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Wave solutions for a discrete reaction-diffusion equation

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Motivated by models from fracture mechanics and from biology, we study the infinite system of differential equations

$$u_n' = u_{n-1} - 2u_n + u_{n+1} - A\sin u_n + F, \quad ' = \frac{d}{dt},$$

where A and F are positive parameters. For fixed $A > 0$ we show that there are monotone travelling waves for F in an interval $F_{crit} < F < A$, and we are able to give a rigorous upper bound for F_{crit}, in contrast to previous work on similar problems. We raise the problem of characterizing those nonlinearities (apparently the more common) for which $F_{crit} > 0$. We show that, for the sine nonlinearity, this is true if $A > 2$. (Our method yields better estimates than this, but does not include all $A > 0$.) We also consider the existence and multiplicity of time independent solutions when $|F| < F_{crit}$.

1 Introduction

We consider the following one-dimensional equation for an infinite sequence of unknown functions $\{u_n(t)\}$:

$$u_n' = u_{n+1} - 2u_n + u_{n+1} - A\sin u_n + F, \quad ' = \frac{d}{dt}, \tag{1.1}$$

where A and F are positive parameters. (The signs of the parameters are mathematically unimportant, since the equation is invariant if we change the sign of A and at the same time replace u_n by $u_n + \pi$, or if we change the sign of F and at the same time replace u_n with $-u_n$.)

This equation arises in several applied areas. In fracture mechanics it is related to the Frenkel–Kontorova model [3] for dislocations, while in neurobiology it is a qualitative model for propagation of an impulse along a myelinated nerve axon [1]. Cardiac cells provide another suitable biological context [6].

In both contexts the nonlinearity $-A\sin u + F$ may be replaced by a more general function $f(u)$ with an appropriate 'cubic-like' graph, but here we will concentrate on the particular form (1.1). Indeed, any one-dimensional propagation in a discrete excitable medium may be described qualitatively by (1.1).

In the mechanical interpretation, u_n stands for the displacement of the nth atom in a row, F is the applied stress, and A measures the strength of the underlying sinusoidal force exerted by a substrate. In neurobiology, u_n is the electrical potential, either of the nth nerve cell or the nth node of the axon.

The solutions of most interest are those which are monotone in n for each fixed t and remain in a fixed interval of length 2π. For definiteness, we will assume that u_n is increasing in n and tends to $u_- = \arcsin \frac{F}{A}$ as $n \to -\infty$ and to $u_+ = u_- + 2\pi$ as $n \to +\infty$. Numerical and physical considerations suggest that if the initial data has these properties, which are sometimes called 'kink-like', then as $t \to \infty$, the solution $\{u_n\}$ tends to either a stationary or travelling wave of the form

$$u_n(t) = u(n - ct) \tag{1.2}$$

where the wave speed c is strictly negative (a travelling wave) or 0 (a stationary wave). Which of these occurs depends on the parameters A and F.

An early study was by Keener [6], who was interested in the neurobiological context. Keener considered a cubic-like nonlinearity, but his results apply to (1.1) in the case that $0 < F < A$. To discuss his conclusions, we may write the nonlinearity in the form

$$K(-\sin u + R),$$

where $K > 0$, $0 < R < 1$. Keener considers R fixed and shows that for large K, initial impulses do not propagate, and instead are 'blocked' by a family of complicated non-monotone stationary, or standing wave, solutions ($c = 0$), while for small enough K, propagation does occur. Subsequently, Zinner [10] proved the existence of a monotone travelling wave for K sufficiently small. One of our results below (Theorem 1.2) allows us to estimate the maximum size of K which supports a wave with $c > 0$. It was shown by Hankerson & Zinner [5] that waves exist for a general class of equations which includes (1.1), with no restriction on K. In another paper, in the course of proving a result about stability of travelling waves, Zinner [11] does establish the uniqueness of a travelling wave and the fact that travelling and standing waves cannot exist for the same speed c. However, the result in Hankerson & Zinner [5] says nothing about whether c is non-zero, so that we do not know if the wave is a stationary wave, or moving to the right, or moving to the left. There is no discussion of the existence of a positive K above which the wave speed is zero. There is also no mention of multiplicity of stationary waves. Since these are key questions discussed here, our work complements that in Hankerson & Zinner [5].

A formal study using asymptotic expansions was carried out by Erneux & Nicolis [2], who were interested in the problem of a small 'threshold', which in our context means F close to A. We will point out below how our work includes a rigorous proof of their asymptotic relation.

Very recently, we learned of new work of Mallet-Paret [8], whose paper overlaps ours, particularly in the use of a similar continuation method to obtain smooth one-parameter families of solutions. The continuation result given below (Theorem 3.1) was independently proved by Mallet-Paret, using a similar Fredholm alternative method. He also obtained the description of the 'bifurcation diagram' (c vs. F) which is a consequence of the work described below, and discussed other problems which we do not address. On the other hand, our existence result is an improvement on those in Zinner [10] and Mallet-Paret [8] because we can give an estimate on how small K must be. In fact, K does not have to be particularly small in the specific example of the sine function.

Another way in which our work differs from that of Mallet-Paret is that we are more interested in the behaviour of the stationary solutions, and in the question of whether they exist away from the symmetric, or equal area, case, i.e. for $F > 0$ in the case of (1.1). Generically, it is not easy to decide this for any particular nonlinearity, especially if it is small in a certain sense. We consider this problem here.

Keener also initiated a study of the initial value problem for (1.1) by considering appropriate upper and lower solutions. In this paper, we study primarily the existence of travelling and stationary wave solutions of (1.1). One can go further to establish results about stability and the long term behaviour of solutions to the initial value problem, again by using upper and lower solutions, but we do not pursue those ideas very far here. (Upper and lower solutions do play a role in one proof below.)

