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Wavefronts for Discrete Two-Dimensional Nonlinear Diffusion Equations

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Abstract—Existence of wavefronts for discrete two-dimensional evolution models involving diffusive terms with a nonlinear dependence on the discrete 'gradients' is studied. For small values of a control parameter, we prove existence of steady wavefronts. We investigate numerically the transition of steady fronts to traveling waves as the control parameter increases.

Keywords—Discrete wavefronts, Discrete nonlinear diffusion equations, Two dimensions, Dislocations.

1. INTRODUCTION

In this paper, we investigate the existence of wave solutions for some two-dimensional discrete diffusion models depending nonlinearly on the 'discrete gradients'.

We will study a simple model for the evolution of edge dislocations in crystals (see [1,2] for details on the derivation and physics of these models)

$$\frac{\partial u_{i,j}}{\partial t} = u_{i-1,j} - 2u_{i,j} + u_{i+1,j} + A((\sin(u_{i,j-1} - u_{i,j}) + \sin(u_{i,j+1} - u_{i,j})). \tag{1}$$

Here, $u_{i,j}(t)$ represents the displacement of the atom (i,j) in the x direction and the parameter $A > 0$ measures the width of the dislocation core. Many models with a similar structure can be found in the physical literature, (see, for instance, [3,4]).

Real dislocations move only when the applied force is larger than a critical value. For small forces, dislocations remain stationary. We prove that model (1) admits stationary solutions corresponding to steady dislocations under a small (or zero) applied force in the x direction. When we increase the value of this shear force, numerical simulations show that the dislocation pattern moves at a certain speed in the x direction. This suggests the existence of discrete travelling waves for (1).

Most known results on wavefront solutions in diffusion systems refer to reaction diffusion systems. In these cases, fronts or pulses are trajectories in an appropriate phase space which join

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zeros of the nonlinear reactive source terms. For continuous systems, many analytical results are available: see [5] for traveling fronts for the heat equation with source term nonlinearities, [6] for traveling fronts in the FitzHugh Nagumo system, [7] for two-dimensional fronts in combustion systems .... For discrete systems, few analytical results are available and evidence of the existence of discrete traveling waves is mainly provided by numerical simulations. In [8], some criteria for pinning or propagation of fronts are stated for discrete one-dimensional heat equations with cubic nonlinearites. In [9], Zinner proves existence of traveling fronts for such equations. In [10], pinning and propagation phenomena are studied for discrete drift diffusion equations modeling semiconductor superlattices.

Whereas in reaction diffusion systems, source terms are responsible for the existence of nonlinear wavefronts, in (1), fronts arise due to the nonlinearity in the gradients. We are not aware of any previous results on wavefront existence for this kind of diffusion model.

The paper is organized as follows. In Section 2, we prove comparison principles and establish existence results for (1). In Section 3, we apply these results to the study of pinning and propagation phenomena when external shear forces are applied.

2. MAXIMUM PRINCIPLES AND GENERAL EXISTENCE RESULTS

Let us denote $l^2 = \{\gamma_{i,j} \in \mathbb{R} \mid \sum_{i,j\in\mathbf{Z}} |\gamma_{i,j}|^2 < \infty\}$. The following existence result holds.

THEOREM 2.1. *Let $\beta_{i,j}$ be such that*

$$\beta_{i+1,j} - 2\beta_{i,j} + \beta_{i-1,j} \in l^2, \qquad \sin(\beta_{i,j+1} - \beta_{i,j}) + \sin(\beta_{i,j-1} - \beta_{i,j}) \in l^2. \tag{2}$$

Let $\alpha_{i,j}$ be such that $\alpha_{i,j} = \beta_{i,j}$ when $|i|+|j| \geq k$, for some k, and $\alpha_{i,j}$ is bounded for $|i|+|j| < k$. Then, there is a unique global solution $u_{i,j}(t)$ to (1) such that $u_{i,j}(0) = \alpha_{i,j}$ and $u_{i,j}(t) - \beta_{i,j} \in l^2$, $\forall\, t > 0$.

