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Long-time Behaviour for Solutions of the Vlasov–Poisson–Fokker–Planck Equation

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We study the long-time behaviour of solutions of the Vlasov–Poisson–Fokker–Planck equation for initial data small enough and satisfying some suitable integrability conditions. Our analysis relies on the study of the linearized problems with bounded potentials decaying fast enough for large times. We obtain global bounds in time for the fundamental solutions of such problems and their derivatives. This allows to get sharp bounds for the decay of the difference between the solutions of the Vlasov–Poisson–Fokker–Planck equation and the solution of the free equation with the same initial data. Thanks to these bounds, we get an explicit form for the second term in the asymptotic expansion of the solutions for large times. © 1998 B. G. Teubner Stuttgart—John Wiley & Sons, Ltd.

KEY WORDS: Vlasov–Poisson–Fokker–Planck; long-time behaviour; fundamental solutions

0. Introduction

The simplest mathematical description of the state of a stellar system or a rarefied plasma is based on collisionless kinetic models, the Liouille–Newton system or the Vlasov–Poisson system in case the induced magnetic fields vary slowly.

The model of collisionless plasmas, specially in controlled fusion or laser fusion, is too idealized and collisional effects need to be incorporated. A way to do that is to model the motion of an individual particle as Brownian motion caused by collisions with the background. The resulting system of mathematical equations is the stochastic Langevin system

$$\mathrm{d}x = v\,\mathrm{d}t,$$

$$\mathrm{d}v = (E(x,t) - \beta v)\,\mathrm{d}t + \sqrt{2\sigma}\,\mathrm{d}b,$$

where E is the electrostatic or gravitational field, β is a viscosity parameter, σ a thermal diffusion parameter and b denotes the standard N-dimensional Brownian motion.

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The Vlasov–Poisson–Fokker–Planck equations result when one incorporates the above Langevin system into the Vlasov equation in order to determine the dynamic behaviour of the expected distribution of particles with respect to position and momentum. Letting $f(x, v, t)$ denote this distribution at time t we get the Vlasov–Poisson–Fokker–Planck system

$$\text{(VPFP)} \quad \begin{cases} f_t - \sigma\Delta_v f + v\nabla_x f + (E(f) - \beta v)\nabla_v f - NBf = 0, & \text{in } \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^+, \\ f(x, v, 0) = f_0(x, v), & \text{in } \mathbb{R}^N \times \mathbb{R}^N, \end{cases}$$

where $E(\rho(f)) = \varepsilon(K *_x \rho(f))$ with $\rho(f) = \int_{\mathbb{R}^N} f(x, v, t)\,\mathrm{d}v$, and $K(x) = x/S_N|x|^N$, S_N being the area of the sphere in $\mathbb{R}^N$. We shall be concerned with the case $N = 3$, $\beta = 0$. The parameter $\varepsilon = \pm 1$ depending on whether the interaction between the particles is electrostatic or gravitational.

Some existence results for (VPFP) are known. Let us review the main ones. Degond proved in Reference 9 existence of smooth global solutions in dimensions one and two. Later, Triolo [16] proved global existence of smooth solutions for small initial data in the class $f_0 \in L^1 \cap L^\infty$, $0 \leqslant f_0 \leqslant Ah(x)g(v)$ in dimension $N \geqslant 3$. For small f_0 he proved global existence and some decay estimates

$$\|E(f)\|_{L^\infty} \leqslant \frac{C}{(1+t)^{N-1}}, \qquad \|\rho(f)\|_{L^\infty} \leqslant \frac{C}{(1+t)^N}.$$

The proof uses an iterative method and relies on the decay given by the dispersive part of the equation to get uniform bounds.

Victory and O'Dwyer [18] proved global existence of classical solutions when $N \leqslant 2$ and local existence when $N \geqslant 3$ for initial data

$$f_0 \in C_b^1 \cap L_{xv}^1, \; (1 + |v|^2)^{\gamma/2}(f_0 + |\nabla_x f_0|_2 + |\nabla_v f_0|_2) \in L_{xv}^\infty, \quad \gamma > 3.$$