Substituting (1.2) into (1.1) gives the principal equation of interest:

$$-cu'(x) = u(x-1) - 2u(x) + u(x+1) - A\sin u(x) + F \tag{1.3}$$

with the boundary conditions

$$\lim_{x\to\pm\infty} u(x) = u_\pm. \tag{1.4}$$

Our point of view generally will be that $A > 0$ is fixed and F varies in the range $0 \leqslant F < A$. We will only consider solutions such that $u(x) \in (u_-, u_+)$ for all x.

When $c = 0$, (1.3) is a difference equation. In this case, solutions may be discontinuous, for there is nothing that connects values of $u(x)$ with values of $u(x+\delta)$ for $|\delta| < 1$. Hence, for $c = 0$ we will generally consider only the discrete version, i.e.

$$u_{n+1} - 2u_n + u_{n-1} - A\sin u_n + F = 0. \tag{1.5}$$

We will show that for a given value of u_0 there is at most one solution $\{u_n\}$ of (1.5) tending to u_- at $-\infty$ and u_+ at $+\infty$. To say that (1.3) has a continuous steady solution is simply to say that (1.5) has such a solution for each $u_0 \in (u_-, u_+)$, and that this solution depends continuously upon u_0. We expect that this does not happen for nonlinearities of the form in (1.3) or (1.5), but have not been able to prove this for small A. Continuous solutions can occur only if $F = 0$, and they do occur if $-A\sin u$ is replaced by a suitable alternative $f(u)$, but we believe that they are found only for very special functions f. Further work will discuss these and their relation to other problems considered below [4].

We will show that for each F in the range $[0, A)$, there is either a unique travelling wave (modulo translation), or else one or more stationary waves. We will see also that stationary and travelling waves cannot co-exist for the same (A, F). (As mentioned earlier, this was also shown by Zinner [11].) The existence proof for travelling waves is fairly complicated, and is only outlined here (see Hastings & McLeod [4]). Most of this paper is devoted to proving results about stationary waves.

The following theorem summarizes the basic relationship between travelling and stationary waves. As stated above, similar results have been obtained by Mallet-Paret [8].

Theorem 1.1 *For F sufficiently close to A there is a unique $c = c(F) < 0$ such that* (1.3) *has a smooth monotonic solution u with $u(-\infty) = u_-$, $u(+\infty) = u_+$, and this solution is* *unique modulo translation. As F decreases,* $|c|$ *decreases, and there is an* $F_{crit} \geqslant 0$ *such that* $\lim_{F\to F_{crit}^+} c(F) = 0$. *At* $F = F_{crit}$ *there is a monotonic solution of* (1.5) *such that*

$$\lim_{n\to\pm\infty} u_n = u_\pm. \tag{1.6}$$

If $0 < F < F_{crit}$*, then there is at least one such solution* $\{u_n\}$*, and if* $F = 0$*, then there are at least two such solutions. For a given value of* u_0*, there is at most one monotonic solution of* (1.5)–(1.6)*. Finally, stationary waves and travelling waves* (*with* $c \neq 0$) *cannot coexist for the same value of F.*

We should mention that one implication of Theorem 1.1 is that for any F with $|F| < A$ there is some kind of wave, either travelling or stationary. However, this is not new, having been proved by Hankerson & Zinner [5]. We believe (although we do not currently have a proof) that, if $F_{crit} > 0$, then there is just one monotonic solution of (1.5)–(1.6) (modulo translation) when $F = F_{crit}$, and at least two for $0 < F < F_{crit}$.

Most of this description carries over to any 'cubic-like' function $f(u)$ [4]. However, if we take A large, we can prove more as stated below.

Theorem 1.2 *If A is sufficiently large* ($A > 2$ *will do*)*, then* $F_{crit} > 0$ *and also the equations* (1.3)–(1.4) *have no continuous* ($c = 0$) *monotone solution.*

We expect the results in Theorem 1.2 to hold for all A in the case of the sine nonlinearity, but have not proved this. Theorem 1.2 is not true for all 'cubic-like' nonlinearities which might be substituted for the sine function in (1.1), but we believe that it is true except in very special cases.

A formal asymptotic analysis for small A [7] is in agreement with Theorems 1.1 and 1.2, and moreover indicates that for $F < F_{crit}$ there are exactly two solutions to equations (5)–(6). The leading-order behaviour of F_{crit} is found by these authors to be

$$F_{crit} \sim C\mathrm{e}^{-\pi^2/\sqrt{A}},$$

where C is a constant (which can be determined), and it is found that the leading-order speed of the travelling wave for $F > F_{crit}$ and of order F_{crit} is given by

$$c \sim \frac{D}{\sqrt{A}}(F^2 - F_{crit}^2)^{1/2},$$

where D is another constant, independent of A and F (which can also be determined).

In this paper we will outline the proof of the statements in Theorems 1.1 and 1.2 about travelling waves, with $c < 0$, and prove completely most of the statements about stationary waves, $c = 0$. (The remaining, more technical, parts will appear elsewhere, in a more general setting.) Theorem 1.1 will be proved in several steps. First, in §2 we prove the results on stationary waves with $F = 0$. In §3, we sketch the main ideas in the proof of existence of travelling waves for F close to A. In §4 we discuss the remainder of Theorem 1.1. Finally, in §5 we prove Theorem 1.2.

