Comm. P.D.E. 19(5-6)
827-872 (1994)

ASYMPTOTIC BEHAVIOR FOR THE VORTICITY EQUATIONS IN DIMENSIONS TWO AND THREE

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Abstract

We establish the selfsimilar behavior of the solutions of the two and three dimensional vorticity equations for some classes of initial data. More precisely, any solution v of the two dimensional vorticity equation taking as initial data a finite Radon measure v_0 is shown to be asymptotically equivalent to the fundamental solution of the heat equation with mass $M = \int_{\mathbb{R}^2} v_0$ provided that $|M|$ is small enough, in the following sense:

$$t^{1-\frac{1}{p}} \|v(t) - MG(t)\|_{L^p(\mathbb{R}^2)} \to 0$$

when t tends to ∞ for all $1 \leq p \leq \infty$. This extends a result due to Giga and Kambe where the total variation of v_0 was assumed to be small.

If v is a solution of the three dimensional vorticity equation with initial data v_0 in the Morrey space $(\overline{M}^{\frac{3}{2}}(\mathbb{R}^3))^3$ with zero divergence and small norm, such that $\lambda^2 v_0(\lambda x) \to \mu$ in the sense of measures as $\lambda \to \infty$ and $\|v_0\|_{\overline{M}^{\frac{3}{2}}(|x|>R)} \to 0$ when $R \to \infty$ then

$$\lim_{t\to\infty} t^{1-\frac{3}{2p}} \|v(t) - \nu(t)\|_p = 0$$

for all $\frac{3}{2} \leq p \leq \infty$, where ν is the unique solution of the vorticity equation with initial data μ, which has been proved to be selfsimilar by Giga and Miyakawa.

AMS Classification: 35B40, 35K05, 35K55, 76Rxx, 35Q35, 76D05.

KEY WORDS: vorticity equations, measure data, aymptotic behavior, heat equation, selfsimilar solutions, Morrey spaces, Navier-Stokes equations.

ASYMPTOTIC BEHAVIOR FOR THE VORTICITY EQUATIONS
IN DIMENSIONS TWO AND THREE

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0. INTRODUCTION

We are concerned with the study of the asymptotic behavior as $t \to \infty$ of the solutions of the vorticity equations in dimensions two and three. We shall assume that we deal with an incompressible viscous fluid filling the whole space $\mathbb{R}^n$ whose density ρ and kinematical viscosity ν are equal to one. Noting by u the velocity of the fluid and by ∂_i the partial derivative with respect to x_i, the equations for the vorticity $v = curl\ u$ read:

$$(V2) \qquad \begin{cases} v_t - \Delta v + u^i \partial_i v = 0 & \text{in } \mathbb{R}^2 \times \mathbb{R}^+ \\ v(x,0) = v_0 & \text{in } \mathbb{R}^2 \end{cases}$$

in two dimensions and

$$(V3) \qquad \begin{cases} v_t - \Delta v + \partial_i(u^i v - v^i u) = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}^+ \\ div\ v = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}^+ \\ v(x,0) = v_0,\ div\ v_0 = 0 & \text{in } \mathbb{R}^3 \end{cases}$$

in dimension 3, with $v = (v^1, v^2, v^3)$. Let us remark that the condition $div\ v = 0$ disappears when $n = 2$ since v is a scalar.

We can express here the velocity as a function of the vorticity by means of the Biot-Savart laws:

$$u(x,t) = K * v(x,t) = \begin{cases} \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(-y_2, y_1)}{|y|^2} v(x-y,t) dy & \text{if } n = 2 \\ \frac{-1}{4\pi} \int_{\mathbb{R}^3} \frac{(y_1, y_2, y_3)}{|y|^3} \times v(x-y,t) dy & \text{if } n = 3 \end{cases}$$

where

$$K(x) = \begin{cases} \frac{1}{2\pi} \frac{(-x_2, x_1)}{|x|^2} & = (K^1, K^2) & \text{if } n = 2 \\ \frac{-1}{4\pi} \frac{(x_1, x_2, x_3)}{|x|^3} & = (K^1, K^2, K^3) & \text{if } n = 3 \end{cases}$$

so that, in fact, we are dealing with integral-differential equations.

Let us first consider the case $n = 2$. By integrating the equation we see that the mass $\int_{\mathbb{R}^2} v(t)$ does not change with time, so that a natural setting to study this equation is furnished by $L^1(\mathbb{R}^2)$ or even $M(\mathbb{R}^2)$, the space of finite Radon measures. The most general results on existence and uniqueness of solutions with initial data in this spaces have been obtained in [7] (where further references on the subject may be found). When $v_0 \in M(\mathbb{R}^2)$ there exists a solution v of $(V2)$ which satisfies the following estimates:

$$\|v(t)\|_{L^p(\mathbb{R}^2)} \ \leq \ C(|v_0|, p) t^{-1+\frac{1}{p}} \quad t > 0$$

for all $p \in [1, \infty]$, where C is a constant depending on p and the total variation of the data $|v_0|$. The uniqueness of the solution is guaranteed if the atomic part of v_0 is small enough. In particular, we have uniqueness provided that $v_0 \in L^1(\mathbb{R}^2)$ (see [7]). It is equally known (see [4]) that if the total variation of the initial data $|v_0|$ is small enough, then

$$t^{-1+\frac{1}{p}} \|v(t) - MG(t)\|_{L^p} \to 0 \quad \text{as } t \to \infty$$

for $1 \leq p \leq \infty$, where $G(t)$ denotes the fundamental solution of the heat equation and $M = \int dv_0$ the mass of the initial data.

This restriction on the data appears to be conected to the technique, which consists in estimating the decay by using the integral equation. In section 1 we shall obtain the same result by only assuming that the fundamental solution of $(V2)$ with initial data $M\delta$ is unique, which is true at least when $|M|$ is small. Uniqueness for large mass data remains an open question.

Our proof relies on the invariance of the equation under the scaling transformation

$$v_\lambda(x,t) = \lambda^2 v(\lambda x, \lambda^2 t) \quad \lambda > 0$$

and the fact that $w = MG(t)$ is a fundamental selfsimilar solution of (V2), that is, a solution satisfying

$$w_\lambda(x,t) = w(x,t) \quad \forall \lambda > 0$$

with initial data $M\delta$. To prove it, it suffices to remark that the nonlinear term vanishes on radial functions. Observe that

$$\|v_\lambda(1)\|_{L^p(\mathbb{R}^2)} = \lambda^{2(1-\frac{1}{p})}\|v(\lambda^2)\|_{L^p(\mathbb{R}^2)}$$

so that the estimates on the decay of $\|v(t)\|_{L^p(\mathbb{R}^2)}$ and the boundedness of $\|v_\lambda(1)\|_{L^p(\mathbb{R}^2)}$ are equivalent. The invariance of the equation under these transformations means that all the terms in the equation decay at the same speed when $t \to \infty$. The fact that when $t \to \infty$ the nonlinear term disappears and the solutions behave like the solutions of the linear heat equation with the same initial data is a consequence of the solutions becaming radial at infinity.

The idea of the proof is the following. Since the initial data $\lambda^2 v_0(\lambda x)$ converge to $M\delta$ in the weak sense of measures as $\lambda \to \infty$, in case the solution w of (V2) with initial data $M\delta$ is unique one can hope the whole family $v_\lambda(t)$ to converge to $w = MG(t)$.

In view of the identity

$$\|v_\lambda(1) - w(1)\|_{L^p(\mathbb{R}^2)} = \lambda^{2(1-\frac{1}{p})}\|v(\lambda^2) - w(\lambda^2)\|_{L^p(\mathbb{R}^2)} \to 0,$$

which holds for any selfsimilar w, our problem is reduced to getting on v_λ enough estimates to prove the strong convergence to $MG(t)$.

We get the following result :

Theorem 1

There exists a constant $A > 0$ such that, if $v_0 \in M(\mathbb{R}^2)$ and $|M| = |\int_{\mathbb{R}^2} v_0| < A$ the solution v of (V2) satisfies :

$$\lim_{t\to\infty} \; t^{1-\frac{1}{p}} \; \|v(t) - w(t)\|_{L^p} \; = \; 0$$

for every $1 \le p \le \infty$, where w is the unique solution of (F2):

$$(F2) \qquad \begin{cases} w_t - \Delta w + (K * w)\cdot \nabla w = 0 & in\ \mathbb{R}^2 \times \mathbb{R}^+ \\ w(x,0) = M\delta & in\ \mathbb{R}^2 \end{cases}$$

which is given explicitely by the formula :

$$w(x,t) = \frac{M}{4\pi t} e^{\frac{-|x|^2}{4t}}$$

We turn now to the case $n = 3$. A convenient framework for the study of the system $(V3)$ is provided by the Morrey spaces of measures, more precisely, by the space $(\overline{M}^{\frac{3}{2}}(\mathbb{R}^3))^3$. We define the Morrey space $\overline{M}^p(\mathbb{R}^3),\ 1 \le p \le \infty$ as being the space of Radon measures v such that:

$$\|v\|_p = \mathop{\mathrm{Sup}}_{x\in\mathbb{R}^3,\ r>0} \; r^{\frac{-3}{p'}} |v|(B(x,r)) \; < \; \infty$$

$\overline{M}^p$ is a Banach space endowed with the norm $\|v\|_p$. Since $L^1(\mathbb{R}^2) \subset \overline{M}^{\frac{3}{2}}(\mathbb{R}^3)$, the results can be particularized to bidimensional fluids by thinking of them as tridimensional plane fluids. On the other hand, several types of initial data, for instance the measures supported by curves (vortex rings, vortex filaments) belong precisely to this space.

In [6] the existence of global solutions for initial data $v_0 \in (\overline{M}^{\frac{3}{2}}(\mathbb{R}^3))^3$ with $div\ v_0 = 0$ such that is small $\|v_0\|_{\frac{3}{2}}$ enough is proved. Under some additional restriction on the data, we shall establish the selfsimilar behavior of such solutions when $t \to \infty$ by using the same scaling technique as in the case $n = 2$. Remark that the functions

$$v_\lambda(x,t) = \lambda^2 v(\lambda x, \lambda^2 t) \quad \lambda > 0$$

satisfy (V3) with initial data $\lambda^2 v_0(\lambda x)$. For a general $v_0 \in (\overline{M}^{\frac{3}{2}}(\mathbb{R}^3))^3$ the limit of the initial data when $\lambda \to \infty$ may fail to exist, so that we must assume the convergence to some limit μ. This limit satifies the homogeneity condition $\mu_\lambda = \mu$. The following technical condition

$$\|v_0\|_{\overline{M}^{\frac{3}{2}}(|x|>R)} \to 0$$

when $R \to \infty$ is also needed to ensure the convergence of v_λ to the selfsimilar solution ν of (V3) with data μ. Since $\mu_\lambda = \mu$ this solution turns out to be selfsimilar. Taking into account that

$$\|(v_\lambda - \nu)(1)\|_{M^p(\mathbb{R}^2)} = \lambda^{2(1-\frac{3}{2p})}\|(v-\nu)(\lambda^2)\|_{M^p(\mathbb{R}^2)}$$

we get the following result :

Theorem 2

There exists a constant $A > 0$ such that if v_0 satisfies the following conditions :

i) $v_0 \in (\overline{M}^{\frac{3}{2}}(\mathbb{R}^3))^3$, $\quad$ *div* $v_0 = 0$, $\quad \|v_0\|_{\frac{3}{2}} \leq A$,

ii) $\lambda^2 v_0(\lambda x) \to \mu$ *on balls in the weak topology of measures when* $\lambda \to \infty$,

iii) $\lambda^2 v_0(\lambda x)\chi_R(x) \to 0$ *in* $\overline{M}^{\frac{3}{2}}(\mathbb{R}^3)$ *when* $R \to \infty$ *uniformly with respect to* λ, *where* χ_R *stands for the characteristic function of* $B(0,R)^c$, $R > 0$,

then,

$$\lim_{t \to \infty} t^{1-\frac{3}{2p}} \|v(t) - \nu(t)\|_p = 0$$

for every $\frac{3}{2} \leq p \leq \infty$, *where* ν *is the only solution of (F3)*

$$(F3) \qquad \begin{cases} \nu_t - \Delta\nu + \partial_i((K*\nu)^i\nu - \nu^i(K*\nu)) = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}^+ \\ \text{div } \nu = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}^+ \\ \nu(x,0) = \mu & \text{in } \mathbb{R}^3 \end{cases}$$

For instance, if $v_0 \in (L^{\frac{3}{2}}(\mathbb{R}^3))^3$, then $\lambda^2 v_0(\lambda x) \rightharpoonup 0$ in $(M(\mathbb{R}^3))^3$. In these conditions we have :

$$\lim_{t\to\infty} t^{1-\frac{1}{p}} \|v(t)\|_{M^p} = 0$$

when $\frac{3}{2} \leq p \leq \infty$, provided that the $L^{\frac{3}{2}}$ norm of the initial data is small. Other examples will be given at the end of Section 2.

This paper is organized as follows. In Section 1 we deal with the two dimensional case. We first prove Theorem 1 and then obtain a more precise result for the case in which the mass of the initial data is zero. We use those results on the asymptotic behavior of the vorticity to get some information on the behavior of the solutions of Navier-Stokes when $t \to \infty$. Section 2 is devoted to the study of the three dimensional case. After proving Theorem 2 we give some exemples in order to make its meaning clear.

1. THE TWO DIMENSIONAL CASE

1.1 Initial data with zero mass

Let us consider the vorticity equation in two dimensions :

$$(V2) \qquad \begin{cases} v_t - \Delta v + u^i \partial_i v = 0 & \text{in } \mathbb{R}^2 \times \mathbb{R}^+ \\ v(x,0) = v_0 & \text{in } \mathbb{R}^2 \end{cases}$$

where the velocity vector $u = (u^1, u^2)$ is given by :

$$u(x,t) = K * v(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(-y_2, y_1)}{|y|^2} v(x-y,t)dy$$

with :

$$K(x) = \frac{1}{2\pi} \frac{(-x_2, x_1)}{|x|^2} = (K^1, K^2)$$

We note $u^i \partial_i v = \sum_{i=1}^{2} u^i \partial_i v$ with $\partial_i v = \frac{\partial\, v}{\partial x_i}$.

In the sequel we shall assume that $v_0 \in M(\mathbb{R}^2)$ where $M(\mathbb{R}^2)$ denotes the space of finite Radon measures, that is, the topologic dual of $C_0(\mathbb{R}^2)$, the space of continuous functions on $\mathbb{R}^2$ vanishing at infinity. $M(\mathbb{R}^2)$ is a Banach space endowed with the norm :

$$|v| = \sup_{\substack{\phi \in C_0(\mathbb{R}^2) \\ \|\phi\|_\infty \leq 1}} | \int_{\mathbb{R}^n} \phi(x) dv|$$

where $\int_{\mathbb{R}^n} \phi(x)dv = \langle v, \phi \rangle$ denotes the duality product. $|v|$ is called the total variation of the measure v.

It is known (see [11], vol 1, p.22, teor. 1.13) that any finite Radon measure v can be decomposed uniquely as: $v = v_c + v_a$ where v_c is the continuous part, which verifies $v_c(\{x\}) = 0$ for every $x \in \mathbb{R}^2$ and v_a is the atomic part, that is, v_a is a pure point measure of the form $v_a = \sum_{j=1}^\infty \alpha_j \delta_{x-x_j}$, $\alpha_j \in \mathbb{R}$, $x_j \in \mathbb{R}^2$. The continuous part can further be decomposed uniquely in the absolutely continuous part, which is a measure defined by a locally integrable function, and the continuous singular part. The continuous singular part and the atomic part together form the singular part of the measure, that is, a measure that vanishes on some set S such that S^c has Lebesgue measure zero.