PROOF. If $u_{i,j}$ is to be a solution of (1) with initial datum $\alpha_{i,j}$, then $v_{i,j} = u_{i,j} - \beta_{i,j}$ must satisfy $v_{i,j}(0) = \alpha_{i,j} - \beta_{i,j} \in l^2$ and

$$\begin{aligned} v'_{i,j} = {} & v_{i+1,j} - 2v_{i,j} + v_{i-1,j} + (\beta_{i+1,j} - 2\beta_{i,j} + \beta_{i-1,j}) + A\,(\sin(v_{i,j+1} - v_{i,j})\cos(\beta_{i,j+1} - \beta_{i,j}) \\ & + \sin(v_{i,j-1} - v_{i,j})\cos(\beta_{i,j-1} - \beta_{i,j}) + (\cos(v_{i,j+1} - v_{i,j}) - 1)\sin(\beta_{i,j+1} - \beta_{i,j}) \\ & + (\cos(v_{i,j-1} - v_{i,j}) - 1)\sin(\beta_{i,j-1} - \beta_{i,j})\sin(\beta_{i,j+1} - \beta_{i,j}) + \sin(\beta_{i,j-1} - \beta_{i,j}))\,. \end{aligned} \tag{3}$$

This is an initial value problem in l^2 with a Lispchitz nonlinearity in l^2. Thus, a unique global solution $v_{i,j}(t) \in l^2$ exists and $u_{i,j} = v_{i,j} + \beta_{i,j}$ is a solution of (1) with $u_{i,j}(t) - \beta_{i,j} \in l^2$. ∎

For solutions $u_{i,j}$ of (1) such that $(u_{i,j+1} - u_{i,j})(t)$ takes values in the intervals where the function $\sin(x)$ is increasing, that is

$$(u_{i,j+1} - u_{i,j})(t) \in \bigcup_{n\in\mathbf{Z}} \left[-\frac{\pi}{2} + 2n\pi,\ \frac{\pi}{2} + 2n\pi\right], \qquad \forall i,j \in \mathbb{Z}, \quad t \geq 0, \tag{4}$$

we prove energy identities and maximum principles. Existence of steady solutions is as follows.

LEMMA 2.1. *Let $u_{i,j}(t)$ be a solution to (1) with $\alpha_{i,j}$,$\beta_{i,j}$ as in Theorem 2.1 and such that (4) holds. As $t \to \infty$, $u_{i,j}(t)$ tends to a limit $s_{i,j}$ which is a steady solution to (1).*

PROOF. Define $w_{i,j}(t) = u_{i,j}(t+\tau) - u_{i,j}(t)$ for some $\tau > 0$. Then,

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}\sum_{i,j} |w_{i,j}(t)|^2\right) = & -\sum_{i,j}((w_{i+1,j} - w_{i,j})(t))^2 - \sum_{i,j}(\sin((u_{i,j+1} - u_{i,j})(t+\tau)) \\ & - \sin((u_{i,j+1} - u_{i,j})(t)))((u_{i,j+1} - u_{i,j})(t+\tau)) - (u_{i,j+1} - u_{i,j})(t)) \leq 0. \end{aligned} \tag{5}$$

This implies $w_{i,j}(t) \longrightarrow 0$ as $t \to \infty$ for every i,j. In conclusion, $u_{i,j}(t)$ tends to a limit $s_{i,j}$ as $t \to \infty$ for every i,j and $s_{i,j}$ is a steady solution to (1). ∎

To ensure the existence of nontrivial steady solutions, we need to control the behavior for i,j large. We can do so by means of sub and supersolutions.

LEMMA 2.2. *Let $u_{i,j}(t)$ be a solution to (1) satisfying (4). Let us assume that a subsolution $\underline{u}_{i,j}(t)$ and a supersolution $\overline{u}_{i,j}(t)$ for (1) satisfying (4) exist*

$$\underline{u}'_{i,j} \le \underline{u}_{i+1,j} - 2\underline{u}_{i,j} + \underline{u}_{i-1,j} + A\left(\sin\left(\underline{u}_{i,j+1} - \underline{u}_{i,j}\right) + \sin\left(\underline{u}_{i,j-1} - \underline{u}_{i,j}\right)\right), \tag{6}$$

$$\overline{u}'_{i,j} \ge \overline{u}_{i+1,j} - 2\overline{u}_{i,j} + \overline{u}_{i-1,j} + A\left(\sin\left(\overline{u}_{i,j+1} - \overline{u}_{i,j}\right) + \sin\left(\overline{u}_{i,j-1} - \overline{u}_{i,j}\right)\right), \tag{7}$$