For such data $E(f) \in L_{xt}^\infty$, $\nabla E(f) \in L_{xt}^\infty$. The key point in their proof is the construction of a fundamental solution for the linear, degenerate, parabolic like problem. In Reference 17, Victory proved the existence of global weak solutions when $n \leqslant 3$ and

$$f_0 \in L^1 \cap L^\infty, \quad \int (|x|^2 + |v|^2) f_0 < \infty.$$

Weckler and Rein [15] gave the sufficient conditions for the solutions constructed in Reference 18 to be global: assuming the field $E(f)$ associated to f_0 to decay fast enough (like $(1+t)^{-\alpha}$ with $\alpha > \frac{1}{2}$), if for f_0 we have a global solution and $f_0 - g_0$ is 'small' enough in an appropriate norm, then we get for g_0 a global solution with E decaying like $(1+t)^{-1}$. The decay rate for E is therefore improved. The proof relies on an estimate for large times of the fundamental solution in terms of the fundamental solution of the linear problem with $E = 0$ and a perturbation technique.

Bouchut [2] constructed global solutions when $N = 3$ and

$$f_0 \in L^1 \cap L^\infty, \quad \int |v|^m f_0 \,\mathrm{d}v\,\mathrm{d}x < \infty, \; m > 6.$$

Moreover, $E(f) \in L_{xt}^\infty$. This gives conditions for the solutions in Reference 18 to be global. In Reference 3 some further results on the smoothness of solutions are given.

Carrillo and Soler [5, 6] proved the existence of global weak solutions for initial data in $L^1 \cap L^p$ and, when $N = 3$, for small data in Morrey spaces. For initial data f_0 in L^1, $f_0 \geqslant 0$ with $S_{9/4}(f_0) = \sup_{\lambda > 0} \|\rho_{0,\lambda}\|_{L^{9/4}} < \infty$ and $\mathrm{Max}(\sup_{\lambda > 0} \|\rho_{0,\lambda}\|_{L^{9/4}}, \|f_0\|_1)$ small enough they get a unique global weak solution satisfying

$$\text{(H)} \quad S_{9/4}(f(t)) \leqslant C, \qquad \|E(t)\|_{L^\infty_x} \leqslant Ct^{-1/2}.$$

Carillo, Soler and Vazquez were the first to determine the asymptotic profile of the solutions f to (VPFP) for large times. In Reference 8 they prove that the long-time behaviour of weak solutions f satisfying condition (H) for large times is given by the fundamental solution of the linear problem with $E = 0$ with mass $\int f_0 \, dx \, dv$ provided that

$$f_0(1 + |x|^2 + |v|^2 + |\log f_0|) \in L^1(\mathbb{R}^6), \qquad E(f_0) \in L^2(\mathbb{R}^3)^3$$

and $f_0 \in L^{9/7}_{xv}$ in the case of gravitational forces, i.e. $\varepsilon = -1$. Therefore, the first term in the asymptotic expansion of f for large times is determined by the free equation and the mass of the inital data. Their study relies on scaling techniques.

We shall study here the long-time behaviour of solutions to (VPVP) when $N = 3$ from another point of view. Our analysis is based on the obtainment of global estimates on the fundamental solutions of linearized problems and its derivatives in which the fundamental solutions and its derivatives are bounded in terms of the fundamental solutions of the linear problem where the potential $E = 0$. Once these estimates are obtained we can get optimal decay rates on the remainder $r(t)$ in the expansion $f(t) = G(f_0)(t) + r(t)$, where $f(t)$ is the solution of (VPFP) with datum f_0 and $G(f_0)$ is the solution of the free equation with datum f_0, provided that f_0 satisfies some integrability conditions. Then, we can find explicit expressions for the second term in the expansion for large times. In this term the influence of the non-linearity appears. This result clarifies the deviation of 'strong' solutions of (VPFP) from the solutions of the free equation.

The paper is organized as follows. In section 1 we get the decay estimates for the solutions of the lineaerized (VPFP) problem with $E = 0$. In section 2 we get the decay estimates for the solutions of the linearized (VPFP) problem with a potential $E \in W^{1,\infty}$. In section 3 we study the long-time behaviour of solutions to (VPFP) for initial data small enough and satisfying suitable integrability conditions. For such data we improve the results obtained in Reference 8. Our study relies mainly in the use of the estimates on fundamental solutions obtained in section 2.2. In the following we shall take $\sigma = 1$.