When $v_0 \in M(\mathbb{R}^2)$ theorems 2.5, 4.2, 4.3 in [7] furnish the following existence result :

Proposition 1.1

For all $v_0 \in M(\mathbb{R}^2)$ there exists a global solution $v \in C((0,\infty); L^p(\mathbb{R}^2))$ of (V2) such that:

$$\begin{array}{lll}
\|v\|_p(t) \ \leq \ C(K) t^{-1+\frac{1}{p}} |v_0|, & t > 0, & 1 \leq p \leq \infty \\
\|u\|_p(t) \ \leq \ C(p,K) t^{\frac{-1}{2}+\frac{1}{p}} |v_0|, & t > 0, & 2 < p \leq \infty \\
\sup_{\varepsilon \leq s \leq T} \|\partial_x^\alpha \partial_t^k v\|_{L^p(\mathbb{R}^n)} < \infty, & \varepsilon, T > 0, & \alpha \in \mathbb{N}^2, k \in \mathbb{N}, 1 \leq p \leq \infty \\
\int_{\mathbb{R}^2} v(x,t) dx = \int_{\mathbb{R}^2} dv_0 & t > 0 &
\end{array}$$

where K is such that $|v_0| \leq K$. Therefore, v is a smooth function for $t > 0$. If $v_0 \in L^1(\mathbb{R}^2)$ the same estimates are valid replacing $|v_0|$ by $\|v_0\|_{L^1(\mathbb{R}^2)}$.

The initial condition is verified in the sense that $v(x,t)$ *converges to* v_0 *when* t *tends to zero in the weak topology of measures, that is,*

$$\lim_{t\to 0} \int_{\mathbb{R}^2} \phi(x)v(x,t) = \int_{\mathbb{R}^2} \phi(x)dv_0$$

for every ϕ *continuous and bounded in* $\mathbb{R}^2$.

Moreover, if the atomic part $v_{0,a}$ *of the measure* v_0 *is small (in the sense that* $|v_{0,a}|$ *is so), the solution* v *is unique in a certain class of functions (see remark below). In particular, if* $v_0 \in L^1(\mathbb{R}^2)$ *the solution is unique.*

Remark 1.1

The solutions of (V2) are obtained from solutions u *of the Navier-Stokes equations with initial velocity* $u_0 = K * v_0$ *by means of the formula* $v = curl\ u$. *All but the estimates on the* L^p *norms of the derivates when* $1 \leq p \leq 2$ *have been obtained in [7]. The remaining estimates follow from the equation.*

We note by $L^{2,\infty}$ *the Lorentz space of the measurable functions* f *such that*

$$\|f\|_{L^{2,\infty}} = \sup_{r>0} r(m\{x/|f(x)| > r\})^{\frac{1}{2}} < \infty$$

In view of the relation existing between u *and* v, *the uniqueness of* v *(which can be proved in a straightforward way) is an inmediate consequence of the following uniqueness result for Navier-Stokes (Theorem 4.5 of [7]):*

'If $v_0 = curl\ u_0 \in M(\mathbb{R}^2)$, $u_0 \in L^{2,\infty}$ *with* $div\ u_0 = 0$ *and the total variation of the atomic part of* v_0 *is small enough, the solution of the Navier-Stokes system is unique in the class of functions* w *such that:*

- $w : [0,\infty) \to L^{2,\infty}(\mathbb{R}^2)$ *is continuous in the weak star topology and* $w(.,0) = u_0$.

- $w : (0,\infty) \to L^p(\mathbb{R}^2)$ *is continuous and satisfies:*

$$\lim_{t\to 0} \sup t^{\frac{1}{2}-\frac{1}{p}} \|w\|_p(t) \leq C|v_{0,a}|$$

if $p > 2$, *where* C *is a constant depending on* p *and on a constant* m *such that* $|v_{0,a}| \leq m$.

- w *is a solution of the integral equation in* $L^{2,\infty}$.*'*

We deduce that v is the only function in

$$C_{weak}([0,\infty); M(\mathbb{R}^2)) \cap C((0,\infty); L^p(\mathbb{R}^2))$$

(for every $p \geq 1$) verifying (V2) in $M(\mathbb{R}^2)$ and

$$\limsup_{t\to 0} t^{1-\frac{1}{p}} \|v\|_p(t) \leq C|v_{0,a}|$$

for $p > 1$, provided that the atomic part of v_0 is small.

Let us assume that $v_0 \in L^1(\mathbb{R}^2)$. For any $\lambda > 0$ the rescaled function $v_\lambda = \lambda^2 v(\lambda x, \lambda^2 t)$ satisfies :

$$\begin{array}{ll} v_{\lambda,t} - \Delta v_\lambda + (K^i * v_\lambda)\partial_i v_\lambda = 0 & \text{in } \mathbb{R}^2 \times \mathbb{R}^+ \\ v_\lambda(x,0) = \lambda^2 v_0(\lambda x) = v_{0,\lambda} & \text{in } \mathbb{R}^2 \end{array}$$

or, equivalently,

$$\begin{aligned} v_\lambda(t) = &\quad G * v_{0,\lambda}(t) - \textstyle\int_0^t G(t-s) * ((K^i * v_\lambda)\partial_i v_\lambda)(s) ds \\ = &\quad G * v_{0,\lambda}(t) - \textstyle\int_0^t \partial_i G(t-s) * ((K^i * v_\lambda) v_\lambda)(s) ds \end{aligned}$$

where G denotes the kernel of the heat equation.

Let us remark that for $t > 0$ the solution is a function even if the initial data is a measure. Therefore, the above definition of v_λ makes sense for $t > 0$. The expression $v_\lambda(x,0) = \lambda^2 v_0(\lambda x) = v_{0,\lambda}$ is valid when v_0 is a function. In case v_0 is a measure the scaling is given by :

$$v_{0,\lambda}(E) \;=\; v_0(\lambda E)$$

for any measurable subset E of $\mathbb{R}^2$ or

$$\langle v_{0,\lambda}, \phi \rangle \;=\; \langle v_0, \phi^\lambda \rangle$$

where $\phi^\lambda(x) = \phi(\lambda^{-1} x)$ with $\phi \in BC(\mathbb{R}^2)$.

We must obtain on v_λ enough estimates to pass to the limit in the equation. These estimates follow from the integral equation satisfied for the solutions, thanks to some known estimates and some variants of the Gronwall lemma. Let us recall some known estimates on convolutions involving K and the heat kernel.

First, for every $1 \leq q \leq p \leq \infty$ and $a \in L^q(\mathbb{R}^n)$ we have that :

$$\begin{aligned} \|G(t) * a\|_{L^p} &\leq C\|a\|_{L^q} t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})} && t \geq 0 \\ \|\partial_i G(t) * a\|_{L^p} &\leq C\|a\|_{L^q} t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{2}} && t \geq 0,\ 1 \leq i \leq n \end{aligned}$$

In general,

$$\|\partial_\alpha G(t) * a\|_{L^p} \leq C\|a\|_{L^q} t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}} \quad t \geq 0, \quad \alpha \in \mathbb{R}^n,\ |\alpha| = k$$

The same estimates remain valid when a is a finite measure taking $q = 1$ and replacing the L^q norm of a by its total variation. When $a \in L^1(\mathbb{R}^n)$ and $M = \int_{\mathbb{R}^n} a$, it is also known that :

$$t^{\frac{n}{2}(1-\frac{1}{p})}\|G(t) * a - MG(t)\|_{L^p} \to 0 \quad t \to 0$$

On the other hand, since $K \in L^{2,\infty}$ we can apply the generalized Young inequality [TR, p.139] to conclude that :

$$\|K * v\|_{L^r} \leq \|K\|_{L^{2,\infty}}\|v\|_{L^p}$$

if $r > 2,\ 1 < p < 2,\ \frac{1}{r} = \frac{1}{p} - \frac{1}{2}$. As far as ∇K is concerned, it turns out to be a Calderon-Zygmund kernel so that

$$\|\nabla K * v\|_{L^p} \leq C_p\|v\|_{L^p}$$

when $1 < p < \infty$ (theorem 2, Ch. 2, [15]).

Finally, we state a Gronwall type lemma which is used to deduce estimates from the integral equation.

Lemma 1.1

Let $\phi \in C[0,T],\ T > 0$ be a positive function such that:

$$\phi(t) \ \leq\ (C_1 t^{-\alpha} + C_3)\ +\ C_2 \int_0^t (t-s)^{-\beta}\phi(s)ds$$

where $C_1, C_2, C_3 > 0$, $0 \leq \beta, \alpha < 1$. *Then :*

$$\phi(t) \ \leq \ Ct^{-\alpha} \quad 0 < t \leq T$$

for a constant C *depending on* $C_1, C_2, C_3, \alpha, \beta$ *and* T.

In particular, given a family ϕ_λ *satisfying the above inequality with the same* $C_1, C_2, C_3, \alpha, \beta$ *and* T *we obtain a uniform bound with respect to* λ.

Proof

The function $\psi(t) = \phi(t)t^\alpha$ satisfies :

$$\psi(t) \ \leq \ (C_1 + C_2 t^\alpha) \ + \ C_2 t^\alpha \int_0^t (t-s)^{-\beta} s^{-\alpha} \psi(s) ds$$

In the following we shall note by C sucessive constants depending on C_1, C_2, C_3 and T. By making the change of variables $s = \sigma t$ we obtain:

$$\psi(t) \ \leq \ C \ + \ C_2 t^{-\beta+1} \big(\int_0^1 (1-\sigma)^{-\beta} \sigma^{-\alpha} d\sigma \big) \delta(t)$$

where

$$\delta(t) = \sup_{0 \leq s \leq t} \ \psi(s)$$

This implies:

$$\delta(t) \leq M$$

if $t \leq t_0$, for some t_0 small enough, depending on C_2 and a positive constant M depending on C_1. Let $t_1 < t_0$. For any $t > t_1$ we have :

$$\begin{aligned} \psi(t) \ &\leq C + C_2 t^\alpha \left(\int_0^{t_1} (t-s)^{-\beta} s^{-\alpha} \psi(s) + \int_{t_1}^t (t-s)^{-\beta} s^{-\alpha} \psi(s) \right) \\ &\leq C + C_2 t^\alpha \left(M t^{-\alpha-\beta+1} B(-1+\alpha, -1+\beta) + t_1^{-\alpha} \int_{t_1}^t (t-s)^{-\beta} \psi(s) \right) \end{aligned}$$

where $B(x,y)$ stands for the beta function. Given $\varepsilon > 0$ we can write :

$$\begin{aligned} \psi(t) \ &\leq \ C + C_2 t^\alpha t_1^{-\alpha} \left(\int_{t_1}^{t-\varepsilon} (t-s)^{-\beta} \psi(s) + \int_{t-\varepsilon}^t (t-s)^{-\beta} \psi(s) \right) \\ &\leq \ C + C_2 t^\alpha t_1^{-\alpha} \left(\varepsilon^{-\beta} \int_{t_1}^t w(s) + \tfrac{\varepsilon^{-\beta+1}}{-\beta+1} w(t) \right) \end{aligned}$$

if $t \geq t_1 + \varepsilon$, where $w(t) = \sup_{t_1 \leq s \leq t} \psi(s)$. Choosing t_1 and ε in such a way that $t_2 = t_1 + \varepsilon < t_0$ we conclude that :

$$\int_{t_1}^{t} w(s) \ \leq \ \int_{t_2}^{t} w(s) + M\varepsilon$$

Therefore, if $w^*(t) = \sup_{t_2 \leq s \leq t} w(s)$ we obtain :

$$\psi(t) \ \leq \ C + C_2 T^\alpha t_1^{-\alpha} \varepsilon^{-\beta} \int_{t_2}^{t} w(s) + C_2 T^\alpha t_1^{-\alpha} \frac{\varepsilon^{-\beta+1}}{-\beta+1} w^*(t)$$

for $t_2 \leq t \leq T$. Choosing ε small enough we deduce that:

$$w^*(t) \ \leq \ C + C \int_{t_2}^{t} w^*(s)$$

By Gronwall's lemma :

$$w^*(t) \ \leq \ C$$

Taking into account the bound

$$\delta(t) \leq M$$

we conclude that

$$\phi(t) \ \leq \ C(C_1, C_2, C_3, T, \alpha, \beta)$$

Now, we are ready to prove the estimates we need:

Theorem 1.1 : Estimates

Let v the solution of (V2) with initial data $v_0 \in M^1(\mathbb{R}^2)$ given in Proposition 1.1 and let us define $v_\lambda = \lambda^2 v(\lambda x, \lambda^2 t)$. We have the following estimates :

(E_1)	$\|v_\lambda(t)\|_p \ \leq C_1(t,p)$	$1 \leq p \leq \infty$
(E_2)	$\|\nabla v_\lambda(t)\|_p \ \leq C_2(t,p)$	$1 \leq p \leq \infty$
(E_3)	$\|v_{\lambda,t}(t)\|_p \ \leq C_3(t,p)$	$1 \leq p \leq \infty$
(E_4)	$\|v_\lambda(t)\|_{L^1(\mathbb{R}^2 - B(0,R))} \to 0$	*when* $R \to \infty$

uniformly with respect to $\lambda \geq 1,\ t \in [0, t_1]\ t_1 > 0$

where $C_i(t,p),\ i = 1,2,3$ *denote positive constants depending on* t *in a continuous way.*

Proof

(E1) By Theorem 4.3 in [7] we know that :

$$\|v_\lambda(t)\|_{L^p(\mathbb{R}^2)} \ \leq\ C(\|v_\lambda(0)\|_{L^1(\mathbb{R}^2)}, p) t^{(-1+\frac{1}{p})}$$

for every $1 \leq p \leq \infty$. On the other hand,

$$\|v_\lambda(t)\|_{L^p(\mathbb{R}^2)} = \lambda^{2(1-\frac{1}{p})} \|v(\lambda^2 t)\|_{L^p(\mathbb{R}^2)}$$

In particular, $\|v_\lambda(0)\|_{L^1(\mathbb{R}^2)} = \|v(0)\|_{L^1(\mathbb{R}^2)}$. Therefore,

$$\|v_\lambda(t)\|_p \ \leq C(\|v(0)\|_{L^1(\mathbb{R}^2)}, p) t^{(-1+\frac{1}{p})} \quad 1 \leq p \leq \infty$$

(E2) For every $\tau > 0$ fixed, the functions v_λ satisfy the integral equation :

$$v_\lambda(t+\tau) = G(t) * v_\lambda(\tau) - \int_0^t G(t-s) * (\nabla v_\lambda(\tau+s) \cdot (v_\lambda * K(\tau+s)))ds$$

so that,

$$\nabla v_\lambda(t+\tau) = \nabla G(t) * v_\lambda(\tau) - \int_0^t \nabla G(t-s) * (\partial_i v_\lambda(s+\tau)\ (v_\lambda * K^i(\tau+s)))ds$$

where G denotes the fundamental solution of the heat equation.