2 Stationary waves for $F = 0$

We consider the case $F = 0$ and the resulting difference equation

$$u_{n+1} = 2u_n - u_{n-1} + A \sin u_n \tag{2.1}$$

where $A > 0$. We look for monotonic solutions such that

$$\lim_{n\to-\infty} u_n = 0, \lim_{n\to\infty} u_n = 2\pi. \tag{2.2}$$

Lemma 2.1 *There is such a solution with* $u_0 = \pi$ *and* $u_n - \pi = \pi - u_{-n}$ *for all* n.

Proof We fix $u_0 = \pi$ and vary u_1 in the interval $(\pi, 2\pi)$ to find the desired solution. The condition $u_0 = \pi$ ensures that $u_n - \pi$ is an odd function of n. We first choose $\epsilon > 0$ so that $-A \sin u > \epsilon(u - \pi)$ for $\pi < u \leqslant \frac{3}{2}\pi$. Then we choose N so large that $(N-1)\epsilon > 1$. Finally, we choose $u_1 - \pi$ so small that $u_j \leqslant \frac{3}{2}\pi$ for $1 \leqslant j \leqslant N$.

We wish to show that under these conditions, the finite sequence $\{u_1, ..., u_N\}$ is not monotone increasing. It is convenient to let $U_n = u_n - \pi$. If $\{U_1, ..., U_N)$ is monotone increasing, then for $2 \leqslant j \leqslant N$ we have

$$U_j \leqslant (2 - \epsilon)U_{j-1} - U_{j-2}.$$

Adding these inequalities results in

$$U_N - U_{N-1} \leqslant -\epsilon \Sigma_{i=2}^{N-1} U_i + (1 - \epsilon)U_1.$$

Since we assumed that $U_i \geqslant U_1$ for $2 \leqslant i \leqslant N$, our lower bound on N then shows that $U_N < U_{N-1}$, a contradiction.

Therefore, we have shown that for sufficiently small U_1, the sequence starts to decrease before crossing π. On the other hand, we have simply to choose $U_1 > \pi$ to have the sequence cross π before decreasing.

Note that if the sequence increases until some first N such that $U_N = \pi$, then $U_{N+1} > \pi$. If, finally, there is an N such that the sequence increases up to $n = N$, with $U_N < \pi$, and $U_N = U_{N+1}$, then $U_{N+2} < U_{N+1}$ so that the sequence decreases before reaching π. □

In summary, we have shown that there are open sets $\Omega, \Lambda \subset (0, \pi)$ such that $U_1 \in \Omega$ implies that $\{U_n\}$ decreases before reaching π, while $U_1 \in \Lambda$ implies that $\{U_n\}$ crosses π before decreasing. The set $(0, \pi)/(\Omega \cup \Lambda)$ is nonempty, and it is easy to see that for any U_1 in this set, $\{U_n\}$ increases monotonically to π. From (2.1) we see that $U_n = -U_{-n}$ and the solution $\{u_n\} = \{U_n + \pi\}$ satisfies the boundary conditions at $\pm\infty$.

We now discuss the uniqueness of this solution. First, we consider whether there is another solution with $u_0 = \pi$. That there is not follows from the following more general lemma.

Lemma 2.2 *For any* $\alpha \in [\pi, 2\pi)$, *there is a unique* $\beta \in (\alpha, 2\pi)$ *such that if* $u_0 = \alpha, u_1 = \beta$, *then the sequence* $\{u_j\}$ *is increasing and tends to* 2π *as* $j \to \infty$.

Proof Existence of such a β is obtained by the method of Lemma 2.1. The proof of uniqueness depends upon the fact that the graph of $-A \sin u$ is concave down on $(\pi, 2\pi)$, and indeed, all that we have done up to this point applies if $A\sin u$ is replaced by any function f anti-symmetric about π such that $f(\pi) = f(2\pi) = 0$ with $f < 0, f'' > 0$ on $(\pi, 2\pi)$.

For $\beta > \alpha$, let $\{u_j(\beta)\}$ be the unique solution of (2.1) such that $u_0 = \alpha, u_1 = \beta$. We take $\{u_j^*\} = \{u_j(\beta^*)\}$ to be a specific solution of the kind described in the statement of this lemma, and let $v_j = \frac{\partial u_j}{\partial \beta}$. Hence $v_1 = 1$. We wish to show that the sequence $\{v_j(\beta^*)\}$ is increasing. We do this by comparing the sequences $\{(u_1^* - \pi)v_j(\beta^*)\}$ with $\{U_j\} = \{u_j^* - \pi\}$. These sequences satisfy, respectively, the equations

$$v_{j+1} = a_j v_j - v_{j-1}$$
$$U_{j+1} = b_j U_j - U_{j-1}$$

with $a_j = 2 - A\cos(U_j)$ and $b_j = \left(2 - \frac{A\sin U_j}{U_j}\right)$. Note that $b_1 > 1$, for otherwise, $U_2 < U_1$. Therefore, the sequences $\{a_j\}$ and $\{b_j\}$ are increasing, and $a_j > b_j$ for all $j \geqslant 0$. The initial conditions are: $v_0 = 0, U_0 = \alpha - \pi \geqslant 0$, $(\beta^* - \pi)v_1 = U_1 > \alpha - \pi$. The following result will then allow us to complete the proof of Lemma 2.2.

Lemma 2.3 *Suppose that δ and γ are two numbers with $\gamma > \delta \geqslant 0$. For a strictly increasing sequence $\{c_j\}$, $c_1 > 1$, let $\{w_j\}$ be the unique solution of*

$$w_{j+1} = c_j w_j - w_{j-1} \tag{2.3}$$

for $j \geqslant 1$ such that $w_0 = \delta$, $w_1 = \gamma$. Consider $\{w_j\}$ as a sequence of functions of the coefficients c_j, and suppose that for some specific sequence $\{c_j^*\}$, $w_j > 0$ for $1 \leqslant j \leqslant n$. Then for $2 \leqslant j \leqslant n+1$, $\frac{\partial w_j}{\partial c_k}|_{\{c_j^*\}} > 0$ for $1 \leqslant k \leqslant j-1$.