such that $\underline{u}_{i,j}(t) < \overline{u}_{i,j}(t)$ and

$$\underline{u}_{i,j}(0) < u_{i,j}(0) < \overline{u}_{i,j}(0), \qquad \text{when } |i| \le N, \quad |j| \le N, \tag{8}$$

$$\underline{u}_{i,j}(t) < u_{i,j}(t) < \overline{u}_{i,j}(t), \quad \forall t, \qquad \text{when } |i| = N, \quad |j| \le N, \quad \text{and} \quad |j| = N, \quad |i| \le N, \tag{9}$$

for some $N \in \mathbb{N}$. Then

$$\underline{u}_{i,j}(t) < u_{i,j}(t) < \overline{u}_{i,j}(t), \qquad \forall t > 0, \quad |i| \le N, \quad |j| \le N. \tag{10}$$

PROOF. Set $h_{i,j} = u_{i,j} - \underline{u}_{i,j}$. For some $a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j} > 0$, we have

$$h'_{i,j} \ge a_{i,j}(h_{i+1,j} - h_{i,j}) + b_{i,j}(h_{i-1,j} - h_{i,j}) + c_{i,j}(h_{i,j+1} - h_{i,j}) + d_{i,j}(h_{i,j-1} - h_{i,j}),$$

since we work in the region where sinus function is increasing. Thus, $h_{i,j}$ attains its minimum in $\{|i| \le N\} \times \{|j| \le N\} \times \{0 \le t \le T\}$ at the 'parabolic boundary' $\{t = 0\} \cup \{|i| = N\} \cup \{|j| = N\}$, where $h_{i,j} \ge 0$. On the contrary, let us assume that the minimum is attained at $(i_0, j_0, t_0) \in \{|i| < N\} \times \{|j| < N\} \times \{0 < t \le T\}$. Then, $h'_{i_0,j_0}(t_0) \le 0$ and

$$\begin{aligned} &a_{i_0,j_0}(h_{i_0+1,j_0} - h_{i_0,j_0}) + b_{i_0,j_0}(h_{i_0-1,j_0} - h_{i_0,j_0}) \\ &+ c_{i_0,j_0}(h_{i_0,j_0+1} - h_{i_0,j_0}) + d_{i_0,j_0}(h_{i_0,j_0-1} - h_{i_0,j_0}) \ge 0. \end{aligned} \tag{11}$$

If inequality (11) is strict, we have a contradiction. Otherwise, $h_{i_0,j_0+1} = h_{i_0,j_0} = h_{i_0,j_0-1} = h_{i_0+1,j_0} = h_{i_0-1,j_0}$ and we conclude that $h_{i,j}(t)$ is constant. As a consequence, $u_{i,j}(t) > \underline{u}_{i,j}(t)$. In an analogous way, we get $u_{i,j}(t) < \overline{u}_{i,j}(t)$. ∎

REMARK 2.1. Lemma 2.2 extends to infinite ($L = \mathbb{Z}^2$) or exterior ($L = \mathbb{Z}^2 - \{|i| \le N\} \times \{|j| \le N\}$) lattices if (8) holds for all $(i,j) \in L$ and (9) is replaced (respectively, supplemented) with $\underline{u}_{i,j}(t) < u_{i,j}(t) < \overline{u}_{i,j}(t)$ for $|i| + |j| \longrightarrow \infty$, for all $t > 0$. It is enough to apply Lemma 2.2 for finite lattices and let the size of the lattice tend to infinity.

THEOREM 2.2. *Let $u_{i,j}(t)$ be the solution of (1) constructed in Theorem 2.1. Let us assume that a steady subsolution $\underline{u}_{i,j}$ and a steady supersolution $\overline{u}_{i,j}$ for (1) are known such that $\underline{u}_{i,j} < \overline{u}_{i,j}$ and (8) holds. Let us also assume that (4) holds for $u_{i,j}(t)$, $\underline{u}_{i,j}$, $\overline{u}_{i,j}$, and $\underline{u}_{i,j} < \beta_{i,j} < \overline{u}_{i,j}$ for $|i| + |j|$ large and $\beta_{i,j}$ satisfying (2). Then,*

$$\underline{u}_{i,j} < u_{i,j}(t) < \overline{u}_{i,j}, \qquad \forall t > 0. \tag{12}$$

The limiting steady solution $s_{i,j}$ to (1) satisfies $\overline{u}_{i,j} \ge s_{i,j} \ge \underline{u}_{i,j}$. Moreover, if

$$\underline{u}_{i,j} \longrightarrow \beta_{i,j}, \ \overline{u}_{i,j} \longrightarrow \beta_{i,j} \text{ uniformly as } |i| + |j| \longrightarrow \infty, \tag{13}$$

then, $s_{i,j} \longrightarrow \beta_{i,j}$ as $|i| + |j| \longrightarrow \infty$.