Taking L^p norms and having into account that $\|v_\lambda(\tau)\|_1 = \|v(\lambda^2\tau)\|_1 \leq \|v_0\|_1$ (Proposition 3.4 in [7]) we obtain the inequality:

$$\begin{aligned} \|\nabla v_\lambda(t+\tau)\|_p \ \leq & \quad C\|v_0\|_1 t^{(-1+\frac{1}{p}-\frac{1}{2})} \\ + & \quad C \int_0^t (t-s)^{\frac{-1}{2}} \|\nabla v_\lambda(s+\tau)\|_p \|v_\lambda * K(\tau+s)\|_\infty ds \end{aligned}$$

It follows from Gagliardo-Nirenberg's inequalities [10] that:

$$\|v_\lambda * K\|_\infty \ \leq \ \|v_\lambda * K\|_r^{(1-\frac{2}{r})} \|v_\lambda * \nabla K\|_r^{\frac{2}{r}}$$

for $r > 2$. Since $K \in L^{2,\infty}$ the term $\|v_\lambda * K\|_r$ can be estimated by means of the generalized Young inequality (see [16], p. 139) :

$$\|v_\lambda * K\|_r \ \leq \ C\|v_\lambda\|_q$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{2}$ with $1 < q < 2$. In the same way, since ∇K is a Calderón-Zygmund kernel we have (see [15], p. 35) :

$$\|v_\lambda * \nabla K\|_r \ \leq \ C\|v_\lambda\|_r$$

for $1 < r < \infty$. Thus,

$$\|v_\lambda * K\|_\infty \ \leq \ C\|v_\lambda\|_q^{(1-\frac{2}{r})} \|v_\lambda\|_r^{\frac{2}{r}}$$

for $r > 2$.

Thanks to (E_1) we conclude that $\|v_\lambda * K(\tau + s)\|_\infty$ is bounded for every $s \geq 0$ by a constant C_τ only depending on τ. Hence,

$$\begin{aligned} \|\nabla v_\lambda(t+\tau)\|_p \ \leq & \quad C\|v(0)\|_1 t^{(-1+\frac{1}{p}-\frac{1}{2})} \\ + & \quad C_\tau \int_0^t (t-s)^{\frac{-1}{2}} \|\nabla v_\lambda(s+\tau)\|_p ds \end{aligned}$$

By applying lemma 1.1 we deduce that

$$\|\nabla v_\lambda(t+\tau)\|_p \leq C(\tau, \|v(0)\|_1) t^{(\frac{1}{p}-\frac{3}{2})}$$

if $p < 2$ y $0 \leq t \leq 2\tau$. Choosing $t = \tau$ we obtain (E_2) when $p < 2$.

Taking L^q norms for $q \geq p$ in the integral equation we get:

$$\begin{aligned} \|\nabla v_\lambda(t+\tau)\|_q \ \leq & \quad C\|v(0)\|_1 t^{(-1+\frac{1}{q}-\frac{1}{2})} \\ + & \quad C \int_0^t (t-s)^{\frac{-1}{p}+\frac{1}{q}-\frac{1}{2}} \|\nabla v_\lambda(s+\tau)\|_p \|v_\lambda * K(\tau+s)\|_\infty ds \end{aligned}$$

where $1 \leq p < 2$. By using the bound obtained when $p < 2$ it follows that

$$\|\nabla v_\lambda(2\tau)\|_q \leq C_\tau$$

provided that $\frac{-1}{p}+\frac{1}{q}+\frac{1}{2}>0$, that is $1\leq q<\infty$. Taking now the L^∞ norm in both sides of the equation and proceeding in an analogous way we obtain (E_2) when $p=\infty$.

(E3) This bound follows from the equation $v_t-\Delta v+(v*K)\cdot\nabla v=0$ after proving that $\|\Delta v_\lambda(t)\|_p\leq C(\|v(0)\|_1,p,t)$. Differentiating the integral equation we obtain:

$$\partial^2_{i,j}v_\lambda(t+\tau)=\partial_iG(t)*\partial_jv_\lambda(\tau)$$

$$-\int_0^t\partial_iG(t-s)*\Big(\nabla v_\lambda\cdot(v_\lambda*\partial_jK)+\nabla\partial_jv_\lambda\cdot(v_\lambda*K)\Big)(\tau+s)ds$$

for $1\leq i,j\leq 2$.

Taking L^p norms in the equation we obtain the inequality :

$$v_\lambda(t+\tau)|_{2,p}\leq C\|\nabla v_\lambda(\tau)\|_1t^{(-1+\frac{1}{p}-\frac{1}{2})}$$

$$+C\int_0^t(t-s)^{\frac{-1}{2}}\Big(|v_\lambda|_{2,p}\|v_\lambda*K\|_\infty+\|\nabla v_\lambda\|_r\|v_\lambda*\nabla K\|_{r'}\Big)(\tau+s)ds$$

where $|v_\lambda(t)|_{2,p}=\sum\limits_{1\leq i,j\leq 2}\|\partial^2_{i,j}v_\lambda(t)\|_{2,p}$ and $\frac{1}{p}=\frac{1}{r}+\frac{1}{r'}$. Thanks to the estimates $(E_1),(E_2)$ we deduce that :

$$\begin{array}{rcl}|v_\lambda(t+\tau)|_{2,p} & \leq & (C\|\nabla v_\lambda(\tau)\|_1t^{(-1+\frac{1}{p}-\frac{1}{2})}+C_\tau)\\ & + & C_\tau\int_0^t(t-s)^{\frac{-1}{2}}|v_\lambda|_{2,p}(\tau+s)ds\end{array}$$

Proceeding like in the proof of (E_2) we first obtain

$$\|\Delta v_\lambda(2\tau)\|_p\ \leq\ |v_\lambda(2\tau)|_{2,p}\leq\ C_\tau$$

for $1\leq p<2$ and then for every p.

(E4) Let us note $B_R=B(0,R)$. We first remark that :

$$\|v_\lambda(t)\|_{L^1(\mathbb{R}^2-B_R)}=\|v(\lambda^2t)\|_{L^1(\mathbb{R}^2-B_{\lambda R})}$$

We know (Theorem 4.3 of [7]) that v has an integral representation of the form:

$$v(x,\lambda^2 t) = \int_{\mathbb{R}^2} \Gamma(x,\lambda^2 t; y, 0) v_0(y) dy$$

with

$$|\Gamma(x,t;y,0)| \ \leq \ C\, G(x-y, ct)$$

for some positive constants C, c depending on the total variation on the initial data v_0. In the sequel we shall note by C several constants not depending on λ. Therefore,

$$\begin{aligned} \|v(\lambda^2 t)\|_{L^1(\mathbb{R}^2 - B_{\lambda R})} &\leq \|G(\lambda^2 ct) * |v_0(y)|\|_{L^1(\mathbb{R}^2 - B_{\lambda R})} \\ &= \|G(ct) * \lambda^2 |v_0(\lambda y)|\|\|_{L^1(\mathbb{R}^2 - B_R)} \end{aligned}$$

Take $\psi(x) \in C^2(\mathbb{R}^2)$ such that $\psi = 0$ if $|x| \leq 1$ and $\psi = 1$ when $|x| \geq 2$. We note $\psi_R(x) = \psi(\frac{x}{R})$ and $w_{\lambda,R} = w_\lambda \psi_R$, where $w_\lambda(t,x) = G(t) * \lambda^2 |v_0(\lambda x)|$. The function $w_{\lambda,R}$ satisfies the equation :

$$w_{\lambda,R,t} - \Delta w_{\lambda,R} = -\frac{2}{R} \nabla w_\lambda (\nabla \psi)_R - \frac{1}{R^2} (\Delta \psi)_R w_\lambda$$

By writing the integral equation and taking the L^1 norm we obtain :

$$\|w_{\lambda,R}\|_1(t) \leq C \|w_{0,\lambda,R}\|_1 + \frac{C}{R} \int_0^t \|\nabla w_\lambda\|_1(s) ds + \frac{C}{R^2} \int_0^t \|w_\lambda\|_1$$

so that

$$\|w_{\lambda,R}\|_1(t) \leq C \|w_{0,\lambda,R}\|_1 + \frac{C\, t}{R}$$

Since

$$\|w_{0,\lambda,R}\|_{L^1(\mathbb{R}^2)} \leq C \|v_0\|_{L^1(|x| > \lambda R)}$$

we conclude that $\|v_\lambda(t)\|_{L^1(\mathbb{R}^2 - B_R)}$ tends to zero when R tends to infinity, uniformly with respect to $\lambda \geq 1$ and $t \in [0, t_1]$.

Theorem 1.2 : Convergence

There exists a constant $A > 0$ *such that, if* $v_0 \in M^1(\mathbb{R}^2)$ *and* $|M| = |\int_{\mathbb{R}^2} v_0| < A$ *the solution* v *of (V2) satisfies :*

$$\lim_{t\to\infty} t^{1-\frac{1}{p}} \, \|v(t) - w(t)\|_{L^p} \; = \; 0$$

for every $1 \le p \le \infty$*, where* w *is the unique solution of (F2):*

$$(F2) \qquad \begin{cases} w_t - \Delta w + (K * w) \cdot \nabla w = 0 & \text{in } \mathbb{R}^2 \times \mathbb{R}^+ \\ w(x,0) = M\delta & \text{in } \mathbb{R}^2 \end{cases}$$

which is given explicitely by the formula :

$$w(x,t) = \frac{M}{4\pi t} e^{\frac{-|x|^2}{4t}}$$

Remark 1.2

In [4] a similar convergence result is proved for initial data $v_0 \in M(\mathbb{R}^2)$ *whose initial variation is small. We have replaced the restriction on the size of* $|v_0|$ *by the uniqueness of the solution of (F2), which is guaranteed if* $|\int_{\mathbb{R}^2} v_0|$ *is small enough. That is, we have reduced the problem of describing the asymptotic behavior of the solution to the problem of the uniqueness of the fundamental solution of the vorticity equation with mass* $M = \int v_0$*.*

Remark 1.3

Since $\|G(t) * v_0 - MG(t)\|_{L^p} t^{1-\frac{1}{p}} \to 0$ *we also deduce that :*

$$\lim_{t\to\infty} t^{1-\frac{1}{p}} \, \|v(t) - G(t) * v_0\|_{L^p} \; = \; 0,$$

that is, in first approximation, v *is asymptotically equivalent to the solution of the heat equation with the same initial data. On the other hand, since* $\|MG(t)\|_{L^p}$ *decreases exactly like* $t^{-1+\frac{1}{p}}$*, we conclude that the first term in the asymptotic expansion of* $v(t)$ *when* t *tends to* ∞ *is* $MG(t)$*.*

Proof

In view of the estimates established in the previous section we know that for every $t > 0$ there are constants $C_i(t),\ i = 1, 2, 3, 4$ such that:

$$\begin{array}{lll}
(E_1) & \|v_\lambda(t)\|_p \ \le C_1(t,p) & 1 \le p \le \infty \\
(E_2) & \|\nabla v_\lambda(t)\|_p \ \le C_2(t,p) & 1 \le p \le \infty \\
(E_3) & \|v_{\lambda,t}(t)\|_p \ \le C_3(t,p) & 1 \le p \le \infty \\
(E_4) & \|v_\lambda(t)\|_{L^1(\mathbb{R}^2 - B(0,R))} \to 0 & \text{if } R \to \infty \\
 & \text{uniformly with respect to } \lambda \ge 1,\ t \in [0, t_1]\ t_1 > 0 &
\end{array}$$

when $v_0 \in M^1(\mathbb{R}^2)$.

Therefore, we can extract a subsequence, noted again v_λ, such that:

$$\begin{array}{ll}
v_\lambda \to w & \text{in } L^\infty((t_0,t_1), L^r(\mathbb{R}^2)) \text{ weak } * \\
\nabla v_\lambda \to \nabla w & \text{in } L^\infty((t_0,t_1), L^r(\mathbb{R}^2)) \text{ weak } * \\
v_{\lambda,t} \to w_t & \text{in } L^\infty((t_0,t_1), L^r(\mathbb{R}^2)) \text{ weak } *
\end{array}$$

for every $1 < r \le \infty$ and every $(t_0, t_1) \subset \mathbb{R}^+$.

Let $B_R = B(0,R) \subset \mathbb{R}^2$. Since the injection $W^{1,r}(B_R) \to L^q(B_R)$ is compact for every finite q if $r \ge 2$ and for $q < r* = \frac{2r}{2-r}$ if $1 \le r < 2$ and we have the estimate (E_3), it follows that the family v_λ is compact in $C(t_0, t_1; L^q(B_R))$ so that we can assume

$$v_\lambda \to w \quad \text{in } \ C((t_0,t_1), L^q(B_R))$$

for every $R > 0$ and for every $1 \le q \le \infty$.

Given $\psi \in C^1_c(\mathbb{R}^+ \times \mathbb{R}^2)$ and $\lambda > 0$, we have :

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^2} \psi v_{\lambda,t} + \int_{\mathbb{R}^+}\int_{\mathbb{R}^2} \nabla\psi\cdot\nabla v_\lambda + \int_{\mathbb{R}^+}\int_{\mathbb{R}^2} \psi(K * v_\lambda)\cdot\nabla v_\lambda \ = 0$$

In order to pass to the limit in the nonlinear term $I_\lambda = \int_{\mathbb{R}^+}\int_{\mathbb{R}^2} \psi(K * v_\lambda)\nabla v_\lambda$ we split it in the following way :

$$\begin{aligned}
I_\lambda = I^1_{R\lambda} + I^2_{R\lambda} = \ & \int_{\mathbb{R}^+}\int_{\mathbb{R}^2} \psi\Big(\int_{\mathbb{R}^2 - B_R} K(x-y)v_\lambda(t,y)dy\Big)\nabla v_\lambda dxdt \\
+ \ & \int_{\mathbb{R}^+}\int_{\mathbb{R}^2} \psi\Big(\int_{B_R} K(x-y)v_\lambda(t,y)dy\Big)\nabla v_\lambda dxdt
\end{aligned}$$

Taking limits when λ, R $\to \infty$ we deduce that $I^1_{R\lambda} \to 0$ and

$$I^2_{R\lambda} \to I = \int_{\mathbb{R}^+} \int_{\mathbb{R}^2} \psi(K * w) \nabla w$$

Let us first prove that, for a fixed R,

$$I^2_{R\lambda} \to I^2_R = \int_{\mathbb{R}^+} \int_{\mathbb{R}^2} \psi\Big(\int_{B_R} K(x-y) w(t,y) dy \Big) \nabla w dx dt$$

when λ tends to ∞. Indeed, noting by χ_{B_R} the characteristic function of B_R we have that:

$$\begin{aligned} \| \textstyle\int_{B_R} K(x-y)(v_\lambda - w)(t,y) dy \|_{L^r} &= \| K * \chi_{B_R} (v_\lambda - w)(t) \|_{L^r} \\ &\leq C \| (v_\lambda - w)(t) \|_{L^q(B_R)} \end{aligned}$$

when $r > 2$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{2}$. Fixed $R > 0$

$$\int_{B_R} K(x-y) v_\lambda(t,y) dy \to \int_{B_R} K(x-y) w(t,y) dy$$

in $L^\infty_t (L^r)_x$, since $\| (v_\lambda - w)(t) \|_{L^q(B_R)}$ converges to zero when λ tends to ∞, uniformly when t runs over a compact set.