We leave the proof of this to later, and in the meantime complete the proof of Lemma 2.2. In the case $\alpha = \pi$, Lemma 2.3 implies Lemma 2.2. In the case $\alpha > \pi$, letting $\delta = \alpha - \pi$ we note that

$$U_2 = 2U_1 - \delta - A\sin U_1 < U_1\left(2 - \frac{A\sin U_1}{U_1}\right)$$
$$< U_1(2 - A\cos U_1) = U_1 v_2$$

and $U_1 v_1(\beta) = U_1$. Then, starting from u_1, Lemma 2.3 again implies Lemma 2.2. □

Corollary 2.4 *The solution found in Lemma 2.1 is unique among monotone solutions with $u_0 = \pi$.*

Proof of Lemma 2.3 Denote w_j for a given γ, δ by $w_j(\gamma, \delta)$. Let $p_j(c_1, \ldots, c_j) = w_{j+1}(1, 0)$. Then, by explicit calculation,

$$p_1(c_1) = c_1$$
$$p_2(c_1, c_2) = c_2 c_1 - 1$$
$$p_3(c_1, c_2, c_3) = c_3 c_2 c_1 - c_3 - c_1$$

and generally

$$p_j(c_1, \ldots, c_j) = c_j p_{j-1}(c_1, \ldots c_{j-1}) - p_{j-2}(c_1, \ldots, c_{j-2}). \tag{2.4}$$

This, and the linearity of equation (2.3), easily lead to the relation

$$w_{j+1}(\gamma, \delta) = \gamma p_j(c_1, \ldots, c_j) - \delta p_{j-1}(c_2, \ldots, c_j) \tag{2.5}$$

provided we set $p_0 = 1$.

From equation (2.5) it is clear that if, for some $\gamma > \delta > 0$, and some increasing sequence c_j, $w_0, w_1, \ldots, w_{j+1}$ are all positive, then $p_0, p_1(c_1), \ldots, p_j(c_1, \ldots, c_j)$ are all positive. In particular, this is true when $c_j = c_j^*$ up to $j = n-1$.

We now wish to show that for $c_j \geqslant c_j^*$, the functions p_j are increasing in all their arguments, up to $j = n$. This follows from the following relations:

$$\frac{\partial p_n}{\partial c_n} = p_{n-1} > 0$$

$$\frac{\partial p_n}{\partial c_{n-1}} = c_n \frac{\partial p_{n-1}}{\partial c_{n-1}} = c_n p_{n-2} = p_1(c_n) p_{n-2}$$

$$\frac{\partial p_n}{\partial c_{n-2}} = c_n \frac{\partial p_{n-1}}{\partial c_{n-2}} - \frac{\partial p_{n-2}}{\partial c_{n-2}} = (c_n c_{n-1} - 1) p_{n-3} = p_2(c_{n-1}, c_n) p_{n-3},$$

and in general, for $1 \leqslant j \leqslant n$,

$$\frac{\partial p_n}{\partial c_j} = p_{n-j}(c_{j+1}, \ldots, c_n) p_{j-1}.$$

Because the sequence $\{c_j\}$ is increasing, an induction argument and the earlier result that $p_1, \ldots, p_{n-1}$ are all positive imply that these derivatives are all positive.

To show that the w_j are increasing in the c_j, we do essentially the same computation as above to obtain that

$$\frac{\partial w_{n+1}}{\partial c_j} = p_{n-j}(c_{j+1}, \ldots, c_n) w_j.$$

Since w_2 is clearly increasing as a function of c_1, it follows by induction on k that the w_k are increasing in $c_1, \ldots, c_{k-1}$ for $1 \leqslant k \leqslant n+1$. This proves Lemma 2.3. □

Our next result shows that there is at least one other solution of (2.1)–(2.2) besides the one with $u_0 = \pi$ and its translates. Again let $U = u - \pi$.

Lemma 2.5 *There is a monotone solution u_n of* (2.1)–(2.2) *such that $u_0 > \pi$, $u_{-1} = 2\pi - u_0$.*

Proof By Lemma 2.2, for each $\delta \in (0, \pi)$ there is a $\gamma \in (\delta, \pi)$ such that $\{U_j\}$, defined by

$$U_{j+1} = 2U_j - U_{j-1} + A \sin U_j, U_0 = \delta, U_1 = \gamma,$$

is increasing for $j \geqslant 0$ and $U_j \to \pi$. Lemma 2.2 also shows that γ is unique, for given δ, implying also that γ varies continuously with δ. Letting $\{U_j^*\}$ denote the solution found in Lemma 2.1, we see that when $\delta = 0$, $\gamma = U_1^*$ and $U_{-1} = -\gamma$, while for $\delta = U_1^*$, $\gamma = U_2^*$ and $U_{-1} = 0$. There must be a $\delta \in (0, U_1^*)$ such that $U_{-1} = -U_0$. This solution then satisfies the 'near symmetry' condition $U_{-j} = -U_{j-1}$ for $j > 0$, and therefore $\{u_j\}$ satisfies (2.1)–(2.2). This proves Lemma 2.5. □

Our next result is about the local uniqueness of the solution found in Lemma 2.1 if we allow u_0 to vary. We have not been able to prove this for all A.