1. Linear problem with $E=0$

It is useful to think about VPFP as a perturbation of the linear problem with $E = 0$, that is,

$$(\mathscr{P}_0) \begin{cases} g_t - \Delta_v g + v \nabla_x g = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+, \\ g(x, v, 0) = g_0(x, v) & \text{in } \mathbb{R}^3 \times \mathbb{R}^3, \end{cases}$$

This equation is hyperbolic in the x variable and parabolic in the v variable. An explicit fundamental solution is known (see References 13 and 14)

$$G(x, v, t; \xi, \nu, \tau) = G(x - \xi, v, \nu, t - \tau)$$
$$= \frac{3^{3/2}}{(2\pi)^3 (t-\tau)^6} \exp\left\{ -\frac{3|x - \xi - [(t-\tau)/2](v+\nu)|^2}{(t-\tau)^3} - \frac{|v-\nu|^2}{4(t-\tau)} \right\}$$

with $x, v, \xi, \nu \in \mathbb{R}^3$ and $t > \tau \geqslant 0$.

1.1. *Some known properties of the fundamental solution*

Lemma 1. (i) *For some positive constants m_1, m_2, M_1, M_2*

$$\int_{\mathbb{R}^3} G(x, v, t; \xi, \nu, \tau)\, \mathrm{d}v = \frac{M_1}{(t-\tau)^{9/2}} \exp\left\{ -\frac{x - \xi - (t-\tau)\nu|^2}{m_1 (t-\tau)^3} \right\},$$

$$\int_{\mathbb{R}^3} G(x, v, t; \xi, \nu, \tau)\, \mathrm{d}x = \frac{M_2}{(t-\tau)^{3/2}} \exp\left\{ -\frac{|v-\nu|^2}{m_2 (t-\tau)} \right\},$$

$$\int_{\mathbb{R}^6} G(x, v, t; \xi, \nu, \tau)\, \mathrm{d}x\, \mathrm{d}v = \int_{\mathbb{R}^6} G(x, v, t; \xi, \nu, \tau)\, \mathrm{d}\xi\, \mathrm{d}\nu = 1,$$

$$\int_{\mathbb{R}^6} G(x, v, t; \xi', \nu', \tau') G(\xi', \nu', \tau'; \xi, \nu, \tau)\, \mathrm{d}\xi'\, \mathrm{d}\nu' = G(x, v, t; \xi, \nu, \tau)$$

with $x, v, \xi, \nu, \xi', \nu' \in \mathbb{R}^3$ and $t > \tau' > \tau \geqslant 0$.

(ii) *The following estimates hold for some positive constant C:*

$$|\nabla_v G(x, v, t; \xi, \nu, \tau)| \leqslant C \frac{G(x/2, v/2, t; \xi/2, \nu/2, \tau)}{(t-\tau)^{1/2}},$$

$$|\nabla_\nu G(x, v, t; \xi, \nu, \tau)| \leqslant C \frac{G(x/2, v/2, t; \xi/2, \nu/2, \tau)}{(t-\tau)^{1/2}},$$

$$|\nabla_x G(x, v, t; \xi, \nu, \tau)| \leqslant C \frac{G(x/2, v/2, t; \xi/2, \nu/2, \tau)}{(t-\tau)^{3/2}},$$

$$|\nabla_\xi G(x, v, t; \xi, \nu, \tau)| \leqslant C \frac{G(x/2, v/2, t; \xi/2, \nu/2, \tau)}{(t-\tau)^{3/2}},$$

$$|\Delta_v G(x, v, t; \xi, \nu, \tau)| \leqslant C \frac{G(x/2, v/2, t; \xi/2, \nu/2, \tau)}{(t-\tau)},$$

$$|\nabla_\nu \nabla_v G(x, v, t; \xi, \nu, \tau)| \leqslant C \frac{G(x/2, v/2, t; \xi/2, \nu/2, \tau)}{(t-\tau)},$$

$$|\nabla_x \nabla_v G(x, v, t; \xi, \nu, \tau)| \leqslant C \frac{G(x/2, v/2, t; \xi/2, \nu/2, \tau)}{(t-\tau)^2}$$

and so on.

Proof. Part (i) is obtained by integrating the explicit expression for G. Part (ii) follows estimating the explicit expressions for the derivatives.