PROOF. To get the bound in terms of sub and supersolutions, we solve (1) in finite lattices $L^N = \{|i| \le N\} \times \{|j| \le N\}$, with N large. Let $u^N_{i,j}$ be the solution of (1) in L^N with initial data $\alpha_{i,j}$ and boundary conditions $u^N_{i,j} = \beta_{i,j}$ for $|i| = N$ or $|j| = N$. By Lemma 2.2, $\underline{u}_{i,j} < u^N_{i,j}(t) < \overline{u}_{i,j}$, for $\{|i| < N\} \times \{|j| < N\}$ and $t > 0$. Thus, $|u^N_{i,j}(t)| < K_{i,j}$, $|\frac{du^N_{i,j}}{dt}(t)| < K_{i,j}$ for $t > 0$, $N \in \mathbb{N}$, $i, j \in \mathbb{Z}$ and some $K_{i,j} > 0$ independent of N, t. For (i,j) fixed, the sequence $u^N_{i,j}$

is equicontinuous in $t \in [0,T]$ for all $T > 0$ and $\{u_{i,j}^N(t)\}_{N\in\mathbb{N}}$ is relatively compact for every t. By Ascoli-Arzela's theorem, we extract a subsequence $u_{i,j}^{N_k(i,j)}(t)$ tending pointwise to some $u_{i,j}(t)$ as $N_k(i,j) \to \infty$. To get a uniformly valid N_k, we use a diagonal extraction procedure. The set $\mathbb{N}\times\mathbb{N}$ is countable. Thus, we replace the set of indexes $(i,j) \in \mathbb{N}\times\mathbb{N}$ with $m \in \mathbb{N}$ and we have a subsequence $u_m^{N_k(m)}(t)$ tending to $u_m(t)$ as $N_k(m) \to \infty$. We take each $N_k(m)$ to be a subsequence of $N_k(m-1)$. Choosing the diagonal sequence $N_k(k)$, we get $u_{i,j}^{N_k(k)}(t) \to u_{i,j}(t)$ as $N_k(k) \to \infty$. This limit $u_{i,j}$ is the solution to (1) constructed in Theorem 2.1 and satisfies $\underline{u}_{i,j} < u_{i,j}(t) < \overline{u}_{i,j}$ for all t. Note that (13) implies that for any $\epsilon > 0$, one can choose $M(\epsilon)$ large enough to have $|u_{i,j}^N(t) - \beta_{i,j}| < \epsilon$ uniformly in $|i| + |j| > M(\epsilon)$, $N > M(\epsilon)$, $t > 0$. As in Lemma 2.1, we pass to the limit as $t \to \infty$ and obtain the results for the steady solution $s_{i,j}$. ∎

We want to solve (1) with initial data behaving at infinity like the elastic far field of an edge dislocation: the angle function, in our context. We choose the branch θ with values in $[0, 2\pi)$ and set $\theta_{i,j}^A = \theta(i + 0.5, (j + 0.5)/\sqrt{A})$. $\beta_{i,j}(A,F) = \theta_{i,j}^A + Fj$ satisfies (2). Physical restrictions impose that A is a parameter of order one, say $A \in (0.1, 10)$. When $A > 1$ and F is small,

$$\beta_{i,j+1}(A,F) - \beta_{i,j}(A,F) \in \left(-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right), \qquad \text{if } j \neq -1 \text{ or } j = -1, \quad i \leq 0, \tag{14}$$

$$\beta_{i,j+1}(A,F) - \beta_{i,j}(A,F) + 2\pi \in \left(-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right), \qquad \text{if } j = -1, \qquad i > 0, \tag{15}$$

for some $\delta > 0$ and we choose $\alpha_{i,j}(A,F) = \beta_{i,j}(A,F)$. When $A \leq 1$, (14),(15) fail at a few central points and we slightly modify $\beta_{i,j}(A,F)$ in that region to obtain an initial data $\alpha_{i,j}(A,F)$ satisfying (14),(15) everywhere, for F small. For such initial data, we are able to prove that $(u_{i,j+1} - u_{i,j})(t)$ takes values in the region where the sinus function is increasing and all the previous results in this section apply. The main difficulties come from the fact that our data generate a jump from 0 to 2π at $j = 0$. The fact that $\sin x$ is periodic allows us to overcome this difficulty.