Lemma 2.6 *Let* $\{u_n^*\}$ *be the monotone solution of* (2.1)–(2.2) *for some* $A > 2$, *with* $u_0^* = \pi$. *Then there is a* $\rho > 0$ *such that if* $\{u_n\}$ *is a monotone solution of* (2.1) *which satisfies* (2.2) *and* $|u_0 - \pi| + |u_1 - u_1^*| < \rho$, *then* $u_0 = \pi$, $u_1 = u_1^*$.

Proof We consider, sufficiently, solutions to (2.1) satisfying the initial conditions

$$u_0 = \alpha > \pi$$
$$u_1 = \beta.$$

The shooting and uniqueness arguments above show that for each small $\alpha - \pi$ there is a unique $\beta = \beta(\alpha)$, continuous as a function of α, such that $\{u_n\}$ is monotone on $0 \leqslant n < \infty$ with $u_n \to 2\pi$ as $n \to \infty$.

We then use the following further result:

Lemma 2.7 *For small* $\alpha - \pi > 0$, $\beta > \beta^*$.

Proof For this proof it is convenient to set $U_n = u_n - \pi$, $U_n^* = u_n^* - \pi$, $\delta = \alpha - \pi$, $\gamma = \beta - \pi$, $\gamma^* = \beta^* - \pi$. We saw above that for $A > \frac{\pi}{2}$, $\gamma^* > \frac{\pi}{2}$, and so for sufficiently small δ, $\gamma > \frac{\pi}{2}$. If $U_1 = \gamma < \gamma^*$, then $\frac{\sin U_1}{U_1} > \frac{\sin U_1^*}{U_1^*}$, so that

$$U_2 = \left(2 - \frac{A \sin U_1}{U_1}\right) U_1 - U_0 < U_2^*.$$

In fact,

$$U_2 - U_1 = \left(1 - \frac{A \sin U_1}{U_1}\right) U_1 - U_0 < U_2^* - U_1^*,$$

and in particular, $U_2 < U_2^*$.

Further, we have that

$$U_3 - U_2 = U_2 - U_1 - A \sin U_2$$

while

$$U_3^* - U_2^* = U_2^* - U_1^* - A \sin U_2^*,$$

and since $\pi > U_2^* > U_2 > \frac{\pi}{2}$ (if $U_n \to \pi$), we have

$$U_3 - U_2 < U_3^* - U_2^*.$$

Continuing, we find that

$$U_{n+1} - U_n < U_{n+1}^* - U_n^*$$

for all $n \geqslant 0$. Since $\Sigma_{n=1}^{\infty}(U_{n+1}^* - U_n^*) = \pi - \gamma^*$, it follows that we could not have $U_n \to \pi$, a contradiction which proves Lemma 2.7. □

However, we then have

$$U_{-1} = \left(2 - \frac{A \sin U_0}{U_0}\right) U_0 - U_1.$$

Since $A > 2$, we see that for sufficiently small U_0, we must have $U_{-1} < -U_1 \leqslant -U_1^*$. However, then a similar argument to that just above with now necessarily $\beta > \beta^*$ shows that U_{-n} must fall below $-\pi$, so that $\{u_n\}$ cannot satisfy (2.2). This proves Lemma 2.6. □

3 Travelling waves for F close to A

The key to discussing the existence of travelling waves with $c < 0$ is the following continuation theorem, which is much the same as that in (MP). We consider systems of the form

$$-cu'(x) = u(x-1) - 2u(x) + u(x+1) + f(u(x), \lambda) \tag{3.1}$$

where f is continuously differentiable in both its arguments on a set $(a, b) \times \Lambda$, Λ being some open connected set of m-dimensional parameters. Further, letting f' denote $\partial f/\partial u$, we have the following additional conditions for each λ:

(1) $f(u_-, \lambda) = f(u_+, \lambda) = 0$ and $f(\hat{u}, \lambda) = 0$ for exactly one point $\hat{u} \in (u_-, u_+)$.

(2) $f(u, \lambda) < 0$ on $(u_-, \hat{u})$ and $f(u, \lambda) > 0$ on $(\hat{u}, u_+)$,

(3) $f'(u_-, \lambda) < 0$, $f'(u_+, \lambda) < 0$.

Here, the roots $u_-, u_+, \hat{u}$ can depend smoothly upon λ.

Theorem 3.1 *Suppose that for some $c_0 \neq 0$, and some $\lambda_0 \in \Lambda$, the problem* (3.1), (1.4) *with $c = c_0$ has a monotone solution u_0. Then there is also a monotone solution for λ sufficiently close to λ_0, with speed c close to c_0.*

This theorem is proved by setting the perturbation problem in $L^2(-\infty, \infty)$ and applying a Fredholm alternative argument. It leads to an existence proof for travelling waves for our specific function in (1.1), or indeed, any specific function satisfying the conditions above, by proving the following additional results. In the first of these we consider any fixed $u_- < u_+$ and the more general class of nonlinear equations (3.1), except that we do not consider λ dependence. We show that there are f's for which solutions exist, by an explicit construction.

Lemma 3.2 *For any $c < 0$ there exists a function f satisfying points* (1)–(3) *above, such that* (3.1) *has a monotone solution satisfying* (1.4) *and approaching the limits exponentially fast at $\pm\infty$.*

The proof of this is simple enough to give immediately. Choose $c < 0$ and some continuously differentiable monotone function u satisfying (1.4) and approaching the limits exponentially. Then define $f(u)$ by (3.1). The monotonicity of $u(x)$ ensures that $f(u)$ is defined uniquely as a function of u. We also obtain $f'(u_\pm)$ from (3.1) and the exponential rate of approach of u to u_- and u^+. For example, if $u \sim u_- + e^{\gamma x}$ as $x \to -\infty$, then

$$f'(u_-) = -c\gamma - 4\sinh^2 \frac{\gamma}{2}.$$

Therefore, to ensure that $f'(u_-) < 0$ we must make sure that γ is not too small. This is accomplished by replacing $u(x)$ with $u(\rho x)$ for a sufficiently large ρ.