Now, we let R tend to ∞ to get :

$$I^2_R \to I^2 = \int_{\mathbb{R}^+} \int_{\mathbb{R}^2} \psi\Big(\int_{\mathbb{R}^2} K(x-y) w(t,y) dy \Big) \nabla w dx dt$$

It suffices to remark that, for almost every t:

$$\alpha_R(t) = \| \int_{\mathbb{R}^2} K(x-y)(w \chi_{B_R} - w)(t,y) dy \|_{L^r} \leq C \| (w \chi_{B_R} - w)(t) \|_{L^q(\mathbb{R}^2)}$$

tends to zero when R tends to ∞ and that $|\alpha_R(t)|$ is bounded by a constant. Since constant functions belong to $L^s(t_0, t_1) \quad 1 \leq s \leq \infty$, we get

$$\| \int_{\mathbb{R}^2} K(x-y)(w \chi_{B_R} - w)(t,y) dy \|_{L^s(t_0,t_1; L^r(\mathbb{R}^2))} \to 0$$

when R tend to ∞.

It remains to prove that $I^1_{R\lambda} \to 0$ when R tend to ∞ uniformly with respect to $\lambda \geq 1$. Since

$$\| \int_{\mathbb{R}^2 - B_R} K(x-y) v_\lambda(t,y) dy \|_{L^r} \; \leq \; C \| v_\lambda(t) \|_{L^q(\mathbb{R}^2 - B_R)}$$

when $r > 2$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{2}$, it suffices to prove that $\|v_\lambda(t)\|_{L^q(\mathbb{R}^2 - B_R)}$ converges to zero when R tends to ∞, uniformly with respect to λ and $t \in (t_0, t_1)$ for some $q \in (1,2)$. This follows from (E_1) and (E_4) by using the interpolation inequality :

$$\|v_\lambda(t)\|_{L^q(\mathbb{R}^2 - B_R)} \; \leq \; \|v_\lambda(t)\|^{\alpha}_{L^1(\mathbb{R}^2 - B_R)} \; \|v_\lambda(t)\|_r^{1-\alpha}$$

with $\frac{1}{q} = \alpha + \frac{(1-\alpha)}{r}$, $0 \leq \alpha \leq 1$. Since:

$$\|v_\lambda(t)\|_{L^\infty(\mathbb{R}^2 - B_R)} \; \leq \; C \|v_\lambda(t)\|^{1-\frac{n}{p}}_{L^p(\mathbb{R}^2 - B_R)} \; \|\nabla v_\lambda(t)\|_p^{\frac{n}{p}}$$

if $n < p < \infty$, we also obtain the convergence in L^∞.

We can now pass to the limit in the equation. If we let λ tend to ∞ and then $R \to \infty$ we obtain :

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}^2} \psi w_t + \int_{\mathbb{R}^+} \int_{\mathbb{R}^2} \nabla \psi \nabla w + \int_{\mathbb{R}^+} \int_{\mathbb{R}^2} \psi (K * w) \nabla w \; = 0$$

so that w satisfies :

$$w_t - \Delta w + (K * w) \cdot \nabla w = 0 \quad \text{in } \mathbb{R}^2 \times \mathbb{R}^+$$

As far as the initial condition is concerned, since $|v_\lambda(0)|$ does not depend on λ, the functions $v_\lambda(t)$ are equicontinuous for the weak topology (see [7]) and :

$$\int_{\mathbb{R}^2} v_\lambda(t,x) \psi(x) dx \to \int_{\mathbb{R}^2} \lambda^2 v_0(\lambda x) \psi(x) dx \quad \text{ if } t \to 0$$

for every $\psi \in BC(\mathbb{R}^2)$, uniformly in λ. Now, $\lambda^2 v_0(\lambda x) \rightharpoonup M\delta$ in $M(\mathbb{R}^2)$ so that :

$$\int_{\mathbb{R}^2} w(t,x) \psi(x) dx \to M\psi(0) \quad \text{ if } t \to 0$$

We can arrive to the same conclusion in another way. Given $\psi \in C_c^2(\mathbb{R}^2)$ we have :

$$\int_{\mathbb{R}^2} w(t)\psi - \int_{\mathbb{R}^2} w(s)\psi = \int_s^t \int_{\mathbb{R}^2} w\Delta\psi + \int_s^t \int_{\mathbb{R}^2} (K * w)w \cdot \nabla w$$

for $t, s > 0$. Since $\|K * w(t)\|_\infty \leq Ct^{\frac{-1}{2}}$, $\|w(t)\|_1 \leq C$ and $\|\nabla w(t)\|_1 \leq Ct^{\frac{-1}{2}}$ we conclude that $\int_{\mathbb{R}^2}(w(t) - w(s))\psi \to 0$ if $t, s \to 0$ so that the limit

$$\lim_{s\to 0} \int_{\mathbb{R}^2} w(s)\psi$$

exists and

$$\int_{\mathbb{R}^2} w(t)\psi - \lim_{s\to 0} \int_{\mathbb{R}^2} w(s)\psi = \int_0^t \int_{\mathbb{R}^2} w\Delta\psi + \int_0^t \int_{\mathbb{R}^2} (K * w)w \cdot \nabla w$$

On the other hand,

$$\int_{\mathbb{R}^2} v_\lambda(t)\psi - \int_{\mathbb{R}^2} \lambda^2 v(\lambda x)\psi = \int_0^t \int_{\mathbb{R}^2} v_\lambda \Delta\psi + \int_0^t \int_{\mathbb{R}^2} (K * v_\lambda)v_\lambda \cdot \nabla v_\lambda$$

Passing to the limit when λ tends to infinity we obtain

$$\int_{\mathbb{R}^2} w(t)\psi - M\psi(0) = \int_0^t \int_{\mathbb{R}^2} w\Delta\psi + \int_0^t \int_{\mathbb{R}^2} (K * w)w \cdot \nabla v_\lambda$$

so that

$$\lim_{s\to 0} \int_{\mathbb{R}^2} w(s)\psi = M\psi(0)$$

when $\psi \in C_c^2$. In order to extend this result to $\psi \in BC(\mathbb{R}^2)$ it suffices to prove

$$\int_{|x|>R} v_\lambda(s)\psi \to 0$$

when $R \to \infty$ uniformly with respect to λ and $s \in [0, t_1]$ which is an inmediate consequence of $(E4)$.

It remains to prove that w solves the integral equation in $M(\mathbb{R}^2)$. For $t > 0$ we know that $w(t) \in L^1(\mathbb{R}^2)$ so that:

$$w(t) = G(t-\varepsilon) * w(\varepsilon) + \int_\varepsilon^t \partial_i G(t-s) * (K * w)^i w(s) ds$$

Since the integral term tends to $\int_0^t \partial_i G(t-s)*(K*w)^i w(s)ds$ in $L^1(\mathbb{R}^2)$ and $G(t-\varepsilon)*w(\varepsilon) \to G(t)*w(0) = MG(t)$ in $M(\mathbb{R}^2)$ when $\varepsilon \to 0$ the integral equation is satisfied.

Now, we apply the uniqueness result in [7], which guarantees the uniqueness of w when $|M| < A$. Thus, the whole sequence v_λ converges to $w = MG(t)$ in $L^\infty((t_0,t_1), W^{1,p}(\mathbb{R}^2))$ weak * and in $L^\infty((t_0,t_1), L^p(B_R))$ strong for every $R > 0$.

On the other hand, the fact that $\|v_\lambda(t)\|_{L^q(\mathbb{R}^2 - B_R)}$ converges to zero when R tends to ∞ uniformly with respect to λ and $t \in (t_0,t_1)$ for every $1 \le q \le \infty$ allows to prove the strong convergence in $L^q(\mathbb{R}^2)$, taking into account that :

$$\begin{aligned}\|v_\lambda(1) - MG(1)\|_{L^q(\mathbb{R}^2)} &\le \|v_\lambda(1) - MG(1)\|_{L^q(B_R)} \\ &+ \|v_\lambda(1)\|_{L^q(\mathbb{R}^2 - B_R)} + \|MG(1)\|_{L^q(\mathbb{R}^2 - B_R)}\end{aligned}$$

Therefore,

$$\|v_\lambda(1) - w_\lambda(1)\|_{L^q(\mathbb{R}^2)} = \lambda^{2(1-\frac{1}{q})} \|v(\lambda^2) - w(\lambda^2)\|_{L^q(\mathbb{R}^2)} \to 0$$

when λ tends to ∞.

Corollary 1.1

Under the hypotheses of the above theorem, if $u = K*v$ and $u_0 = K*v_0$ then

$$t^{\frac{1}{2}-\frac{1}{r}} \|u - G*u_0\|_{L^r} \to 0 \quad 2 < r \le \infty$$

Proof

It follows from Theorem 1.1 having into account that $G*(K*v_0) = K*(G*v_0)$ and that :

- if $2 < r < \infty$:

$$\|K*(v - G*v_0)\|_{L^r} \ \le \ \|K\|_{L^{2,\infty}} \|v - G*v_0\|_{L^q}$$

with $\frac{1}{r} = \frac{1}{q} - \frac{1}{2}$

- if $r = \infty$

$$\|K * (v - G * v_0)\|_{L^\infty} \ \leq \ C\|K * (v - G * v_0)\|_{L^q}^{1-\frac{2}{q}} \|v - G * v_0\|_{L^q}^{\frac{2}{q}}$$

for every $q > 2$.

Remark 1.4

The convergence

$$t^{\frac{1}{2}-\frac{1}{r}} \|u(t) - G(t) * u_0\|_{L^r(\mathbb{R}^2)} \to 0 \quad 2 < r < \infty$$

was proved in [4] for $u_0 = K * v_0$, v_0 being a measure of small total variation. Previously, Kato proved in [8] that

$$\|u\|_{L^p} \leq Ct^{-\frac{1}{q}+\frac{1}{p}}$$

$$\|\nabla u\|_{L^p} \leq Ct^{-\frac{1}{2}-\frac{1}{q}+\frac{1}{p}}$$

for $p > q$ when the initial data belongs to $L^2 \cap L^q$ with $1 < q \leq 2$ and its L^2 norm is small, provided that the exponents are less than one. The asymptotic behavior in L^2 norm has been studied for instance in [8], [13], [9] and [17]. The most precise results are contained in the following theorem by Wiegner [17] :

$$\|u(t) - G(t) * u_0\|_{L^2}^2 \leq h(t)(1+t)^{-d}$$

with $d = 2 - 2max(1 - \alpha, 0)$ and

$$h(t) \ = \ \begin{cases} \varepsilon(t) \to 0 \quad t \to \infty & \text{if} \quad \alpha = 0 \\ Cln^2(t+c) & \text{if} \quad \alpha = 1 \\ C & \text{if} \quad \alpha \neq 0, 1 \end{cases}$$

if $\|G(t) * u_0\|_2^2 \leq C(1+t)^{-\alpha}$. Therefore, if $u_0 \in L^p \cap L^2,\ 1 \leq p < 2$ we have

$$t^{\frac{1}{p}-\frac{1}{2}} \|u(t) - G(t) * u_0\|_{L^2(\mathbb{R}^2)} \to 0$$

1.2 Initial data with zero mass

We shall show here that for smooth regular data with zero mass the result obtained in the previous section, that is,

$$t^{-1+\frac{1}{r}}\ \|v(t)\|_{L^r(\mathbb{R}^2)} \to 0 \quad \text{as} \quad t \to \infty$$

can be improved to yield a decay rate of order $t^{\frac{-3}{2}+\frac{1}{r}}$. A similar phenomena has been observed in [1] for some diffusion-convection equations. Remark that the solutions of the heat equation with zero mass data also exhibit this decay rate. This fact allow to improve the decay estimates on the velocity u. We get a decay rate of order $t^{1-\frac{1}{p}}$ for $\|u(t)\|_{L^p}$ when $p > 2$.

Let us state all this in a more precise way.

Theorem 1.3

Set $P = e^{\frac{|x|^2}{4}}$. *If* $v_0 \in L^2(\mathbb{R}^2, P) \cap L^\infty(\mathbb{R}^2)$ *and* $M = \int_{\mathbb{R}^2} v_0 = 0$, *then*

$$\|v(t)\|_{L^r(\mathbb{R}^2)} \leq C t^{\frac{-3}{2}+\frac{1}{r}}$$

if $1 \leq r \leq \infty$.

Proof

The proof needs the change to selfsimilar variables and the introduction of some weigthed Sobolev spaces. It follows exactly the proof of Theorem 1 in [1] so that we shall merely scketch it. For more details see [1].

Passing to selfsimilar variables $(y, s) = (\frac{x}{(t+1)^{\frac{1}{2}}}, ln(t+1))$ we deduce that $w(s, y) = e^s v(e^s - 1, e^{\frac{s}{2}} y)$ satisfies:

$$(A2) \qquad \begin{cases} w_s - \Delta w - \frac{y}{2}\nabla w - w + (K^i * w)\ \partial_i w = 0 & \text{in } \mathbb{R}^2 \times \mathbb{R}^+ \\ w(x, 0) = v_0 & \text{in } \mathbb{R}^2 \end{cases}$$

We know that $w \in C^\infty((0,\infty) \times \mathbb{R}^2)$ and that $w \in C((0,\infty); W^{2,p}(\mathbb{R}^2)) \cap C([0,\infty); L^p(\mathbb{R}^2))$. On the other hand the relation

$$\|w(s)\|_{L^p} = e^s \|u(e^s - 1)\|_{L^p}$$

for $1 \leq p \leq \infty$ and the estimates established in Theorem 1.1 imply that :

$$\|w(s)\|_{2,p} \leq C(p, s_0) \quad s \geq s_0,\ 1 \leq p \leq \infty$$
$$\|w(s)\|_p \leq C(p) \quad s \geq 0,\ 1 \leq p \leq \infty$$

The operator $Lw = -\Delta w - \frac{y}{2}\nabla w$ is a selfadjoint operator in $L^2(\mathbb{R}^2, P)$ with domain $H^2(\mathbb{R}^2, P)$. Its inverse L^{-1} defines a bounded compact operator from $L^2(\mathbb{R}^2, P)$ in itself. The eigenvalues of L are $\lambda_k = \frac{1+k}{2}, k = 1, 2, 3...$ and the associated eigenspaces E_k are spanned by :

$$\phi_1 = \tfrac{1}{4\pi}P^{-1}, \quad k = 1$$
$$\phi_{\alpha(k)} = \partial_{\alpha(k)}\phi_1, \quad \alpha(k) \in \mathbb{N}^2,\ |\alpha(k)| = k - 1,\ k > 1$$

By Theorem 1.2 we know that $\|w(s)\|_p \to 0$ when s tends to infinity for every $p \geq 1$ which implies $\|(K^i * w)\|_{L^\infty} \to 0 \quad t \to \infty$. Therefore, given $\varepsilon > 0$ there exists s_ε such that

$$\|(K^i * w)\partial_i w(s)\|_{L^2(P)} \leq \varepsilon\|\nabla w(s)\|_{L^2(P)}$$

when $s \geq s_\varepsilon$. Proceeding as in the Proposition 2 of [1] we prove that

$$w \in C([0, \infty); L^2(P)) \cap C((0, \infty); H^1(P))$$

and as in Proposition 3 :

$$w \in L^\infty([1, \infty); H^1(P))$$

If we note by $E_1^\perp$ the orthogonal of E_1 in $L^2(P)$ we have :

$$\langle (L - I)v, v\rangle \geq \frac{1}{2}\|v\|^2_{H^1(P)}$$

for every $v \in E_1^\perp \cap H^1(P)$. Since

$$\int_{\mathbb{R}^2} w(s) = 0 \quad \forall s \geq 0$$

we can take $v = w$ in the above inequality.

