0$ and using the convergence to zero of h we obtain that $|N(y) \cap S(y_n)| \to 0$ as $n \to \infty$. The proof that $|N(y_n) \setminus N(y)|$ goes to zero follows by similar arguments. □

It is clear that we can separate the study of the continuous dependence in measure of the support in two different cases: a) we are just interested in the support of the solution restricted to a compact subset of Ω, or b) we are interested to its behavior on the whole Ω. The first case is simpler and we have already all the instruments to state a result. By the contrary, the second one needs some further hypotheses on the data which is what is assumed in Theorem 2.1.

Proof of Theorem 2.1. By well known results on the continuous dependence of the solution with respect to the data we obtain that $y_n \to y$ in $L^\infty(D)$. Combining Theorem 2.4 and Corollary 1, and applying Lemma 2.5 with $h_\infty(t) = \sup(\bar{h}(t), k\epsilon^{1/\beta})$, we obtain for a subsequence of y_n (still denoted with y_n) that $(N(y_n) \div N(y)) \cap D \to 0$ in measure. □

To handle the case with the whole Ω, we know that all the solutions of problem (1) related to the family of control $U_{\rm ad}$ satisfy $||y||_{L^\infty(\Omega)} \leq Y$ for some $Y > 0$ (take, for instance, $Y = M^{1/q}$). If we assume Y to be sufficiently small we can suppose that all the supports are contained in the same compact in Ω. In fact from Díaz [16] we have that if $||y||_{L^\infty(\Omega)} \leq Y$, then

$$N(y) \supset \{x \in N(u) \ : \ dis(x, S(u)) \geq \epsilon + W(\epsilon) \text{ for some } \epsilon > 0\}$$

where the constant $W(\epsilon)$ depends on the L^∞ norm of y. Of course this condition makes sense when

$$dis(\partial S(u), \partial\Omega) > \epsilon + W(\epsilon).$$

At this point we apply Theorem 2.1 with the set D given by the one which contains all the supports of the sequence of solutions and we obtain, in this way, the global continuity in measure of the support.

Now we are ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\{u_n\} \subset U_{\rm ad}$ be a minimizing sequence for J. As $U_{\rm ad}$ is bounded in $H^1(\omega) \cap L^\infty(\omega)$ there exists a subsequence (which we still denote with $\{u_n\}$) which converges weakly in $H^1(\omega)$ and weakly star in $L^\infty(\omega)$ to a function u, and hence (passing to another subsequence) also strongly in $L^2(\omega)$. Thus, from the convergence a.e. we obtain that u satisfies condition (6). Hence u belongs to $U_{\rm ad}$. We will check now the continuity of J with respect to the $L^2(\omega)$ norm. The function $\frac{1}{G(y)}$ is uniformly bounded for all $0 \leq y \in L^2(\Omega)$ because $G(0) > 0$ and G is increasing. So, using the Lebesgue dominate convergence theorem, the functional

$$u \mapsto \int_\Omega \frac{1}{G(y(x;u))}\, dx$$

is continuous in $L^2(\omega)$. From the previous results we already know that

$$\int_\Omega \chi_{S(y(x;u))\cap B}(x)\, dx$$

is continuous in $L^2(\omega)$ (again passing to a subsequence) and so, finally, we obtain that

$$J(u_n) \to J(u) = \min .$$

□

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