1.2. *Decay of solutions*

Once the fundamental solution is known, for any solution of the linear problem with initial datum $g_0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$

$$(\mathscr{P}_0)\ \begin{cases} g_t - \Delta_v g + v\nabla_x g = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+, \\ g(x, v, 0) = g_0(x, v) & \text{in } \mathbb{R}^3 \times \mathbb{R}^3, \end{cases}$$

we have the integral expression

$$g(x, v, t) = \iint G(x, v, t; \xi, \nu, 0) g_0(\xi, \nu)\, \mathrm{d}\xi\, \mathrm{d}\nu = G(g_0).$$

We are interested in positive solutions. Since $G \geqslant 0$, we have a maximum principle. Thus, for positive initial data we get positive solutions. The decay estimates can be extended to negative data changing signs.

Lemma 2. *The following decay estimates hold*:

(E1) $\|G(g_0)\|_{L^p_v(L^1_x)} \leqslant Ct^{-(3/2)(1/r - 1/p)} \|g_0\|_{L^p_v(L^1_x)}, \quad p \geqslant r,$

(E2) $\|G(g_0)\|_{L^p_v(L^q_x)} \leqslant \dfrac{C}{t^{(3/2)(-3/q + 3/r - 1/p + 1/\alpha)}} \|g_0\|_{L^\alpha_v(L^r_x)}, \quad p \geqslant \alpha,\ q \geqslant r,$

(E3) $\|G(g_0)\|_{L^p_{xv}} \leqslant \|g_0\|_{L^p_{xv}}, \quad 1 \leqslant p \leqslant \infty,$

(E4) $\|G(g_0)\|_{L^p_x(L^1_v)} \leqslant Ct^{-(9/2)(1/r - 1/p)} \operatorname{Sup}_{\lambda > 0} \|\rho_{0,\lambda}\|_{L^r_x}$
$\qquad \leqslant Ct^{-(9/2)(1/r - 1/p)} \|g_0\|_{L^1_v(L^r_x)}, \quad p \geqslant r.$

Proof. The estimates follow from the integral formula taking into account Lemma 1(i):

(E1):

$$\begin{aligned} \int G(g_0)(x, v, t)\, \mathrm{d}x &= \iint\left(\int G(x, v, t, \xi, \nu, 0)\, \mathrm{d}x \right) g_0(\xi, \nu)\, \mathrm{d}\xi\, \mathrm{d}\nu \\ &= \frac{M_2}{t^{3/2}} \iint \exp\left(-\frac{|v - \nu|^2}{m_2 t} \right) g_0(\xi, \nu)\, \mathrm{d}\xi\, \mathrm{d}\nu \\ &= \frac{M_2}{t^{3/2}} \exp\left(-\frac{|\cdot|^2}{m_2 t} \right) *_v \left(\int g_0(\xi, \cdot)\, \mathrm{d}\xi \right), \end{aligned}$$

where $1/r + 1/r' = 1 + 1/p$. Therefore,

$$\begin{aligned} \|G(g_0)\|_{L^p_v(L^1_x)} &\leqslant \left\| \frac{C}{t^{3/2}} \exp\left(-\frac{|\cdot|^2}{m_2 t} \right) \right\|_{L^{r'}_v} \|g_0\|_{L^r_v(L^1_x)} \\ &\leqslant Ct^{-(3/2)(1/r - 1/p)} \|g_0\|_{L^r_v(L^1_x)}, \quad p \geqslant r, \end{aligned}$$

(E2):

$$\|G(g_0)\|_{L^p_v(L^q_x)} \leqslant \left\| \int \mathrm{d}v \, \| G(\cdot, v, \mathrm{v}, t) *_x g_0(\cdot, \mathrm{v}) \|_{L^q_x} \right\|_{L^p_v}$$

$$\leqslant \left\| \int \mathrm{d}v \, \| G(\cdot, v, \mathrm{v}, t) \|_{L^{r'}_x} \| g_0(\cdot, \mathrm{v}) \|_{L^r_x} \right\|_{L^p_v}$$

$$\leqslant \left\| \frac{C}{t^{(3/2)(4-3/r')}} \exp\left\{ -\frac{|v-\mathrm{v}|^2 r'}{m_2 t} \right\} *_v \| g_0(\cdot, \mathrm{v}) \|_{L^r_x} \right\|_{L^p_v}$$

$$\leqslant \frac{C}{t^{(3/2)(-3/q+3/r-1/p+1/\alpha)}} \| g_0 \|_{L^\alpha_v(L^r_x)}, \quad p \geqslant \alpha, \; q \geqslant r.$$