Multiplying the equation by wP and integrating on $\mathbb{R}^2$ we obtain :

$$\frac{1}{2}\frac{d}{ds}\|w(s)\|^2_{L^2(P)} + \frac{1}{2}\|w(s)\|^2_{H^1(P)} \le \varepsilon(\frac{2}{3})^{\frac{1}{2}}\|w(s)\|^2_{H^1(P)} \quad s \ge s_\varepsilon$$

so that

$$\frac{d}{ds}\|w(s)\|^2_{L^2(P)} + (1-\varepsilon)\|w(s)\|^2_{L^2(P)} \le 0 \quad s \ge s'_\varepsilon$$

for a certain s'_ε. Therefore,

$$\|w(s)\|_{L^2(P)} \le C_\varepsilon e^{\frac{-(1-\varepsilon)s}{2}}$$

for every $s \ge 0$. From the integral equation we get:

$$\|w(s)\|_{H^1(P)} \le C_\varepsilon e^{\frac{-(1-\varepsilon)s}{2}}$$

for every $s \ge s_0$. Multiplying the equation by wP and integrating we obtain:

$$\frac{d}{ds}\|w(s)\|^2_{L^2(P)} + \|w(s)\|^2_{L^2(P)} \le 2g(s) \quad s \ge s_0$$

where $g(s) = \int_{\mathbb{R}^2} P(y)w(s,y)(K^i * w)(s,y)\partial_i w(s,y)dy$. Now, we have

$$|g(s)| \le C\|w(s)\|^2_{H^1(P)}\|K * w\|_\infty(s) \le C_\varepsilon e^{-s+\varepsilon s}e^{\frac{-(1-\varepsilon)s\beta}{2}}$$

for some $0 < \beta < 1$. Choosing ε small enough $\int e^s g(s) < \infty$ so that by integrating the inequality it follows

$$\|w(s)\|_{L^2(P)} \le Ce^{\frac{-s}{2}} \quad s \ge 0$$

which implies :

$$\|w(s)\|_{L^r} \le Ce^{\frac{-s}{2}} \quad s > 0$$

for every $1 \le r \le 2$, that is

$$\|v(t)\|_{L^r} \le Ct^{-\frac{3}{2}+\frac{1}{r}} \quad t > 0$$

From the integral equation we get

$$\|w(s)\|_{H^1(P)} \leq C_\varepsilon e^{\frac{-(1-\varepsilon)s}{2}}$$

for every $s \geq 1$ and by interpolation we obatin the result for for $r > 2$.

Corollary 1.2

If $u_0 \in W^{1,1}(\mathbb{R}^2)$ *and* $v_0 = curl\ u_0 \in L^2(\mathbb{R}^2, P) \cap L^\infty(\mathbb{R}^2)$ *then the following estimate holds for the solution* u *of (NS)*

$$t^{1-\frac{1}{p}} \|u(t)\|_{L^p} \leq C$$

when $p > 2$, *where* C *denotes a positive constant.*

Proof

In this case $M = \int_{\mathbb{R}^2} curl\ u_0 = 0$ so that it suffices to apply the previous theorem and the same argument as in Corollary 1.1

2. THE THREE DIMENSIONAL CASE

Let us consider the system satisfied by the vorticity $v = (v^1, v^2, v^3)$ in dimension $n = 3$:

$$(V3) \qquad \begin{cases} v_t - \Delta v + \partial_i (u^i v - v^i u) = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}^+ \\ div\ v = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}^+ \\ v(x,0) = v_0,\ div\ v_0 = 0 & \text{in } \mathbb{R}^3 \end{cases}$$

where the velocity $u = (u^1, u^2, u^3)$ is given by :

$$u(x,t) = K * v(x,t) = \frac{-1}{4\pi} \int_{\mathbb{R}^3} \frac{(y_1, y_2, y_3)}{|y|^3} \times v(x-y,t) dy$$

with :

$$K(x) = \frac{-1}{4\pi} \frac{(x_1, x_2, x_3)}{|x|^3} = (K^1, K^2, K^3)$$

We note $\partial_i (u^i v - v^i u) = \sum_{i=1}^3 (u^i v - v^i u)$ with $\partial\, v = \frac{\partial\, v}{\partial x_i}$.

In the sequel we shall work with solutions defined in spaces of measures of Morrey. For every $1 \leq p \leq \infty$ we define $\overline{M}^p$ to be the space formed by the measures $\mu \in (M(\mathbb{R}^3))^3$ which verify :

$$|\mu|(B(x,r)) \leq C r^{\frac{3}{p'}}, \quad p' = \frac{p}{p-1}$$

where $B(x,r)$ represents the open ball centered at x with radius r in $\mathbb{R}^3$ and C is a positive constant independient of x and r. $\overline{M}^p$ is a Banach space endowed with the norm

$$\|\mu\|_p = \sup_{x \in \mathbb{R}^3, r>0} r^{-\frac{3}{p'}} |\mu|(B(x,r))$$

Let us note by M^p the closed subspace $\overline{M}^p \cap L^1_{loc}$. It can be proved that $\overline{M}^1$ is the space of finite measures, $M^1 = L^1$and $\overline{M}^\infty = M^\infty = L^\infty$. When $1 < p < \infty$ we have $L^p \subset M^p$. All the properties we shall need have been proved in [6].

Let us assume $v_0 \in (\overline{M}^{\frac{3}{2}})^3$ and $div\ v_0 = 0$. It is known (Theorems 4.1 and 4.2. of [6]) that, if $\|v_0\|_{\frac{3}{2}} < A$ for some constant A small enough, the system (V3) has a unique solution, defined for every positive t, such that :

i) $v \in BC((0,\infty); (\overline{M}^{\frac{3}{2}}))^3$ y $v(t) \to v_0$ on balls when $t \to 0$ in the weak topology of measures.

ii) $t^{\frac{1}{2}} u \in BC((0,\infty); (L^\infty)^3)$ and $t^{1-\frac{3}{2q}} v \in BC((0,\infty); (M^q)^3)$ for $\frac{3}{2} < q < 6$.

iii) $v \in C^\infty(\mathbb{R}^3 \times (0,\infty))$, verifies a div v $= 0$ and satisfies the equation in the classic sense.

Our goal here is to make precise the behavior of these solutions when t tends to infinity by means of selfsimilar solutions. Let us remark that the equation is invariant under the scaling transformation $v_\lambda(x,t) = \lambda^2 v(\lambda x, \lambda^2 t)$ (which makes sense since we are dealing with measures defined by locally integrable fonctions) in such a way that v_λ is a solution of:

$$\begin{aligned}
&v_{\lambda,t} - \Delta v_\lambda + \partial_i\big((K^i * v_\lambda)v_\lambda - v_\lambda^i (K * v_\lambda)\big) = 0 && \text{in } \mathbb{R}^3 \times \mathbb{R}^+ \\
&div\ v_\lambda = 0 && \text{in } \mathbb{R}^3 \times \mathbb{R}^+ \\
&v_\lambda(x,0) = \lambda^2 v_0(\lambda x) = v_{0,\lambda},\ div\ v_{0,\lambda} = 0 && \text{in } \mathbb{R}^3
\end{aligned}$$

In the case of a measure μ the scaling is given by :

$$\mu_\lambda(E) = \mu(\lambda E)\lambda^{-1}$$

Let us consider any open ball $B(x,r)$. Since $\|v_{0,\lambda}\|_{M^{\frac{3}{2}}} = \|v_0\|_{M^{\frac{3}{2}}}$, the total variation of the measures $v_{0,\lambda}$ in that ball is uniformly bounded. Therefore, we can extract a subsequence v_{0,λ_i} converging to a measure μ in the weak topology of measures on balls. Besides,

$$|\mu|(B(x,r)) \leq \liminf \int_{B(x,r)} |v_{0,\lambda_i}| \leq Cr$$

so that $\mu \in (\overline{M}^{\frac{3}{2}})^3$ and $\|\mu\|_{\overline{M}^{\frac{3}{2}}} \leq \|v_0\|_{M^{\frac{3}{2}}}$.

Nevertheless, since we want μ to satisfy the homogeneity condition $\mu_\lambda = \mu$ we need the convergence of the whole family, that is, $\lambda^2 v_0(\lambda x) \to \mu$ when $\lambda \to \infty$. This is verified in some particular cases. For instance, when $v_0 \in (L^{\frac{3}{2}})^3$ we shall prove that $v_{0,\lambda}$ converges weakly to zero in $(L^{\frac{3}{2}})^3$. In the general case, we can construct families converging to a non zero limit. It suffices to consider functions whose coordinate functions have the form $v^{i_3}(x_{i_1}, x_{i_2}, x_{i_3}) = \phi(x_{i_1}, x_{i_2}) \in L^1(\mathbb{R}^2)$. By Proposition 2.2 in [6] we know that $L^1(\mathbb{R}^2) = M^1(\mathbb{R}^2)$ is embedded in $M^{\frac{3}{2}}(\mathbb{R}^3)$ with continuous embedding, so that such functions belong to $M^{\frac{3}{2}}(\mathbb{R}^3)$. On the other hand, it is clear that $\lambda^2 v^{i_3}(\lambda x) \to (\int \phi)\, \delta_{x_{i_1}, x_{i_2}}$ in $M(\mathbb{R}^3)$.

However, as far as we know, there is no characterization of the set of $v_0 \in (M^{\frac{3}{2}})^3$ such that $div\ v_0 = 0$ and $\lambda^2 v_0(\lambda x) \to \mu \in (\overline{M}^{\frac{3}{2}})^3$ when $\lambda \to \infty$.

If we analyse the way in which solutions of (V3) are constructed in [6] we realize that this construction furnish uniform estimates on v_λ thanks to the fact that the $\overline{M}^{\frac{3}{2}}$ norm of the initial data is independent of λ. Those estimates, together with a uniform estimate with respect to λ when $|x| \to \infty$ allow to pass to the limit in the equation and conclude that v_λ converges to a selfsimilar solution of:

$$\begin{aligned}
&w_t - \Delta w + \partial_i\big((K^i * w)w - w^i(K * w)\big) = 0 && \text{in } \mathbb{R}^3 \times \mathbb{R}^+ \\
&div\ w = 0 && \text{in } \mathbb{R}^3 \times \mathbb{R}^+ \\
&w(x,0) = \mu,\ div\ \mu = 0 && \text{in } \mathbb{R}^3
\end{aligned}$$

(which exists and is unique in view of the properties inherited by the limit μ and Theorem 5.1 in [6]). This implies that :

$$\lim_{t\to\infty} t^{1-\frac{3}{2p}} \|v(t) - w(t)\|_{M^p} = 0$$

for every $\frac{3}{2} \le p \le \infty$.

Theorem 2.1 : Estimates

There exists a constant $A > 0$ such that, if v is a solution of (V3) with $v_0 \in (\overline{M}^{\frac{3}{2}}(\mathbb{R}^3))^3$ such that $\|v_0\|_{\frac{3}{2}} \le A$, we have the following estimates:

$$\begin{array}{lll} (E_1) & \|v_\lambda(t)\|_{M^p} \le C_1(t,p) & \frac{3}{2} \le p \le \infty \\ (E_2) & \|\nabla v_\lambda(t)\|_{M^p} \le C_2(t,p) & \frac{3}{2} \le p \le \infty \\ (E_3) & \|v_{\lambda,t}(t)\|_{M^p} \le C_3(t,p) & \frac{3}{2} \le p \le \infty \end{array}$$

for some positive constants $C_i(t,p)$, $i = 1,2,3$ which depend on t in a continuous way. Moreover, if $\lambda^2 v_0(\lambda x)\chi_R(x) \to 0$ in $\overline{M}^{\frac{3}{2}}(\mathbb{R}^3)$ when $R \to \infty$ uniformly in $\lambda \ge 1$, where χ_R denotes the characteristic function of $B(0,R)^c$, $R > 0$ then :

$$(E_4) \quad \|v_\lambda(t)\chi_R\|_{\overline{M}^{\frac{3}{2}}} \to 0$$

when $R \to \infty$, uniformly with respect to $\lambda \ge 1$ and $t \in [0,t_1]$, $t_1 > 0$.

Proof

Let us consider the following iterative sheme :

$$\begin{array}{ll} v_{\lambda,j+1,t} - \Delta v_{\lambda,j+1} = -\partial_i\big((K^i * v_{\lambda,j})v_{\lambda,j} - v^i_{\lambda,j}(K * v_{\lambda,j})\big) & \text{in } \mathbb{R}^3 \times \mathbb{R}^+ \\ div\ v_{\lambda,j+1} = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}^+ \\ v_{\lambda,j+1}(x,0) = \lambda^2 v_0(\lambda x) = v_{0,\lambda},\ div\ v_{0,\lambda} = 0 & \text{in } \mathbb{R}^3 \end{array}$$

for $j \ge 0$, starting from the solution of the heat equation with the same initial data:

$$v_{\lambda,0}(t,x) = G * v_{0,\lambda}(t,x)$$

To estimate the norm of the solutions we write this system in integral form :

$$v_{\lambda,j+1}(t,x) = G * v_{0,\lambda}(t,x) + \int_0^t \partial_i G(t-s) * \big((K^i * v_{\lambda,j})v_{\lambda,j}(s) - v^i_{\lambda,j}(K * v_{\lambda,j})(s)\big)(x)ds$$

In the following we shall note $\|\|_p$ the norm in M^p. Let us first remark that the norms of $G * v_{0,\lambda}$ and their derivatives can be bounded uniformly with respect to λ. This fact follows by estimating the norms of the convolutions by means of the inequalities established in the Proposition 3.2 of [6], which generalize to Morrey spaces the usual estimates involving L^p norms. Indeed,

$$\|G(t) * v_{0,\lambda}\|_p \leq C_p t^{-1+\frac{3}{2p}} \|v_{0,\lambda}\|_{\frac{3}{2}} \quad p \geq \frac{3}{2},\ t > 0$$

In the sequel we note by C_p any constant depending only on p. Now,

$$\begin{aligned} \|v_{0,\lambda}\|_p &= \sup_{\substack{x\in\mathbb{R}^3\\ r>0}} r^{\frac{-3}{p'}} \int_{B(x,r)} \lambda^2 |v_0(\lambda y)|dy \\ &= \sup_{\substack{\lambda x\in\mathbb{R}^3\\ \lambda r>0}} \lambda^{\frac{3}{p'}} (\lambda r)^{\frac{-3}{p'}} \int_{B(\lambda x,\lambda r)} \lambda^{-1}|v_0(z)|dz = \quad \lambda^{\frac{3}{p'}-1}\|v_0\| \end{aligned}$$

so that $\|v_{0,\lambda}\|_{\frac{3}{2}} = \|v_0\|_{\frac{3}{2}}$. If $p < \frac{3}{2}$ we deduce that $\|v_{0,\lambda}\|_p \to 0$ and if $p > \frac{3}{2}$ then $\|v_{0,\lambda}\|_p \to \infty$.

Therefore,

$$\|G(t) * v_{0,\lambda}\|_p \leq C_p t^{-1+\frac{3}{2p}} \|v_0\|_{\frac{3}{2}} \quad p \geq \frac{3}{2},\ t > 0$$

Similarly, we have :

$$\|\partial_i G(t) * v_{0,\lambda}\|_p \leq C_p t^{-1+\frac{3}{2p}-\frac{1}{2}} \|v_0\|_{\frac{3}{2}} \quad p \geq \frac{3}{2},\ t > 0$$

$$\begin{aligned} \|\partial_i G(t+\tau) * v_{\lambda,0}\|_p &\leq C_p t^{-\frac{3}{2q}+\frac{3}{2p}} \|\partial G(\tau) * v_{\lambda,0}\|_q \\ &\leq C_p t^{-\frac{3}{2q}+\frac{3}{2p}} \tau^{-1+\frac{3}{2q}-\frac{1}{2}} \|v_0\|_{\frac{3}{2}} \qquad p \geq q \geq \tfrac{3}{2},\ t > 0 \end{aligned}$$

Analogous estimates hold for higher order derivatives.