LEMMA 2.3. *Let $u_{i,j}(t)$ be a solution of (1) with initial data $\alpha_{i,j}(A,F)$ defined above. Then $u_{i,j}(t)$ satisfies (14),(15) for every $t > 0$.*

PROOF. Our $\alpha_{i,j}(A,F)$ has been selected ensuring that the hypotheses in Theorem 2.1 and conditions (14),(15) hold. Thus, a solution $u_{i,j}$ to (1) with $u_{i,j}(t) - \beta_{i,j}(A,F) \in l^2$ exist. The equations for $w_{i,j} = u_{i,j+1} - u_{i,j}$ are

$$\begin{aligned} w_{i,j}' &= (w_{i+1,j} - w_{i,j}) + (w_{i-1,j} - w_{i,j}) + A\left((\sin(w_{i,j+1}) - \sin(w_{i,j}))\right.\\ &\quad \left.+(\sin(w_{i,j-1}) - \sin(w_{i,j}))\right), \\ w_{i,j}(0) &= \alpha_{i,j+1}(A,F) - \alpha_{i,j}(A,F). \end{aligned} \tag{16}$$

Set $I = (-\pi/2 + \delta, \pi/2 - \delta)$. We need to check

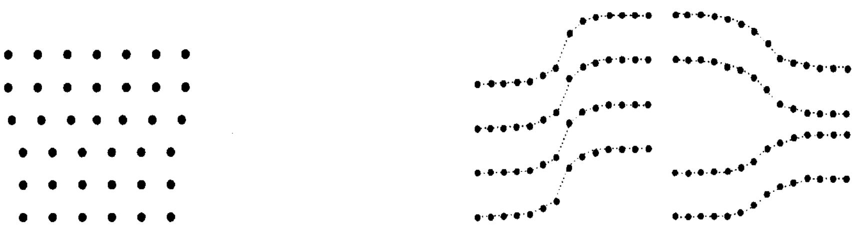
$$w_{i,-1} + 2\pi \in I, \qquad \text{if } i > 0, \quad w_{i,j} \in I \text{ elsewhere.} \tag{17}$$

Assume that at time $t_0 > 0$, either the value $\pi/2 - \delta$ or $-\pi/2 + \delta$ is attained for the first time at a point (i_0, j_0) which must, therefore, be a maximum or a minimum, except for $j_0 = -1, 0$, $i_0 \geq 0$. For simplicity, we assume that the neighbours satisfy (17). To fix ideas, suppose that the conditions fail at (i_0, j_0) because the value $\pi/2 - \delta$ is attained. If $j_0 \neq 0$ and $j_0 \neq -1$, (16) implies $w_{i_0,j_0}' < 0$. The same holds for $j_0 = 0$ or $j_0 = -1$ when $i_0 \leq 0$. If $j_0 = 0$ with $i_0 > 0$, $w_{i_0,-1}$ enters the equation through the sinus and (17) implies $w_{i_0,j_0}' < 0$. When $j_0 = -1$ with $i_0 > 1$, (17) implies again $w_{i_0,j_0}' < 0$. Thus, (17) cannot fail at those points. Let us assume that (17) hold everywhere for $t \leq t_0$ except at the point $i_0 = 1$, $j_0 = -1$ at $t = t_0$. Then, energy identities and comparison principles analogous to those in Lemmas 2.1 and 2.2 hold. For (i,j) on the boundary of $L^N = \{|i| \leq N \times |j| \leq N\}$, $N > 2$ fixed, and $t \in [t, t_0]$, $m_{i,j} \leq w_{i,j}(t) \leq$

$M_{i,j}$ with $m_{i,j}, M_{i,j}$ taking values in the region where the sinus function is increasing. We construct sub and supersolutions for (16) in L^N of the form $\alpha_{i,j+1}(A,F) - \alpha_{i,j}(A,F) \pm \epsilon(|i|^n + 1)$ for $\epsilon > 0$ small and an integer n to be adequately selected depending on A and ϵ. Then, $|w_{1,-1}(t) - (\alpha_{1,0}(A,F) - \alpha_{1,-1}(A,F))| \leq 2\epsilon$ for $t \leq t_0$. For ϵ small enough, this prevents (17) from failing at t_0.