Therefore,

$$\|G(g_0)\|_{L^p_v(L^q_x)} \leqslant \frac{C}{t^{(3/2)(-3/q+3/r-1/p+1/\alpha)}} \| g_0 \|_{L^\alpha_v(L^r_x)}, \quad p \geqslant \alpha, \; q \geqslant r.$$

(E3): Since

$$\int_{\mathbb{R}^6} G(x, v, t; \xi, \mathrm{v}, \tau) \, \mathrm{d}x \, \mathrm{d}v = \int_{\mathbb{R}^6} G(x, v, t; \xi, \mathrm{v}, \tau) \, \mathrm{d}\xi \, \mathrm{d}v = 1,$$

we have

$$\|G(g_0)\|_{L^1_{xv}} \leqslant \|g_0\|_{L^1_{xv}}, \qquad \|G(g_0)\|_{L^\infty_{xv}} \leqslant \|g_0\|_{L^\infty_{xv}}.$$

Interpolating

$$\|G(g_0)\|_{L^p_{xv}} \leqslant \|g_0\|_{L^p_{xv}},$$

(E4):

$$\|G(g_0)\|_{L^p_x(L^1_v)} \leqslant \int \mathrm{d}v \left\| \frac{M_1}{t^{9/2}} \exp\left\{ -\frac{|\cdot - t\mathrm{v}|^2}{m_1 t^3} \right\} *_x g_0(\cdot, \mathrm{v}) \right\|_{L^p_x}$$

$$\leqslant \int \mathrm{d}v \left\| \frac{C}{t^{9/2}} \exp\left\{ -\frac{|\cdot|^2}{m_1 t^3} \right\} *_x \int g_0(\cdot - t\mathrm{v}, \mathrm{v}) \, \mathrm{d}\mathrm{v} \right\|_{L^p_x}$$

$$\leqslant C t^{-(9/2)(1/r-1/p)} \left\| \int g_0(\cdot - t\mathrm{v}, \mathrm{v}) \, \mathrm{d}\mathrm{v} \right\|_{L^r_x}, \quad p \geqslant r.$$

In view of this, we introduce

$$\mathrm{Sup}_{\lambda>0} \left\| \int g_0(\cdot - \lambda \mathrm{v}, \mathrm{v}) \, \mathrm{d}\mathrm{v} \right\|_{L^r_x} = \mathrm{Sup}_{\lambda>0} \| \rho_{0,\lambda} \|_{L^r_x},$$

where $\rho_{0,\lambda} = \int g_0(\cdot - t\mathrm{v}, \mathrm{v}) \, \mathrm{d}\mathrm{v}$, so that

$$\|G(g_0)\|_{L^p_x(L^1_v)} \leqslant C t^{-(9/2)(1/r-1/p)} \mathrm{Sup}_{\lambda>0} \| \rho_{0,\lambda} \|_{L^r_x}$$

$$\leqslant C t^{-(9/2)(1/r-1/p)} \| g_0 \|_{L^1_v(L^r_x)}, \quad p \geqslant r.$$

Remark 1. Using estimate (E4) we get for $g_0 \geqslant 0$

$$\|\rho(G(g_0))(t)\|_{L^p_x} \leqslant C(\|g_0\|_{L^1_v(L^r_x)})t^{-9/2(1/r-1/p)}, \quad p \geqslant r.$$

Thanks to this estimate and the properties of kernel K (see Reference 12 for the first inequality) we get

$$\|E(\rho(G(g_0)))(t)\|_{L^p_x} \leqslant C\|\rho(G(g_0))(t)\|^{2/3}_{L^\infty_x}\|\rho(G(g_0)))(t)\|^{1/3}_{L^1_x}$$
$$\leqslant C(\|g_0\|_{L^1_{xv}}, \|g_0\|_{L^1_v(L^r_x)})t^{-3/r}, \quad 1 \leqslant r \leqslant \infty.$$

As we can see, the field E decays very fast, it may reach the decay rate t^{-3} for initial data in L^1. When $g_0 \in L^1_v(L^r_x)$ the decay rate for large times is worse but the singularity at zero is smaller. For g_0 satisfying more integrability conditions we can reduce the singularity at time zero.