Therefore, the estimates we want to prove are true for the lineal problem without any restriction on the size of the initial data. Let us remark that

$v_{\lambda,0} = G * v_{0,\lambda}$ are obtained from the solution of the heat equation $v = G * v_0$ by means of the scaling $v_{\lambda,0}(x,t) = \lambda^2 v(\lambda x, \lambda^2 t)$. Assuming that the $\overline{M}^{\frac{3}{2}}$ norm of v_0 is smaller than a constant A (which will be made explicit in the proof) we can use this fact to obtain uniform bounds of the norms of the solutions of the nonlinear problem by using the integral equation.

(E1) For every $T > 0$ we define

$$A_{\lambda,j,r} = \sup_{0 \le t \le T} t^{(1-\frac{3}{2r})} \|v_{\lambda,j}(t)\|_r$$

When $n = 3$

$$\|\partial_\alpha G(t) * v_0\|_p \le C(k)\, t^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})-\frac{k}{2}} \|v_0\|_q$$

for every $\alpha \in \mathbb{R}^3$ with $|\alpha| = k$, $v_0 \in M^{\frac{3}{2}}$ y $p \ge q$.

Set $C_1 = max(C(0), C(1), C(2))$. Then,

$$A_{\lambda,0,r} \le C_1\, \|v_0\|_{\frac{3}{2}}$$

for every $r \ge \frac{3}{2}$. In order to estimate the integral term we use the estimates for Riesz potentials in Morrey spaces established in [6] (Prop. 3.1, ii). Thanks to these estimates we know that $K * \mu \in L^\infty$ when $\mu \in M^p \cap M^q$ with p and q such that $\frac{1}{q} + \frac{2}{3} < 1 < \frac{1}{p} + \frac{2}{3}$. In particular, we can take p, $q = 2p$ with $\frac{3}{2} < p < 3$ and then :

$$\|K * \mu\|_\infty \le C_2 \|\mu\|_p^{\frac{2p}{3}-1} \|\mu\|_{2p}^{2-\frac{2p}{3}}$$

Therefore,

$$\|v^i(K * v) - (K * v)^i v\|_r \le \|K * v\|_\infty \|v\|_r \le C_2 \|v\|_p^{\frac{2p}{3}-1} \|v\|_{2p}^{2-\frac{2p}{3}} \|v\|_r$$

if $\frac{3}{2} < p < 3$. Taking norms in the integral equation we obtain :

$$\|v_{\lambda,j+1}(t)\|_r \le C_1 t^{-1+\frac{3}{2r}} \|v_0\|_{\frac{3}{2}}$$
$$+ \int_0^t C_1 (t-s)^{\frac{-1}{2}} C_2 \|v_{\lambda,j}\|_p^{\frac{2p}{3}-1} \|v_{\lambda,j}\|_{2p}^{2-\frac{2p}{3}} \|v_{\lambda,j}\|_r(s) ds$$

so that:

$$A_{\lambda,j+1,r} \leq C_1\|v_0\|_{\frac{3}{2}} + C_1C_2A_{\lambda,j,p}^{\frac{2p}{3}-1}A_{\lambda,j,2p}^{2-\frac{2p}{3}}A_{\lambda,j,r}\ t^{1-\frac{3}{2r}}\int_0^t (t-s)^{\frac{-1}{2}}s^{\frac{-3}{2}+\frac{3}{2r}}ds$$

By making a change of variables in the integral it follows that :

$$A_{\lambda,j+1,r} \leq C_1\|v_0\|_{\frac{3}{2}} + C_1C_2A_{\lambda,j,p}^{\frac{2p}{3}-1}A_{\lambda,j,2p}^{2-\frac{2p}{3}}A_{\lambda,j,r}B(\frac{1}{2},\frac{-1}{2}+\frac{3}{2r})$$

for $\frac{-1}{2}+\frac{3}{2r} > 0$, that is $r < 3$. We note by $B(x,y)$ the beta function

$$B(x,y) = \int_0^1 (1-\sigma)^{x-1}\sigma^{y-1}d\sigma \quad x > 0, y > 0.$$

Taking $r = p$ and $r = \frac{3}{2}$ we get :

$$A_{\lambda,j+1,p} \leq C_1\|v_0\|_{\frac{3}{2}} + C_1C_2B(\frac{1}{2},\frac{3}{2p}-\frac{1}{2})A_{\lambda,j,p}A_{\lambda,j,p}^{\frac{2p}{3}-1}A_{\lambda,j,2p}^{2-\frac{2p}{3}}$$

$$A_{\lambda,j+1,\frac{3}{2}} \leq C_1\|v_0\|_{\frac{3}{2}} + C_1C_2B(\frac{1}{2},\frac{1}{2})A_{\lambda,j,\frac{3}{2}}A_{\lambda,j,p}^{\frac{2p}{3}-1}A_{\lambda,j,2p}^{2-\frac{2p}{3}}$$

Similarly, from :

$$\|v_{\lambda,j+1}\|_{2p} \leq C_1t^{-1+\frac{3}{4p}}\|v_0\|_{\frac{3}{2}}$$
$$+\int_0^t C_1(t-s)^{\frac{-1}{2}-\frac{3}{2p}}C_2\|v_{\lambda,j}\|_p^{\frac{2p}{3}-1}\|v_{\lambda,j}\|_{2p}^{2-\frac{2p}{3}}\|v_{\lambda,j}\|_p(s)ds$$

we deduce that :

$$A_{\lambda,j+1,2p} \leq C_1\|v_0\|_{\frac{3}{2}} + C_1C_2B(\frac{1}{2}-\frac{3}{4p},\frac{3}{2p}-\frac{1}{2})A_{\lambda,j,p}A_{\lambda,j,p}^{\frac{2p}{3}-1}A_{\lambda,j,2p}^{2-\frac{2p}{3}}$$

if $p > \frac{3}{2}$.

Set $B = max(B(\frac{1}{2}-\frac{3}{4p},\frac{3}{2p}-\frac{1}{2}), B(\frac{1}{2},\frac{3}{2p}-\frac{1}{2}), B(\frac{1}{2},\frac{1}{2}))$ and $A_{\lambda,j} = max(A_{\lambda,j,p}, A_{\lambda,j,2p})$. Then,

$$\begin{aligned} A_{\lambda,j+1,\frac{3}{2}} &\leq C_1\|v_0\|_{\frac{3}{2}} + (C_1C_2BA_{\lambda,j})\ A_{\lambda,j,\frac{3}{2}} \\ A_{\lambda,j+1} &\leq C_1\|v_0\|_{\frac{3}{2}} + C_1C_2BA_{\lambda,j}^2 \end{aligned}$$

Assuming that $C_1C_2BA_{\lambda,j} \leq S$, we conclude that $C_1C_2BA_{\lambda,j+1} \leq S$ if

$$C_1^2C_2B\|v_0\|_{\frac{3}{2}} + S^2 \leq S$$

This is posible by taking $\|v_0\|_{\frac{3}{2}}$ small enough in order to have $C_1^2C_2B\|v_0\|_{\frac{3}{2}} \leq \frac{1}{4}$ and $S = \frac{1-(1-4C_1^2C_2B\|v_0\|_{\frac{3}{2}})^{\frac{1}{2}}}{2}$. It follows that :

$$A_{\lambda,j} \leq \frac{1}{2C_1C_2B}$$
$$A_{\lambda,j+1,\frac{3}{2}} \leq 2A_{0,\frac{3}{2}}$$

for every $j \geq 1$. In the same way as in [6] we deduce that, for a fixed λ, $v_{\lambda,j}$ converges to a limit noted v_λ in the weak topology of measures and in M^p, $\frac{3}{2} \leq p < 6$ when $j \to \infty$ (it suffices to estimate the norm of the diference $v_{\lambda,j+1} - v_{\lambda,j}$ and conclude that we have a Cauchy sequence). The limit v_λ satisfies the integral equation and we have the estimates :

$$\|v_\lambda(t)\|_{\frac{3}{2}} \leq 2\|v_0\|_{\frac{3}{2}}$$

$$\|v_\lambda(t)\|_r \leq \frac{1}{2C_1C_2B}t^{(-1+\frac{3}{2r})}$$

with $r = p, 2p$ provided $\|v_0\|_{\frac{3}{2}} \leq \frac{1}{4C_1^2C_2B} = A$ and $3 > p > \frac{3}{2}$. In order to bound the M^3 norm it suffices to remark that :

$$\|v_{\lambda,j+1}\|_3 \leq \|v_{\lambda,j+1}\|_{3+\varepsilon} + \|v_{\lambda,j+1}\|_{3-\varepsilon}$$

for every $\varepsilon > 0$ or to use an interpolation inequality (Prop. 2.3 i in [6])

If now we consider $r \geq q \geq \frac{3}{2}$ we have :

$$\|v_{\lambda,j+1}(t+\tau)\|_r \leq C_1t^{-1+\frac{3}{2r}}\|v_{\lambda,0}(\tau)\|_{\frac{3}{2}}$$

$$+\int_0^t C_1(t-s)^{\frac{-1}{2}-\frac{3}{2q}+\frac{3}{2r}}C_2\|v_{\lambda,j}\|_p^{\frac{2p}{3}-1}\|v_{\lambda,j}\|_{2p}^{2-\frac{2p}{3}}\|v_{\lambda,j}\|_q(s+\tau)ds$$

In view of the remarks we made for the heat equation we know that the norms $\|v_{\lambda,0}(\tau)\|_{\frac{3}{2}}$ are bounded uniformly with respect to λ and τ. Taking $3 > p > \frac{3}{2}$

and $6 > q > 3$ we know that $\|v_{\lambda,j}\|_p(s+\tau),\ \|v_{\lambda,j}\|_{2p}\ (t+\tau),\ \|v_{\lambda,j}\|_q(s+\tau)$ are bounded, uniformly with respect to λ and j by $\frac{1}{(2C_1C_2B)^2}\tau^{\frac{-3}{2}+\frac{3}{2q}}$. Therefore :

$$\|v_{\lambda,j+1}(2\tau)\|_r \leq C_1\tau^{-1+\frac{3}{2r}}\|v_{\lambda,0}(\tau)\|_q + \tau^{-\frac{3}{2}+\frac{3}{2q}}C_3\int_0^\tau(\tau-s)^{\frac{-1}{2}-\frac{3}{2q}+\frac{3}{2r}}ds$$

If $\frac{1}{2}-\frac{3}{2q}+\frac{3}{2r} > 0$, that is, if $\frac{1}{3}+\frac{1}{r} > \frac{1}{q}$, what is always true when $q > 3$ the integral is finite and we deduce that :

$$\|v_{\lambda,j+1}(2\tau)\|_r \leq C_4\tau^{-1+\frac{3}{2r}}$$

for every $r > 3$.

(E2) It can be proved in in the same way as the previous one. For every $T, \tau > 0$ we define

$$A_{\tau,\lambda,j,r} = \sup_{0\leq t\leq T} t^{(1-\frac{3}{2r})}\|\nabla v_{\lambda,j}(t+\tau)\|_r$$

Taking norms in the integral equation

$$\nabla v_{\lambda,j+1}(t) = G * \nabla v_{0,\lambda}(t) + \int_0^t \partial_i G(t-s) * \big((K^i * \nabla v_{\lambda,j})v_{\lambda,j}(s) \\ +(K^i * v_{\lambda,j})\nabla v_{\lambda,j}(s) - v^i_{\lambda,j}(K*\nabla v_{\lambda,j})(s) - \nabla v^i_{\lambda,j}(K * v_{\lambda,j})(s)\big)ds$$

we obtain :

$$\|\nabla v_{\lambda,j+1}(t+\tau)\|_r \leq C_1 t^{-1+\frac{3}{2r}}\|\nabla v_{\lambda,0}(\tau)\|_{\frac{3}{2}} \\ + \int_0^t C_1(t-s)^{\frac{-1}{2}}\|\nabla v_{\lambda,j}\|_r\|K * v_{\lambda,j}\|_\infty(s+\tau) \\ + \int_0^t C_1(t-s)^{\frac{-1}{2}}C_2\|\nabla v_{\lambda,j}\|_p^{\frac{2p}{3}-1}\|\nabla v_{\lambda,j}\|_{2p}^{2-\frac{2p}{3}}\|v_{\lambda,j}\|_r(s+\tau)ds$$

Taking $r = p, 2p$ with $p \in (\frac{3}{2}, 3)$ and using the previous estimate (E1) we have :

$$\begin{aligned} A_{\tau,\lambda,j+1,p} \leq\ & C_1C_2T^{\frac{1}{2}}A_{\tau,\lambda,j,p}\ \frac{\tau^{\frac{-1}{2}}}{2C_1C_2B}\ B(\tfrac{1}{2},\tfrac{3}{2p}) \\ +\ & C_1\|\nabla v_{\lambda,0}(\tau)\|_{\frac{3}{2}} + C_1C_2T^{1-\frac{3}{2p}}A_{\tau,\lambda,j,p}^{\frac{2p}{3}-1}A_{\tau,\lambda,j,2p}^{2-\frac{2p}{3}}\frac{\tau^{-1+\frac{3}{2p}}}{2C_1C_2B}B(\tfrac{1}{2},\tfrac{1}{2}) \end{aligned}$$

$$A_{\tau,\lambda,j+1,2p} \leq \quad C_1C_2T^{\frac{1}{2}}A_{\tau,\lambda,j,2p}\ \frac{\tau^{\frac{-1}{2}}}{2C_1C_2B}\ B(\tfrac{1}{2},\tfrac{3}{4p})$$

$$+ \quad C_1\|\nabla v_{\lambda,0}(\tau)\|_{\frac{3}{2}} + C_1C_2T^{1-\frac{3}{4p}}A_{\tau,\lambda,j,p}^{\frac{2p}{3}-1}A_{\tau,\lambda,j,2p}^{2-\frac{2p}{3}}\frac{\tau^{-1+\frac{3}{4p}}}{2C_1C_2B}B(\tfrac{1}{2},\tfrac{1}{2})$$

Set $B' = max(B(\frac{1}{2},\frac{3}{2p}), B(\frac{1}{2},\frac{3}{4p}), B(\frac{1}{2},\frac{1}{2}))$ and $A_{\tau,\lambda,j} = max(A_{\tau,\lambda,j,p},$ $A_{\tau,\lambda,j,2p})$. Then, if $T = a\tau,\ a \leq 1$ we have

$$A_{\tau,\lambda,j+1} \leq C_1\|\nabla v_{\lambda,0}(\tau)\|_{\frac{3}{2}} + (\frac{a^{\frac{1}{2}}B' + a^{1-\frac{3}{2p}}B}{2B})A_{\tau,\lambda,j}$$

In $(E1)$ we can replace B by $B+B'$. Therefore we can assume that $B > B'$and take $a = 1$. Or either we choose a small enough to have :

$$A_{\tau,\lambda,j+1} \leq 2C_1\|\nabla v_{\lambda,0}(\tau)\|_{\frac{3}{2}}$$

Thus,

$$\|\nabla v_{\lambda,j}(t+\tau)\|_r \leq 2C_1 t^{\frac{-3}{2}+\frac{3}{2r}}\|v_0\|_{\frac{3}{2}}$$

for $r = p, 2p$ with $\frac{3}{2} < p < 3$ and $0 < t \leq a\tau$. Taking limits it follows that :

$$\|\nabla v_\lambda((1+a)\tau)\|_r \leq c\tau^{\frac{-3}{2}+\frac{3}{2r}}\|v_0\|_{\frac{3}{2}}$$

The M^r norms with $r > 3$ and $r = 3, \frac{3}{2}$ are bounded like in the previous case.