Having constructed a travelling wave for a particular function f, we now use Theorem 3.1 to show that this wave exists for functions f_1 sufficiently close to f. Suppose also that $f_1 > f$.

Lemma 3.3 *If $f_1 > f$ and f_1 is sufficiently close in C^1 to f, then the travelling wave still exists, with speed $c_1 \leqslant c$.*

The first assertion is a special case of Theorem 3.1, while the second, $c_1 \leqslant c$, will be proved below.

Proof We have to prove that if $f_1 > f$ then $c_1 \leqslant c$. Since f_1 is C^1—close to f, and f vanishes at u_- and u_+, with $f' < 0$ at both points, f_1 will vanish at points $v_- > u_-$ and $v_+ > u_+$, and there will be, by Theorem 3.1, a travelling wave u_1 connecting v_- to v_+, with speed c_1. Let $\phi(x,t) = u_1(x - c_1 t) - u(x - ct)$. Then

$$\begin{aligned}\phi_t &= \phi(x+1,t) - 2\phi(x,t) + \phi(x-1,t) + f_1(u_1) - f(u)\\ &> \phi(x+1,t) - 2\phi(x,t) + \phi(x-1,t) + f(u_1) - f(u)\\ &= \phi(x+1,t) - 2\phi(x,t) + \phi(x-1,t) + f'(\zeta)\phi,\end{aligned}$$

where ζ is some value (depending on (x,t)) between u and u_1. Also, $\phi(-\infty,t) > 0$, $\phi(\infty,t) > 0$, and we can translate u_1 to the left if necessary to ensure that $\phi(x,0) > 0$ everywhere. Suppose that $0 > c_1 > c$. (Both waves are moving to the left.) Then eventually u will 'catch up' to u_1, and there will be a first $t > 0$ where $\phi(x,t) = 0$ for some x with $\phi \geqslant 0$ everywhere. This implies that $\phi(x-1,t) = 0$, and then $\phi(x-2,t) = 0$, and so forth: $\phi(x-n,t) = 0$ for all n, contradicting the fact that $\phi(-\infty,t) = v_- - u_- > 0$ for all t. This proves Lemma 3.3. □

Corollary 3.4 *The problem* (1.3)–(1.4) *has a monotone solution if F is sufficiently close to A.*

(Just after the proof we give a specific upper bound for F_{crit}. Thus we have a rigorous estimate of what 'sufficiently close' means in Corollary 3.4. It is the fact that we can do this that marks an improvement in our method for proving existence over previous approaches.)

Proof We first note a further point about the proof of Lemma 3.2. We ensured, for a fixed $c < 0$, that $f'(u_\pm)$ was negative by a scaling $x \to \rho x$ where ρ was sufficiently large. However, we can also achieve this for arbitrarily small ρ, simply by lowering c. But by choosing ρ, and therefore c, small, we can ensure that $\frac{d}{dx}u(\rho x)$ is uniformly small, and this implies that $\max_u |f(u)|$ can be made as small as we wish.

Then, by raising F, and perhaps shifting $-A\sin u + F$ along the u axis, we can ensure that for some $F < A$, $-A\sin u + F$ lies above the function f constructed above. The continuation result, Theorem 3.1, then implies the existence of a travelling wave with speed $c_1 \leqslant c < 0$. This, with Lemma 3.3, leads to the family of travelling waves with increasing c as F decreases. □

We note the following specific function $u(x)$ which could be used in the construction of f. Let $u(x) = \pi \tanh(\alpha x)$ for any $\alpha \in (0, 1)$. The factor π is chosen for comparison with the sine nonlinearity. Substituting in (3.1), we find that

$$f(u) = 2\pi \tanh^2 \alpha \frac{v(1 - v^2)}{1 - \tanh^2(\alpha)v^2} + O(c),$$

where $v = \frac{u}{\pi}$. This enables us easily to get the estimate that when $A = 1$, $F_{crit} < 0.4$. We have made no attempt to optimize this estimate.

At this stage we can explain further the relation of our work to the asymptotic result of Erneux & Nicolis [2]. They consider the equation $u_n' = d(u_{n+1} - 2u_n + u_{n-1}) + f(u_n)$, with a cubic (FitzHugh–Nagumo type) nonlinearity $f(u) = u(1 - u)(u - a)$. The middle root, a, is taken to be small, and for a given a they look for the smallest d, called $d^*(a)$, which allows propagation of a wave. Using formal asymptotic analysis they obtain a relation $d^*(a) = O(a^2)$ as a tends to zero. To compare this with our result, divide equation (1.1) by A and let $d = \frac{1}{A}$. The construction by which we obtain travelling waves shows that there is some $\delta > 0$, independent of A for large A, such that travelling waves exist if $A - \delta < F < A$, i.e. if $d > \frac{1}{\delta}(1 - \frac{F}{A}) > 0$. On the other hand, letting $\tilde{a}$ denote the distance between the lowest two roots of $-A \sin u + F = 0$ in $(-\pi, \pi)$, it is easily shown that $1 - \frac{F}{A} = O(\tilde{a}^2)$. This proves the asymptotic result of Erneux & Nicolis [2].