3. PINNING AND PROPAGATION PHENOMENA

In this section, we generate steady solutions to (1) representing steady edge dislocations (see Figure 1a).



(a) Schematic edge dislocation. (b) Left, usual two dimensional.

Figure 1.

We are interested in fronts with the following structure: for $j \geq 0$, $u_{i,j}$ decreases from π to 0 as i goes from $-\infty$ to ∞ but, for $j < 0$, $u_{i,j}$ increases from π to 2π as i goes from $-\infty$ to ∞. Note that in reaction diffusion two-dimensional equations, one usually looks for fronts which look like a collection of similar kinks, see Figure 1b.

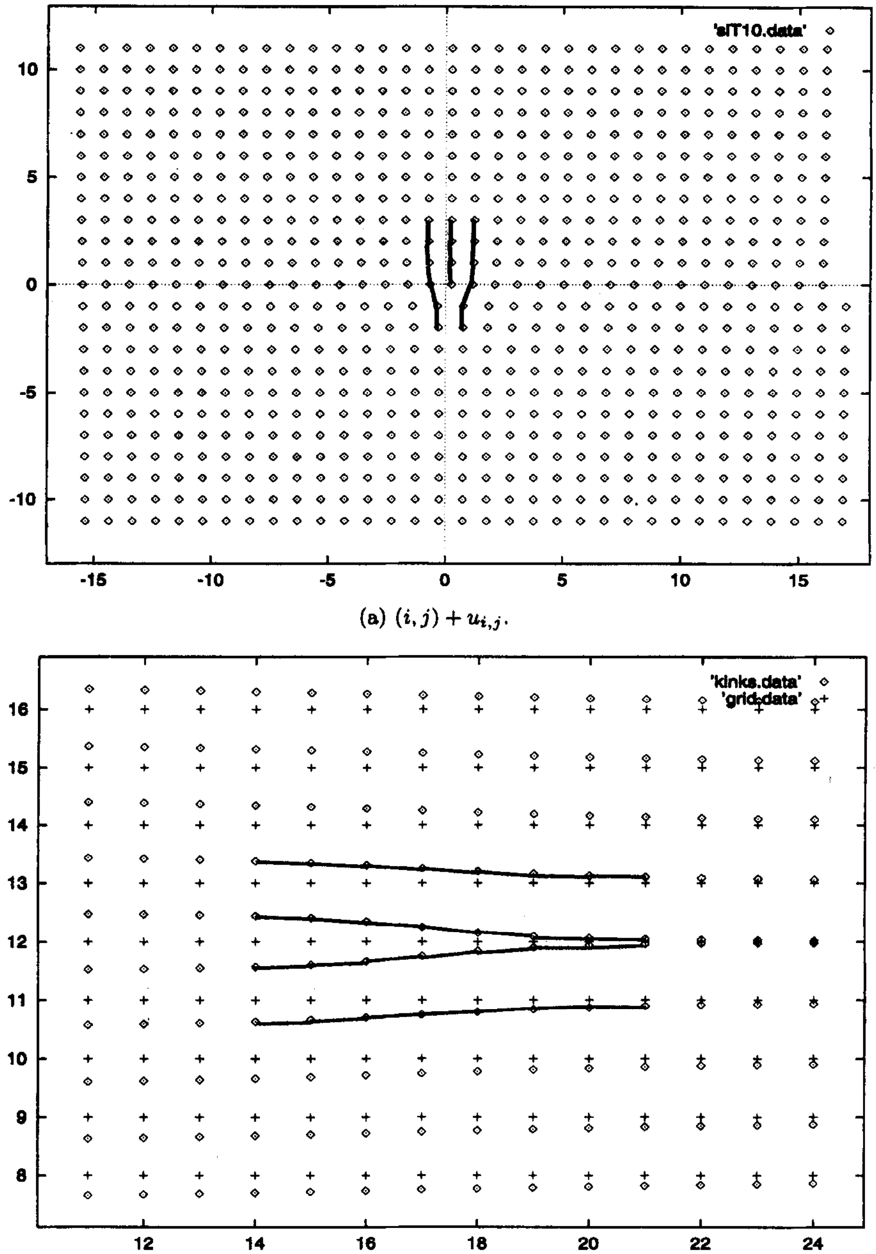
To generate our fronts, we solve the evolution problem (1) with initial conditions $\alpha_{i,j}(A,F)$ defined in the previous section. $\theta + Fj$ is the elastic far field corresponding to an edge dislocation bearing an applied shear force in the x direction. This continuous field has a singularity in the gradients, which is smoothed out by the discrete evolution model. Steady front solutions are expected for F small (corresponding to steady edge dislocations) and travelling wave solutions for F large (corresponding to moving dislocation). The corresponding long time limits are trivial piecewise constant solutions when the profile $u_{i,j}(t)$ propagates in the x direction. To avoid propagation, we need to find convenient sub and supersolutions, which guarantee a nontrivial behavior at infinity. For $F = 0$ or F small, functions of the form