(E3) By using the equation, to obtain estimates on the norm $v_{\lambda,t}$ it suffices to estimate Δv_λ. For every $T, \tau > 0$ we define again:

$$A_{\tau,\lambda,j,r} = \sup_{0\leq t\leq T} t^{(1-\frac{3}{2r})}\|\partial_{l,k}v_{\lambda,j}(t+\tau)\|_r$$

where $\|\partial_{l,k}v_{\lambda,j}(t)\|_r = \sum_{l,k}\|\partial_{l,k}v_{\lambda,j}(t)\|_r$. Taking norms in the integral equation

$$\partial_{l,k}v_{\lambda,j+1}(t) = G * \partial_{l,k}v_{0,\lambda}(t)$$
$$+ \int_0^t \partial_i G(t-s) * (K^i * \partial_{l,k}v_{\lambda,j})v_{\lambda,j}(s) + (K^i * v_{\lambda,j})\partial_{l,k}v_{\lambda,j}(s)ds$$
$$- \int_0^t \partial_i G(t-s) * v^i_{\lambda,j}(K * \partial_{l,k}v_{\lambda,j})(s) - \partial_{l,k}v^i_{\lambda,j}(K * v_{\lambda,j})(s)ds$$
$$+ \int_0^t \partial_i G(t-s) * (K^i * \partial_k v_{\lambda,j})\partial_l v_{\lambda,j}(s) - \partial_l v^i_{\lambda,j}(K * \partial_k v_{\lambda,j})(s)ds$$
$$+ \int_0^t \partial_i G(t-s) * (K^i * \partial_l v_{\lambda,j})\partial_k v_{\lambda,j}(s) - \partial_k v^i_{\lambda,j}(K * \partial_l v_{\lambda,j})(s)ds$$

we obtain :

$$
\begin{aligned}
\|\partial_{l,k} v_{\lambda,j+1}(t+\tau)\|_r \ & \leq C_1 t^{-1+\frac{3}{2r}} \|\partial_{l,k} v_{\lambda,0}(\tau)\|_{\frac{3}{2}} \\
+ \int_0^t C_1 (t-s)^{\frac{-1}{2}} \big(C_3 \|\nabla v_{\lambda,j}\|_r \|K * \nabla v\|_\infty & + \|\partial_{l,k} v_{\lambda,j}\|_r \|K * v\|_\infty \\
+ C_2 \|\partial_{l,k} v_{\lambda,j}\|_p^{\frac{2p}{3}-1} \|\partial_{l,k} v_{\lambda,j}\|_{2p}^{2-\frac{2p}{3}} & \|v_{\lambda,j}\|_r \big)(s+\tau) ds
\end{aligned}
$$

Taking $r = p, 2p$ with $p \in (\frac{3}{2}, 3)$ and using the estimates obtained in (E1) and (E2) we get:

$$
\begin{aligned}
A_{\tau,\lambda,j+1,p} \leq \quad & C_1 C_\tau + C_1 C_2 T^{\frac{1}{2}} A_{\tau,\lambda,j,p} \ \frac{\tau^{\frac{-1}{2}}}{2C_1C_2B} \ B(\tfrac{1}{2}, \tfrac{3}{2p}) \\
+ \quad & C_1 C_2 T^{1-\frac{3}{2p}} \big(A_{\tau,\lambda,j,p}^{\frac{2p}{3}-1} A_{\tau,\lambda,j,2p}^{2-\frac{2p}{3}} \ \frac{\tau^{-1+\frac{3}{2p}}}{2C_1C_2B} \ B(\tfrac{1}{2}, \tfrac{1}{2}) \ + \ C_\tau \big)
\end{aligned}
$$

$$
\begin{aligned}
A_{\tau,\lambda,j+1,2p} \leq \quad & C_1 C_\tau + C_1 C_2 T^{\frac{1}{2}} A_{\tau,\lambda,j,2p} \ \frac{\tau^{\frac{-1}{2}}}{2C_1C_2B} \ B(\tfrac{1}{2}, \tfrac{3}{4p}) \\
+ \quad & C_1 C_2 T^{1-\frac{3}{4p}} \big(A_{\tau,\lambda,j,p}^{\frac{2p}{3}-1} A_{\tau,\lambda,j,2p}^{2-\frac{2p}{3}} \ \frac{\tau^{-1+\frac{3}{4p}}}{2C_1C_2B} \ B(\tfrac{1}{2}, \tfrac{1}{2}) \ + \ C_\tau \big)
\end{aligned}
$$

if $0 \leq T \leq \tau$.

Set $A_{\tau,\lambda,j} = max(A_{\tau,\lambda,j,p}, A_{\tau,\lambda,j,2p})$. and $B' = max(B(\frac{1}{2}, \frac{3}{2p}), B(\frac{1}{2}, \frac{3}{4p}), B(\frac{1}{2}, \frac{1}{2}))$ Then, if $T = a\tau,\ a \leq 1$ we have

$$
A_{\tau,\lambda,j+1} \ \leq \ C_\tau + (\frac{a^{\frac{1}{2}} B' + a^{1-\frac{3}{2p}} B}{2B}) A_{\tau,\lambda,j}
$$

As we did in $(E2)$ taking a small enough we conclude :

$$
A_{\tau,\lambda,j+1} \leq C_\tau
$$

so that

$$
\|\partial_{l,k} v_{\lambda,j}(t+\tau)\|_r \leq C_\tau
$$

for $r = p, 2p$ with $\frac{3}{2} < p < 3$ and $0 < t \leq a\tau$. Therefore,

$$
\|\partial_{l,k} v_\lambda((1+a)\tau)\|_r \leq C_\tau
$$

By intepolating we obtain the bound for the M^3 norm. Let us assume that $r \geq q > \frac{3}{2}$. Taking norms in the integral equation :

$$\begin{gathered}\|\partial_{l,k} v_{\lambda,j+1}(t+\tau)\|_r \leq C_1 t^{-1+\frac{3}{2r}} \|\partial_{l,k} v_{\lambda,0}(\tau)\|_{\frac{3}{2}} \\ + \int_0^t C_1 (t-s)^{\frac{-1}{2}-\frac{3}{2q}+\frac{3}{2r}} \|\partial_{l,k} v_{\lambda,j}\|_q \|K * v_{\lambda,j}\|_\infty (s+\tau) ds \\ + \int_0^t C_1 (t-s)^{\frac{-1}{2}} C_2 \|\partial_{l,k} v_{\lambda,j}\|_p^{\frac{2p}{3}-1} \|\partial_{l,k} v_{\lambda,j}\|_{2p}^{2-\frac{2p}{3}} \|v_{\lambda,j}\|_r (s+\tau) ds \\ + \int_0^t C_1 C_3 (t-s)^{\frac{-1}{2}} \|\nabla v_{\lambda,j}\|_r \|K * \nabla v_{\lambda,j}\|_\infty (s+\tau) ds\end{gathered}$$

and choosing $6 > q > 3$ we conclude that :

$$\|\partial_{l,k} v_{\lambda,j}(t)\|_r \leq C_\tau .$$

It remains the case $r = \frac{3}{2}$. We consider the inequality :

$$\begin{gathered}\|\partial_{l,k} v_{\lambda,j+1}(t+\tau)\|_{\frac{3}{2}} \leq C_1 \|\partial_{l,k} v_{\lambda,0}(\tau)\|_{\frac{3}{2}} \\ + \int_0^t C_1 (t-s)^{\frac{-1}{2}} \big(\|\nabla v_{\lambda,j}\|_{\frac{3}{2}} \|K * \nabla v_{\lambda,j}\|_\infty + \|\partial_{l,k} v_{\lambda,j}\|_{\frac{3}{2}} \|K * v_{\lambda,j}\|_\infty \\ + C_2 \|\partial_{l,k} v_{\lambda,j}\|_p^{\frac{2p}{3}-1} \|\partial_{l,k} v_{\lambda,j}\|_{2p}^{2-\frac{2p}{3}} \|v_{\lambda,j}\|_{\frac{3}{2}} \big)(s+\tau) ds\end{gathered}$$

Using the estimates we have just proved it follows that:

$$A_{\tau,\lambda,j+1,\frac{3}{2}} \leq C_\tau + C_1 C_2 \, \frac{a^{\frac{1}{2}}}{2 C_1 C_2 B} B(\frac{1}{2}, 1) A_{\tau,\lambda,j,\frac{3}{2}}$$

if $0 < T \leq a\tau$. Therefore,

$$\|\partial_{l,k} v_\lambda((1+a)\tau)\|_{\frac{3}{2}} \leq C_\tau .$$

(E4) Take $\psi(x) \in C^2(\mathbb{R}^3)$ such that $\psi = 0$ si $|x| \leq 1$ and $\psi = 1$ when $|x| \geq 2$. Let us note $\psi_R(x) = \psi(\frac{x}{R})$ and $v_{\lambda,R} = v_\lambda \psi_R$. The function $v_{\lambda,R}$ satisfies the equation :

$$\begin{gathered}v_{\lambda,R,t} - \Delta v_{\lambda,R} = (v_{\lambda,t} - \Delta v_\lambda)\psi_R - \tfrac{2}{R} \nabla v_\lambda (\nabla\psi)_R - \tfrac{1}{R^2} (\Delta\psi)_R v_\lambda \\ = \partial_i \big((K * v_\lambda)^i v_{\lambda,R} - (K * v_\lambda) v^i_{\lambda,R} \big) + \tfrac{(\partial_i \psi)_R}{R} \big((K * v_\lambda)^i v_\lambda - (K * v_\lambda) v^i_\lambda \big) \\ - \tfrac{2}{R} \nabla v_\lambda (\nabla\psi)_R - \tfrac{1}{R^2} (\Delta\psi)_R v_\lambda\end{gathered}$$

Writing the associated integral equation and taking the norm $M^{\frac{3}{2}}$ we obtain:

$$\begin{aligned}\|v_{\lambda,R}\|_{\frac{3}{2}}(t) &\leq C\|v_{0,\lambda,R}\|_{\frac{3}{2}} + C\int_0^t (t-s)^{\frac{-1}{2}}\|v_{\lambda,R}\|_{\frac{3}{2}}(s)\|K*v_\lambda\|_\infty(s)ds \\ &+\tfrac{C}{R}\int_0^t(\|\nabla v_\lambda\|_{\frac{3}{2}} + \|K*v_\lambda\|_\infty\|v_\lambda\|_{\frac{3}{2}})(s)ds + \tfrac{C}{R^2}\int_0^t\|v_\lambda\|_{\frac{3}{2}}\end{aligned}$$

so that

$$\|v_{\lambda,R}\|_{\frac{3}{2}}(t) \leq C\|v_{0,\lambda,R}\|_{\frac{3}{2}} + \frac{1}{2}\int_0^t (t-s)^{\frac{-1}{2}} s^{\frac{-1}{2}} \|v_{\lambda,R}\|_{\frac{3}{2}}(s)ds + \frac{C}{R}(t^{\frac{1}{2}}+t)$$

It follows that:

$$\|v_{\lambda,R}\|_{\frac{3}{2}}(t) \leq \frac{C}{R}(t^{\frac{1}{2}}+t) + C\|v_{0,\lambda,R}\|_{\frac{3}{2}}$$

which tends to zero when R tends to infinity, uniformly with respect to λ and t, when t runs over a compact interval.

Theorem 2.2 : Convergence

There exists a constant $A > 0$ such that if v_0 satisfies the following conditions :

(2.2.1) $v_0 \in (\overline{M}^{\frac{3}{2}}(\mathbb{R}^3))^3$, *div* $v_0 = 0$, $\|v_0\|_{\frac{3}{2}} \leq A$,

(2.2.2) $\lambda^2 v_0(\lambda x) \to \mu$ *on balls in the weak topology of measures when* $\lambda \to \infty$,

(2.2.3) $\lambda^2 v_0(\lambda x)\chi_R(x) \to 0$ *in* $\overline{M}^{\frac{3}{2}}(\mathbb{R}^3)$ *when* $R \to \infty$ *uniformly with respect to* $\lambda \geq 1$, *where* χ_R *stands for the characteristic function of* $B(0,R)^c$.

then,

$$\lim_{t\to\infty} t^{1-\frac{3}{2p}} \|v(t) - \nu(t)\|_p = 0$$

for every $\frac{3}{2} \leq p \leq \infty$, *where* ν *is the unique solution of (F3)*

$$(F3) \qquad \begin{cases} \nu_t - \Delta\nu + \partial_i((K*\nu)^i\nu - \nu^i(K*\nu)) = 0 & \mathit{in}\ \mathbb{R}^3\times\mathbb{R}^+ \\ \mathit{div}\ \nu = 0 & \mathit{in}\ \mathbb{R}^3\times\mathbb{R}^+ \\ \nu(x,0) = \mu & \mathit{in}\ \mathbb{R}^3 \end{cases}$$

This solution ν *is known to be selfsimilar.*

Remark 2.1

The constant A is choosen in such a way that the estimates of Theorem 2.1 and the uniqueness results obtained in [6] are valid.

It follows from the hypotheses that the limit $\mu \in \overline{M}^{\frac{3}{2}}$ *with* $\|\mu\|_{\frac{3}{2}} \leq A$ *and satisfies the homogeneity condition (H2). In these conditions we know that (F3) has a unique solution, which is also selfsimilar and, therefore, of the form :*

$$u(x,t) = t^{-1} f(\frac{x}{t^{\frac{1}{2}}})$$

where f is a solution of:

$$\begin{array}{ll} -\Delta f - \frac{x_i}{2}\partial_i f - f + \partial_i((K * f)^i f - f^i (K * f)) = 0 & in\ \mathbb{R}^3 \\ div\ f = 0 & in\ \mathbb{R}^3 \end{array}$$

and satisfies :

$$t^{-1} f(\frac{x}{t^{\frac{1}{2}}}) \to \mu \quad if \quad t \to 0$$

Proof

Since $M^\infty = L^\infty$, the estimates obtained in Theorem 2.1 imply that v_λ is bounded in $C^1_{\text{loc}}((0,\infty); L^\infty(\mathbb{R}^3)) \cap C_{\text{loc}}((0,\infty); W^{1,\infty}(\mathbb{R}^3))$. Set $B_R = B(0,R)$. In view of the compactness of the injection $W^{1,p}(B_R) \to C(B_R)$ for p large we deduce that v_λ is compact in $C((0,T) \times B_R)$. Therefore, we can extract a subsequence v_{λ_i}, noted again v_λ such that :

$$\begin{array}{llll} v_\lambda & \to \nu & \text{in } L^\infty(0,T; L^\infty(\mathbb{R}^3)) & \text{weak } * \\ \nabla v_\lambda & \to \nabla\nu & \text{in } L^\infty(0,T; L^\infty(\mathbb{R}^3)) & \text{weak } * \\ v_{\lambda,t} & \to \nu_t & \text{in } L^\infty(0,T; L^\infty(\mathbb{R}^3)) & \text{weak } * \\ v_\lambda & \to \nu & \text{in } C((0,T); L^q(B_R)) & \text{strong} \quad \forall R > 0 \end{array}$$

for some $\nu \in W^{1,\infty}(0,T; L^\infty(\mathbb{R}^3))$. The estimate (E1) implies that $\nu(t) \in M^p$ for every $p \geq \frac{3}{2}$ and almost every $t \in (0,T)$.