4 Remainder of Theorem 1.1

We have seen that steady solutions exist for $F = 0$ and travelling wave solutions exist for F large. We now show that stationary states and travelling waves cannot both exist for the same F. In fact, we prove:

Lemma 4.1 *For a given value of F there exists at most one value of c (possibly 0) and if $c \neq 0$, (modulo translation) at most one travelling wave.*

Proof Suppose that we have

$$-c_1 u' = u(x + 1) - 2u(x) + u(x - 1) - A \sin u + F \tag{4.1}$$

and

$$-c_2 v' = v(x + 1) - 2v(x) + v(x - 1) - A \sin v + F \tag{4.2}$$

and $0 \geqslant c_1 > c_2$. Also, suppose that the rates of decay of u, v to the limit as $x \to \infty$ are $e^{-\gamma_1 x}$ and $e^{-\gamma_2 x}$, with $u \sim u_+ - ke^{-\gamma_1 x}$ and similarly for v. Then,

$$-c_1\gamma_1 = -4 \sinh^2 \frac{1}{2}\gamma_1 - A \cos(u_+),$$

$$-c_2\gamma_2 = -4 \sinh^2 \frac{1}{2}\gamma_2 - A \cos(u_+).$$

Thus

$$-c_1\gamma_1 + 4 \sinh^2 \frac{1}{2}\gamma_1 = -c_2\gamma_2 + 4 \sinh^2 \frac{1}{2}\gamma_2,$$

from which it is clear that $0 \geqslant c_1 > c_2$ implies that $\gamma_2 < \gamma_1$.

Similarly, if the rates of decay as $x \to -\infty$ are $e^{\delta_1 x}$ and $e^{\delta_2 x}$, then we see that $\delta_2 > \delta_1$.

Hence, if we take u, v to be monotonic increasing to the same limits, we see that u is ultimately above v at both ends, and so if we translate u sufficiently to the left, $u > v$ everywhere. Now adjust the translation so that $u \geqslant v$, with equality at at least one point, perhaps several. Subtracting (4.1) from (4.2) at a left-most such point leads to a contradiction. A similar argument shows that we cannot have different solutions for the same $c \neq 0$. This proves Lemma 4.1. □

We have now proved, or outlined the proof of, the first two sentences in the statement of Theorem 1.1. We can conclude, therefore, that there is an interval (F_{crit}, A) of F values where there is a unique travelling wave, with speed $c(F) < 0$ which increases to 0 as F decreases to F_{crit}. Further, a compactness argument can be used to show that at $F = F_{crit}$ there is a stationary wave, with speed $c = 0$. (It must be shown that this stationary wave satisfies the boundary conditions. This is done by integrating the equation from $-\infty$ to x. The argument is short but we will leave it to the subsequent paper [4].) If F_{crit}=0 then there is not much more to say. However (and we believe this is true for the sine nonlinearity for any positive A), if $F_{crit} > 0$, then we wish to show that in the interval $[0, F_{crit}]$ there is a stationary wave.

This is done by first observing that from the symmetry, if we replace F by $-F$, then there are the same travelling or stationary waves, but with $u(x)$ replaced by $-u(-x) + 2\pi$ and c by $-c$. In particular, there is a stationary wave at $-F_{crit}$.

We now use a comparison argument by considering the initial value problem

$$\begin{cases} u_{n,t} = u_{n+1} - 2u_n + u_{n-1} - A \sin u_n + F \\ u_n(0) = u_n^0 \end{cases} \tag{4.3}$$

A subsolution $\underline{u}_n(t)$ is a function satisfying:

$$\underline{u}_{n,t} \leqslant \underline{u}_{n+1} - 2\underline{u}_n + \underline{u}_{n-1} - A \sin \underline{u}_n + F$$

and $\underline{u}_n(0) \leqslant u_n^0$. A supersolution $\overline{u}_n(t)$ is a function satisfying:

$$\overline{u}_{n,t} \geqslant \overline{u}_{n+1} - 2\overline{u}_n + \overline{u}_{n-1} - A \sin \overline{u}_n + F$$

and $\overline{u}_n(0) \geqslant u_n^0$. The following comparison principle holds (see Keener [6] and Walter [9]):

Lemma 4.2 *If $\overline{u}_n$ is a supersolution and $\underline{u}_n$ is a subsolution for the solution u_n of the Cauchy problem* (4.3)*, then $\underline{u}_n \leqslant u_n \leqslant \overline{u}_n$.*

Now consider some $F \in (-F_{crit}, F_{crit})$. Choose as initial condition u_n^0 the stationary wave for $-F_{crit}$. It will then be below some translate of the stationary wave for F_{crit}. Further, since $F > -F_{crit}$, $u_n'(0) > 0$ for each n. It easily follows that $u_n'(t) > 0$ for all t, and since u_n is bounded by the upper stationary wave, it must tend to a limit, which is necessarily also a stationary wave. This completes our proof, partly in outline form, of Theorem 1.1. For further details see Hastings & McLeod [4].

5 Remainder of Theorem 1.2

We first show that $F_{crit} > 0$ if A is sufficiently large. The idea here is to block propagation by constructing a monotone upper solution to the steady state problem. That is, for small $F > 0$ we will construct a monotone sequence $\{u_n\}$ such that

$$u_{n+1} - 2u_n + u_{n-1} - A\sin u + F \leqslant 0. \tag{5.1}$$

We further require that $u_{-\infty} > u_-$ and $u_\infty > u_+$. Then, suppose that for some small $F > 0$ there is a travelling wave solution u, with $u(-\infty) = u_-$, $u(+\infty) = u_+$. By shifting u if necessary we can assume that $u(n) < u_n$ for all n. If the speed c of u is strictly negative, then there will be a first t such that $u(n-ct) = u_n$ for at least one n, $u(n-ct) \leqslant u_n$ for all n. Since $c < 0$, $u' > 0$, we have $u(n-1-ct)-2u(n-ct)+u(n+1-ct)-A\sin u(n-ct)+F > 0$ for all n and as above this leads to a contradiction.