$$\underline{u}^{\epsilon}_{i,j} = \alpha_{i,j}(A,F) - \epsilon\left(1 - \frac{1}{\sqrt{1+i^2+j^2}}\right), \qquad \overline{u}^{\epsilon}_{i,j} = \alpha_{i,j}(A,F) + \epsilon\left(1 - \frac{1}{\sqrt{1+i^2+j^2}}\right) \tag{18}$$

satisfy (6),(7) for $\{i^2 + j^2 \geq N\}$ provided $\epsilon \geq \epsilon(N)$ with $\epsilon(N) \sim 1/N^2$. By slightly modifying the values near $i = 0$, $j = 0$, we get global sub and supersolutions $\underline{u}_{i,j}$, $\overline{u}_{i,j}$ with $\epsilon = 1$. By Theorem 2.2, the solution $u_{i,j}(t)$ of (1) with initial data $\alpha_{i,j}(A,F)$ satisfies (12) and its long time limit $s_{i,j}$ is a steady solution to (1) satisfying $\overline{u}_{i,j} \geq s_{i,j} \geq \underline{u}_{i,j}$. Such sub and supersolutions block propagation and keep the distortion in the lattice in a fixed position. Moreover, for N large and $\{i^2 + j^2 \geq N\}$, $\overline{u}^{\epsilon}_{i,j} \geq s_{i,j} \geq \underline{u}^{\epsilon}_{i,j}$ with $\epsilon \sim 1/N^2$. Therefore, for F small, steady front solutions to (1) exist which can be identified with steady edge dislocations under zero or small shear force in the x direction.

Figure 2b shows a numerical simulation of the steady fronts generated by this procedure when $A = 0.3$ and $F = 0$. Figure 2a shows the corresponding distortion (edge dislocation) in the lattice

When $F \neq 0$, $\beta_{i,j}(A,F)$ is unbounded, but the differences (which are the quantities entering the equations) are just slightly perturbed: $\beta_{i+1,j}(A,F) - \beta_{i,j}(A,F)$ do not depend on F,



(a) $(i,j) + u_{i,j}$.

(b) $u_{i,j}$.

Figure 2.

$\beta_{i,j+1}(A,F) - \beta_{i,j}(A,F) = \theta^A_{i,j+1} - \theta^A_{i,j} + F$. We can solve (1) with such unbounded behavior at infinity or, equivalently, write $u_{i,j} = v_{i,j} + Fj$ and solve

$$v'_{i,j} = v_{i-1,j} - 2v_{i,j} + v_{i+1,j} + A(\sin(v_{i,j-1} - v_{i,j} - F) + \sin(v_{i,j+1} - v_{i,j} + F)) \tag{19}$$

with bounded data. Now, the applied force enters the equation as a small parameter. We choose this second procedure for numerical computations. Numerical simulations of (19) in finite lattices taking as initial data the steady solution for $F = 0$ and boundary condition $\beta_{i,j}(A,F)$ show that for F small, the dislocation in Figure 2 remains steady. For F large enough (depending on A, but not on the size of the lattice), the 'dislocation' glides to the right ($F > 0$) or left ($F < 0$) at a certain speed. The profiles for $u_{i,j}$, j fixed, are increasing kinks when j is negative and decreasing kinks when j is positive. When $F > 0$ is large enough, the profiles in Figure 2b move to the right with a definite speed, uniformly in j. This suggests the existence of travelling waves. When $F < 0$ is large enough, the profiles move uniformly to the left.

As a final remark, we emphasize that real lattices are finite, as the lattices used for simulations. The interest of studying analytically the dynamics of dislocations in infinite lattices comes from the possibility of identifying dislocations with wavefronts and then formulating more complex problems of dislocation interaction as wavefront interaction problems.

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