On the other hand, for almost every t

$$\begin{aligned} \|v_\lambda(t) - \nu(t)\|_p &= \sup_{\substack{x\in\mathbb{R}^3 \\ r>0}} \tfrac{1}{r}\big(\textstyle\int_{B(x,r)\cap B_R} |v_\lambda - \nu|(t) + \int_{B(x,r)\cap B_R^c} |v_\lambda - \nu|(t)\big) \\ &= S_1(R) + S_2(R) \end{aligned}$$

Since

$$\int_{B(x,r)\cap B_R} |v_\lambda - \nu|(t) \le \|v_\lambda(t) - \nu(t)\|_{L^{\frac{3}{2}}(B_R)} m(B(x,R))^{\frac{1}{3}}$$

we have that $S_1(R) \to 0$ when $\lambda \to \infty$ for each fixed R. The estimate $(E4)$ implies that

$$S_2(R) \to 0 \quad R \to \infty$$

uniformly with respect to $\lambda \ge 1$ and $t \in (t_0, t_1)$. Given $\varepsilon > 0$ we choose R such that $S_2(R) < \frac{\varepsilon}{2}$. Fixed R, we choose λ such that $S_1(R) < \frac{\varepsilon}{2}$. Therefore, we have proved that :

$$\|v_\lambda(t) - \nu(t)\|_{\frac{3}{2}} \to 0$$

when $\lambda \to \infty$.

The convergence is extended to $\infty > p \ge \frac{3}{2}$ by means of the interpolation inequality:

$$\|v_\lambda - \nu\|_p(t) \le \|v_\lambda(t) - \nu(t)\|_{\frac{3}{2}}^{\alpha} \|v_\lambda(t) - \nu(t)\|_\infty^{1-\alpha}$$

with $0 \le \alpha = \frac{3}{2p} \le 1$. To prove this inequality it suffices to remark that :

$$\|w\|_p = \sup_{\substack{x\in\mathbb{R}^3\\ r>0}} \frac{1}{r^{\frac{3}{p'}}} \int_{B(x,r)} |w| \le \sup_{\substack{x\in\mathbb{R}^3\\ r>0}} \frac{1}{r^{\frac{3}{p'}}} \int_{B(x,r)} |w|^\alpha \; \|w\|_\infty^{1-\alpha} \le \|w\|_{\frac{3}{2}}^{\alpha} \|w\|_\infty^{1-\alpha}$$

The convergence when $p = \infty$ follows from (Proposition 3.3 *ii* in [6])

$$\|w\|_\infty \le C\|w\|_p^{1-\frac{n}{p}} \|\nabla w\|_p^{\frac{n}{p}}$$

if $w \in M^p,\ \nabla w \in M^p,\ n < p < \infty.$

Once we have proved the strong convergence, it follows inmediatly that the limit ν is a solution of (F3). Taking into account that ν is selfsimilar

$$\|v_\lambda(t) - \nu(t)\|_p = \lambda^{\frac{3}{p'}-1} \|v(\lambda^2 t) - \nu(t)\|_p$$

we deduce that

$$\lim_{t\to\infty} \; t^{1-\frac{3}{2p}} \; \|v(t) - \nu(t)\|_p \; = \; 0$$

Remark 2.2

1) According to this theorem, the asymptotic behavior of the solutions of the three dimensional vorticity system (V3) when t tends to infinity is determined by the initial data we consider.

This remark also holds in two dimensions. In this case the mass of the data is reflected in the asymptotic behavior. Let us recall (see section III.1) that when $v_0 \in L^1(\mathbb{R}^2)$ and $\int_{\mathbb{R}^2} v_0$ is small enough the solution of the vorticity equation in two dimensions (V2) is asymptotically equivalent to the solution of the heat equation with initial data v_0. It is known that the solutions of the heat equation corresponding to initial data with mass M also behave in first approximation like the fundamental solution with mass M, that is, like M G(t). Such solutions are the unique sefsimilar solutions of the heat equation and, due to their radial symmetry they are also solutions of (V2).

2) Turning back to the three dimensional case, we can describe the asymptotic behavior of the solutions of (V3) by means of selfsimilar solutions of (F3), provided that the initial data satisfy the conditions 2.2.1, 2.2.2 and 2.2.3. The limit μ has the homogeneity poperty $\mu(B(\lambda x, \lambda r)) = \lambda \mu B(x, r)$.

Let us consider several particular cases.

Exemple 1

Let $v_0 \in L^{\frac{3}{2}}(\mathbb{R}^3)$ be such that $\|v_0\|_{L^{\frac{3}{2}}} \leq A$. In this case, when $t > 0$, the solution and its derivatives belong to L^p for every $p \geq \frac{3}{2}$ and Theorems 2.1, 2.2 remain true by replacing the M^p norms by L^p norms.

Since $\|\lambda^2 v_0(\lambda x)\|_{L^{\frac{3}{2}}} = \|v_0\|_{L^{\frac{3}{2}}}$, we can extract a subsequence v_{0,λ_i} tending to a limit μ in $L^{\frac{3}{2}}$ weak. Let us identify the limit. Since $v_{0,\lambda}$ is bounded in $L^{\frac{3}{2}}$ norm it suffices to consider test functions in a dense subspace, for instance $C_c^\infty(\mathbb{R}^3)$. Let us take $\phi \in C_c^\infty(\mathbb{R}^3)$. We have :

$$\int_{\mathbb{R}^3} \lambda^2 v_0(\lambda x)\phi(x)dx = \int_{\mathbb{R}^3} v_0(z)\phi(\lambda^{-1}z)\lambda^{-1}dz \to \int_{\mathbb{R}^3} \mu(x)\phi(x)dx$$

On the other hand,

$$\|\phi(\lambda^{-1}z)\lambda^{-1}\|_p = \lambda^{-1+\frac{3}{p}}\|\phi\|_p$$

Taking $p = 3 = (\frac{3}{2})'$ we see that the L^3 norm is bounded so that $\phi(\lambda^{-1}z)\lambda^{-1}$ tens to ν in L^3 weak. Taking into account that $\|\phi(\lambda^{-1}z)\lambda^{-1}\|_p$ tends to zero if $p > 3$ it follows that $\nu = 0$ so that

$$\int_{\mathbb{R}^3} \lambda^2 v_0(\lambda x)\phi(x)dx = \int_{\mathbb{R}^3} v_0(z)\phi(\lambda^{-1}z)\lambda^{-1}dz \to 0 = \int_{\mathbb{R}^3} \mu(x)\phi(x)dx$$

Therefore, $\mu = 0$ and the uniqueness of the limit implies the convergence of the whole family $\lambda^2 v_0(\lambda x) \to 0$. The corresponding solution of (F3) is the zero one and we conclude that :

$$t^{1-\frac{3}{2p}}\|v(t)\|_p \to 0 \quad \text{si } t \to \infty$$

provide that 2.2.3 is true. To prove 2.2.3 it suffices to remark that :

$$\|\lambda^2 v_0(\lambda x)\chi_R\|_{M^{\frac{3}{2}}(\mathbb{R}^3)} \leq C\|v_0(x)\|_{L^{\frac{3}{2}}(|x|>\lambda R)}$$

Exemple 2

Let us take v_0 of the form

$$v_0(x_1, x_2, x_3) = (\phi_1(x_2, x_3), \phi_2(x_1, x_3), \phi_3(x_1, x_2))$$

with $\phi_i \in L^1(\mathbb{R}^2)$. We remark that if $\phi \in L^1$

$$\frac{1}{r}\int_{B(z,r)} |\phi(x_1, x_2)|dx_1dx_2dx_3 \leq C\int_{\mathbb{R}^3} \phi(x)dx$$

so that $v_0 \in (M^{\frac{3}{2}}(\mathbb{R}^3))^3$. Such v_0 satisfies $div\ v_0 = 0$ and its norm is small if the $L^1(\mathbb{R}^2)$ norms of the coordinates also are. On the other hand, if $\psi \in C_c(\mathbb{R}^3)$

$$\int_{\mathbb{R}^3} \lambda^2\phi_3(\lambda x_1, \lambda x_2)\psi(x_1, x_2, x_3)dx_1dx_2dx_3 \to (\int_{\mathbb{R}^2} \phi_1)\int_{\mathbb{R}} \psi(0, 0, x_3)dx_3$$

so that

$$\lambda^2\phi_3(\lambda x_1, \lambda x_2) \to (\int_{\mathbb{R}^2} \phi_3)\delta_{x_1,x_2} = C_3\delta_{x_1,x_2}$$

Analogous results hold for the other components.

Condition 2.2.3 follows from:

$$\frac{1}{r}\int_{B(z,r)\cap B(O,R)^c}\lambda^2\phi(\lambda x_1,\lambda x_2)dx_1dx_2dx_3 \le C\int_{|x|>\lambda R}\phi(x_1,x_2)dx_1dx_2$$

Exemple 3

If we take a initial data of the form $v_0 = \mu + v_0$ where :

- $v_0, div\ \nu = 0$ y $\|v_0\|_{\frac{3}{2}} \le A$,

- μ belongs to the class of measures in $M^{\frac{3}{2}}$ generating selfsimilar solutions with

$$\|\mu\|_{\frac{3}{2}} \le A \quad \text{and} \quad div\mu = 0,$$

we have that $v_{0,\lambda} \rightharpoonup \mu$. However, condition 2.2.3 fails. If we suppose that μ is defined by a function of $L^{\frac{3}{2},\infty}(\mathbb{R}^3)$ the homogeneity condition implies that such a function must be homogeneous of degree -2. For instance, any combination of:

$$\frac{(0,-x_3,x_2)}{|x|^3},\ \frac{(x_3,0,-x_1)}{|x|^3},\ \frac{(-x_2,x_1,0)}{|x|^3}$$

will do. But 2.2.3 forced the $\overline{M}^{\frac{3}{2}}$ norm to be zero at infinity. Another kind of measures we can consider are those supported by rays γ_j emanating from the origin:

$$\langle \mu,\phi\rangle = \sum_{j=1}^{k}\alpha_j\int_{\gamma_j}\phi d\gamma_j \quad \phi \in (\phi^1,\phi^2,\phi^3) \in C_0(\mathbb{R}^3)$$

with $\alpha_j \in \mathbb{R}$, and $\sum \alpha_j = 0$ in order the divergence to be zero.

Remark 2.3

In general, if $v_0 \in L^q(\mathbb{R}^n)$ and $v_{0,\lambda} = \lambda^{\frac{n}{q}} v_0(\lambda x)$ we have :

$$\|v_0(\lambda x)\lambda^{\frac{n}{q}}\|_p = \lambda^{\frac{n}{q}-\frac{n}{p}}\|v_0\|_p$$

The L^q norm is constant, so that we can extract a subsequence converging weakly in L^q if $1 < q < \infty$, in $M(\mathbb{R}^3)$ if $q = 1$. Arguing as we did before we have :

$$\int_{\mathbb{R}^n} v_{0,\lambda}(x)\phi(x)dx = \int_{\mathbb{R}^n} v_0(z)\phi(\lambda^{-1}z)\lambda^{\frac{-n}{q'}}dz \to \int_{\mathbb{R}^n}\mu(x)\phi(x)dx = \langle\mu,\phi\rangle$$

Taking into account that

$$\|\phi(\lambda^{-1}z)\lambda^{\frac{-n}{q'}}\|_p = \lambda^{\frac{-n}{q'}+\frac{n}{p}}\|\phi\|_p$$

we deduce that $\phi(\lambda^{-1}z)\lambda^{\frac{-n}{q'}}$ *tends weakly to zero in* $L^{q'}$ *si* $q' < p$ *for some* p*, that is, if* $q > 1$*. When* $q = 1$ *we know that* $v_{0,\lambda} \rightharpoonup (\int v_0)\delta$ *in* $M(\mathbb{R}^n)$*. We have proved that, if* $\infty > q > 1$*, then* $v_{0,\lambda} \rightharpoonup 0$ *in* $L^q(\mathbb{R}^n)$ *when* $\lambda \to \infty$*.*

Let us now consider the linear heat equation. Let u *be a solution*

$$\begin{aligned} u_t - \Delta u = 0 \qquad & in\ \mathbb{R}^n \times \mathbb{R}^+ \\ u(x,0) = u_0 \qquad & in\ \mathbb{R}^n \times \mathbb{R}^+ \end{aligned}$$

with $u_0 \in L^p$, $p > 1$ *not belonging to* L^q *for* $q < p$*. It is known that:*

$$\|u(t)\|_r \leq Ct^{\frac{n}{2}(\frac{1}{p}-\frac{1}{r})}\|u_0\|_p \quad r \geq q,\ t > 0$$

We consider the family of functions $u_\lambda = \lambda^{\frac{n}{p}}(\lambda x, \lambda^2 t)$*. For these functions we can obtain analogous estimates to those of Theorems 1.1 and 2.1, which allow to prove the convergence to a solution of*

$$\begin{aligned} \nu_t - \Delta \nu = 0 \qquad & in\ \mathbb{R}^n \times \mathbb{R}^+ \\ \nu(x,0) = \mu \qquad & in\ \mathbb{R}^n \times \mathbb{R}^+ \end{aligned}$$

where :

- $\mu = (\int_{\mathbb{R}^2} u_0)\delta$ *if* $p = 1$

- $\mu = 0$ *if* $1 < p < \infty$

- *if* $p = \infty$ *the limit may fail to exist. In this case, unlike the preceeding ones, there exist data* u_0 *which are invariant by the scaling, that is, such that* $u_0(x) = u_0(\lambda x)$ *for every* $\lambda > 0$*. It suffices to take* $u_0 = k$ *or* $u_0 = k\chi_{x^i \geq 0, i=1,\ldots,n}$*, which generate selfsimilar solutions of the form* k *or* $\frac{k}{\pi^{\frac{n}{2}}}\int_{\{\frac{x^i}{2t^{\frac{1}{2}}} \geq z^i\}} e^{-|z|^2}dz$*.*

Therefore,

- *if* $p = 1$

$$t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{r})}\|u(t) - (\int_{\mathbb{R}^2} u_0)G(t)\|_r \to 0$$

when t tends to infinity.

- *if* $1 < p < \infty$

$$t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{r})}\|u(t)\|_r \to 0$$

when t tends to infinity (which is evident if $u_0 \in L^q$ for $q < p$) and $r \geq p$.

- *if* $p = \infty$

$$\|u(t) - \nu(t)\|_r \to 0$$

when t tends to infinity if there exists a function $\mu \in L^\infty$ which is scaling invariant and such that $u_0 - \mu$ belongs to the clousure of C_c^∞ in L^∞ so that $u - \nu$ satisfies $(E4)$.

If we want to consider data such that $u_0(\lambda x) = \lambda^{\frac{N}{p}} u_0(\lambda x)$ with $p \neq \infty$ we must forget the L^p spaces and consider $L^{p,\infty}$. Taking in the heat equation the same data as in Exemple 3 and $\mu \in L^{\frac{3}{2},\infty}$ we deduce that $u - \nu$ satisfies $(E4)$ so that:

$$t^{(1-\frac{3}{2r})}\|u(t) - \nu\|_r \to 0$$

when t tends to infinity and $r \geq \frac{3}{2}$. However, when dealing with $(V3)$, it is not clear how to prove $(E4)$ because of the convolution appearing in the nonlinear term.

ACKNOWLEDGEMENTS

The author would like to thank E. Zuazua for suggesting her this problem. This work has been partially supported by project PB90-0245 of the DGICYT (Spain).

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