Remark 2. Differentiating the integral expression for $G(g_0)$ we get the following integral expression for the derivatives:

$$\int\int \partial_{x_i} G(x, v, t; \xi, v, 0) g_0(\xi, v)\,\mathrm{d}\xi\,\mathrm{d}v = \partial_{x_i} G(g_0).$$

Using the estimates on the derivatives of G (Lemma 1(ii)) we obtain

$$|\partial_{x_i} G(g_0)| \leqslant \int |\partial_{x_i} G(x, v, t; \xi, v, 0)|\,|g_0(\xi, v)|\,\mathrm{d}\xi\,\mathrm{d}v$$
$$\leqslant \int \frac{G(x/2, v/2, t; \xi/2, v/2, 0)}{t^{3/2}} |g_0(\xi, v)|\,\mathrm{d}\xi\,\mathrm{d}v.$$

Thus, we get the same decay estimates as for $\|G(g_0)\|$ with an extra decay factor $t^{-3/2}$. For $\partial_v G(g_0)$ we get an extra decay factor of order $t^{-1/2}$, and so on.

2. Linear problem with $E \neq 0$

We deal here with the problem

$$(\mathscr{P}_E) \begin{cases} g_t - \Delta_v g + v\nabla_x g + E(x, t)\nabla_v g = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+, \\ g(x, v, 0) = g_0(x, v) & \text{in } \mathbb{R}^3 \times \mathbb{R}^3 \end{cases}$$

with $g_0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$.

2.1. *E bounded and E Lipschitz*

When $\|E\|_{L^\infty_{xt}(\mathbb{R}^3 \times [0,T])} \leqslant C$ and $\|\nabla_x E\|_{L^\infty_{xt}(\mathbb{R}^3 \times [0,T])} \leqslant C$ a classical fundamental solution $\Gamma_E(x, v, t; \xi, v, \tau)$ to problem $\mathscr{P}_E$ is known to exist [18]. This function is defined to be the only function such that

(1) For fixed $(\xi, v, \tau) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+$, Γ_E as a function of (x, v, t) satisfies the equation

$$g_t - \Delta_v g + v\nabla_x g + E(x, t)\nabla_v g = 0 \quad \text{in } \mathbb{R}^3 \times \mathbb{R}^3 \times (\tau, T].$$

(2) For every continuous and bounded function $g_0(x, v)$

$$\lim_{t\to\tau}\int\int \Gamma_E(x, v, t; \xi, v, \tau) f_0(\xi, v)\,d\xi\,dv = f_0(x, v).$$

We then have the solution to our problem $\mathscr{P}_E$ given by

$$\text{(I1)}\quad g(x, v, t) = \int\int \Gamma_E(x, v, t; \xi, v, 0) g_0(\xi, v)\,d\xi\,dv$$

in $[0, T]$. Let us recall some known results on the behaviour of the fundamental solution.

Lemma 3. *Let us assume* $\|E\|_{L^\infty_{xt}(\mathbb{R}^3\times[0,T])} \leqslant C$ *and* $\|\nabla_x E\|_{L^\infty_{xt}(\mathbb{R}^3\times[0,T])} \leqslant C$. *Then*

(i) *We have*

$$\text{(FS1)}\quad 0 \leqslant \Gamma_E(x, v, t; \xi, v, \tau) \leqslant C(\|E\|_{L^\infty_{xt}}, T)\, G\left(\frac{x}{2}, \frac{v}{2}, t; \frac{\xi}{2}, \frac{v}{2}, \tau\right),$$

$$\text{(FS2)}\quad |\partial_v \Gamma_E(x, v, t; \xi, v, \tau)| \leqslant C(\|E\|_{L^\infty_{xt}}, T)\frac{G(x/2, v/2, t; \xi/2, v/2, \tau)}{(t-\tau)^{1/2}},$$

$$\text{(FS3)}\quad |\partial_x \Gamma_E(x, v, t; \xi, v, \tau)| \leqslant C(\|E\|_{L^\infty_{xt}}, \|\nabla E\|_{L^\infty_{xt}}, T)\frac{G(x/2, v/2, t; \xi/2, v/2, \tau)}{(t-\tau)^{3/2}},$$

$$\text{(FS4)}\quad |\Delta_v \Gamma_E(x, v, t; \xi, v, \tau)| \leqslant C(\|E\|_{L^\infty_{xt}}, \|\nabla E\|_{L^\infty_{xt}}, T)\frac{G(x/2, v/2, t; \xi/2, v/2, \tau)}{(t-\tau)},$$

for $t-\tau > 0$, $t, \tau \in [0, T]$.