To construct the desired solution to (5.1), we start with $F = 0$ and let $\{w_n\}$ be the unique steady state solution of (1.5)–(1.6) with $w_0 = \pi$. Since $F = 0$ we have $w_{-\infty} = 0$, $w_\infty = 2\pi$. (The existence and uniqueness of $\{w_n\}$ was proved in Lemma 2.1 and Corollary 2.4.) We then set $u_n = w_n + s_n$ so that (5.1) becomes

$$w_{n+1} + s_{n+1} - 2w_n - 2s_n + w_{n-1} + s_{n-1} - A\sin(w_n + s_n) + F \leqslant 0.$$

Therefore, we need

$$-(s_{n+1} - 2s_n + s_{n-1} - Acos(w_n)s_n) \geqslant O(s_n^2) + F. \tag{5.2}$$

Choose M, N such that $w_n \leqslant \frac{\pi}{2}$ for $n \leqslant -M$, $\frac{\pi}{2} < w_n < \frac{3\pi}{2}$ for $-M < n < N$, and $w_n \geqslant \frac{3\pi}{2}$ for $n \geqslant N$. By a translation we can assume that $M = N$ or $M = N + 1$. For definiteness assume that $M = N$. For $n \geqslant 0$ we let

$$s_n = \left\{ \begin{array}{ll} \epsilon - 2j\epsilon & \text{if} \quad n = M - j, 1 \leqslant j \leqslant M \\ \epsilon \quad \text{if} \quad n \geqslant M & \end{array} \right\}$$

and then set $s_n = s_{-n}$. (A slight adjustment is necessary if $M = N + 1$.)

We now check (5.2), and need consider only $n \geqslant 0$. If $1 \leqslant n \leqslant M - 1$ then (5.2) becomes

$$-A\cos(w_n)(2(M-n) - 1)\epsilon \geqslant O(s_n^2) + F,$$

and since $\frac{\pi}{2} < w_n < \frac{3\pi}{2}$, and w_n is independent of ϵ, this is true for some positive F if ϵ is sufficiently small. If $n = M$, then $s_{n+1} - 2s_n + s_{n-1} = -2\epsilon$ and $\cos w_n \geqslant 0$, $s_n = \epsilon$, so that again (5.2) holds. For $n > M$ we have $\cos w_n > 0$ and $s_n = \epsilon$ so that once again (5.2) is verified. Finally, for $n = 0$ we have $\cos w_0 = -1$ and (5.2) becomes

$$(-4 + (2M-1)A)\epsilon \geqslant O(\epsilon^2) + F$$

and for $A > \frac{4}{2M-1}$ and sufficiently small ϵ this holds for some positive F.

This proves the first assertion of Theorem 1.2. For the second part, about the non-existence of a continuous stationary wave for A sufficiently large, we first observe that there cannot be such a wave unless $F = 0$. This is seen by multiplying the equation by $u'(x)$ (in a distributional sense if necessary) and integrating by parts. Finally, from (1.3) with $c = 0$ we see that since $u(x+1) + u(x-1)$ is increasing, the function $-2u - A\sin u$ must be decreasing everywhere in $[0, 2\pi]$, which it will not be if $A > 2$.

6 Conclusion

We have improved previous estimates on the range of parameters for which certain infinite systems of differential equations have travelling waves, and studied the problem of when, as the parameter F decreases the wave speed falls to zero. We have also given results on the existence of two stationary wave solutions in the symmetric case, which is $F = 0$ in our particular model. One unsolved problem is to characterize those special nonlinearities where the travelling wave continues to exist all the way down to the symmetric case.

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References

[1] Bell, J. & Cosner, C. (1984) Threshold behavior and propagation for nonlinear differential-difference systems motivated by modeling myelinated axons. *Quart. J. Appl. Math.* **42**, 1–14.
[2] Erneux, T. & Nicolis, G. (1993) Propagating waves in discrete bistable reaction-diffusion systems. *Physica D*, **67**, 237–244.
[3] Frenkel, J. & Kontorova, T. (1938) On the theory of plastic deformation and twinning. *J. Phys. USSR*, **13**, 1–10.
[4] Hastings, S. P. & McLeod, J. B. In preparation.
[5] Hankerson, D. & Zinner, B. (1992) Wavefronts for a cooperative tridiagonal system of differential equations. *J. Dynamics & Diff. Eq.* **5**, 359–373.
[6] Keener, J. P. (1987) Propagation and its failure in coupled systems of discrete excitable cells. *SIAM J. Appl. Math.* **47**, 556–572.
[7] King, J. R. & Chapman, S. J. Asymptotics beyond all orders and Stokes lines in nonlinear differential-difference equations. Preprint.
[8] Mallet-Paret, J. (1997) The global structure of traveling waves in spatially discrete dynamical systems. Preprint, August.
[9] Walter, W. (1970) *Differential and Integral Inequalities*. Springer-Verlag.
[10] Zinner, B. (1992) Existence of traveling wavefront solutions for the discrete Nagumo equation. *J. Diff. Eq.* **96**, 1–27.
[11] Zinner, B. (1991) Stability of traveling wavefronts for the discrete Nagumo equation. *SIAM J. Math. Anal.* **22**, 1016–1020.