(ii) *It holds*

$$\int_{\mathbb{R}^6} \Gamma_E(x, v, t; \xi, v, \tau)\,dx\,dv = \int_{\mathbb{R}^6} \Gamma_E(x, v, t; \xi, v, \tau)\,d\xi\,dv = 1,$$

$$\int_{\mathbb{R}^6} \Gamma_E(x, v, t; \xi', v', \tau')\Gamma_E(\xi', v', \tau'; \xi, v, \tau)\,d\xi'\,dv' = \Gamma_E(x, v, t; \xi, v, \tau)$$

with $x, v, \xi, v, \xi', v' \in \mathbb{R}^3$ *and* $t > \tau' > \tau \geqslant 0$.

(iii) $\|g(t)\|_{L^p_{xv}} \leqslant \|g_0\|_{L^p}$, $1 \leqslant p \leqslant \infty$.

(iv) *We have an integral expression*

$$\text{(I2)}\quad \Gamma_E(x, v, t; \xi, v, \tau) = G(x, v, t; \xi, v, \tau)$$
$$+ \int_\tau^t\int\int \partial_{v'} G(x, v, t; \xi', v', s) E(s, \xi')\Gamma_E(\xi', v', s; \xi, v, \tau)\,d\xi'\,dv'\,ds$$
$$= G(x, v, t; \xi, v, \tau) - \int_\tau^t\int\int G(x, v, t; \xi', v,' s) E(s, \xi')\partial_{v'}\Gamma_E(\xi', v', s; \xi, v, \tau)\,d\xi'\,dv'\,ds.$$

Proof. Statements (i), (ii) and (iv) were proved in Reference 18. From (ii) it follows that

$$\|g(t)\|_{L^1_{xv}} \leqslant \|g_0\|_{L^1}, \qquad \|g(t)\|_{L^\infty_{xv}} \leqslant \|g_0\|_{L^\infty}.$$

Interpolating we get (iii) for every p.

Remark 3. The second identity in (ii) states that the value of the fundamental solution at time t corresponding to an initial time τ is the same as the value of the solution of (P_E) at time t taking the value of the fundamental solution with singularity at τ as the initial datum at time $\tau' \in (\tau, t)$.

Remark 4. In view of the estimates on Γ_E, we could think of getting for $f(t)$ the same kind of decay estimates we got for $G(f_0)(t)$. However, the fact that the constant $C(\|E\|_{L^\infty_{xt}}, T)$, depends on T makes it impossible at first sight.

Remark 5. The condition $\|\nabla_x E\|_{L^\infty_{xt}(\mathbb{R}^3 \times [0,T])} \leqslant C$ is used in Reference 18 to ensure that the function Γ_E, which is the only solution of the integral equation (iv), is a classical solution of the partial differential equation (it allows to justify the differentiation inside the integral term).

Corollary 1. *Let us assume* $\|E\|_{L^\infty_{xt}(\mathbb{R}^3 \times [0,T])} \leqslant C$. *Then, a generalized fundamental solution* $\Gamma_E(x, v, t; \xi, \nu, \tau)$ *exists. This solution satisfies estimates* (FS1), (FS2) *and parts* (ii)–(iv) *in Lemma* 3. *The integral expression* (I1) *for solutions of* $\mathscr{P}_E$ *also holds.*

Proof. A function $\Gamma(x, v, t; \xi, \nu, \tau)$ solving the integral equation in Lemma 3(iv) is known to exist (see Reference 18). This function satisfies (FS1) and (FS2). We must relate the integral equation to the partial differential equation.

Let us see that solutions f of (P_E) with bounded potentials E are limits of solutions f_δ of mollified problems with potentials $E_\delta \in L^\infty_{xt}$ with $\nabla_x E_\delta$ in L^∞_{xt}. We take $E_\delta = E * \eta_\delta$ where η_δ is a mollifying family. We then have $\|E_\delta\|_{L^\infty_{xt}} \leqslant \|E\|_{L^\infty_{xt}}$ and $E_\delta \to E$ in L^∞_{xt} weak $*$.

For each f_δ we have the corresponding fundamental solution Γ_δ satisfying the estimates in Lemma 3. In particular, we have the bounds

$$(\mathrm{FS1}_\delta)\ \ |\Gamma_\delta(x, v, t; \xi, \nu, \tau)| \leqslant C(\|E_\delta\|_{L^\infty_{xt}}, T)\, G\left(\frac{x}{2}, \frac{v}{2}, t; \frac{\xi}{2}, \frac{\nu}{2}, \tau\right),$$

$$(\mathrm{FS2}_\delta)\ \ |\partial_v \Gamma_\delta(x, v, t; \xi, \nu, \tau)| \leqslant C(\|E_\delta\|_{L^\infty_{xt}}, T)\, \frac{G(x/2, v/2, t; \xi/2, \nu/2, \tau)}{(t-\tau)^{1/2}}$$

and the identities

$$(\mathrm{I1}_\delta)\ \ f_\delta(x, v, t) = \iint \Gamma_\delta(x, v, t; \xi, \nu, 0) f_0(\xi, \nu)\, \mathrm{d}\xi\, \mathrm{d}\nu,$$

$$(\mathrm{I2}_\delta)\ \ \Gamma_\delta(x, v, t; \xi, v, \tau) = G(x, v, t; \xi, v, \tau)$$

$$+ \int_\tau^t \int\int \partial_{v'} G(x, v, t; \xi', v', s) E(s, \xi') \Gamma_\delta(\xi', v', s; \xi, v, \tau)\, \mathrm{d}\xi'\, \mathrm{d}v'\, \mathrm{d}s$$

$$= G(x, v, t; \xi, v, \tau) - \int_\tau^t \int\int G(x, v, t; \xi', v', s) E(s, \xi') \partial_{v'} \Gamma_\delta(\xi', v', s; \xi, v, \tau)\, \mathrm{d}\xi'\, \mathrm{d}v'\, \mathrm{d}s.$$

We assume that f_0 is smooth with compact support. Then, the formulae can be extended to integrable f_0 by density.

Since Γ_δ is bounded (locally in t) in any L^p_{xvt} space a subsequence denoted again Γ_δ, converges weakly (locally in t) in any L^p_{xvt} (weakly * if $p = \infty$) to a function Γ_E and we can pass to the limit in the right-hand side. In the distribution sense, the derivatives of Γ_δ with respect to v converge weakly to the derivatives of Γ_E.

On the other hand, if a sequence g_n converges to g in the sense of distributions and $g_n \leqslant h$ then $g \leqslant h$ in the sense of distributions. Therefore, we can pass to the limit in the bounds (FS1) and (FS2) for Γ_δ and get them for Γ_E in the sense of distributions. By density we can extend the inequalities to test functions decaying at infinity.

It is clear from the estimate ($\mathrm{FS1}_\delta$) and the integral expression ($\mathrm{I1}_\delta$) that f_δ is uniformly bounded in any space L^p_{xvt} with respect to δ and locally in t. Therefore, f_δ converges weakly (locally in t) in any L^p_{xvt} space to a function f and their derivatives also converge in the sense of distributions.

Now, multiplying the equation satisfied for f_δ by f_δ we get a uniform L^2_{xvt} bound on $\nabla_v f_\delta$. If we multiply the equation by $|v|^2$ we get a uniform L^1 bound on $|v|^2 f_\delta$.

We can pass to the limit in the equation satisfied by f_δ except in the term $E_\delta \nabla_v f_\delta$. Let us take a test function $\alpha(x)\beta(v)\gamma(t)$. Then, we must pass to the limit in

$$\int_0^T \int \mathrm{d}x\, \mathrm{d}t\, E_\delta(x, t)\alpha(x)\gamma(t) \int \mathrm{d}v\, \nabla_v \beta(v) f_\delta(x, v, t).$$

We split the integral in two $I_1 + I_2$, where I_1 is the integral when $|v| < R$ and I_2 the integral when $|v| \geqslant R$, $R > 0$. I_2 goes to zero as R tends to ∞ uniformly with respect to δ thanks to the estimate on $|v|^2 f_\delta$. On the other hand, for a.e. fixed (x, t) the function $F_\delta(x, t) = \int_{|v| < R} \mathrm{d}u\, \nabla_v \beta(v) f_\delta(x, v, t)$ tends to $F(x, t) = \int_{|v| < R} \mathrm{d}v\, \nabla_v \beta(v) f(x, v, t)$ thanks to the bounds on $\nabla_v f_\delta$. We have also bound

$$|F_\delta(x, t)| \leqslant C \int_{|v| < R} \mathrm{d}v\, |\nabla_v \beta(v)| \int\int G\left(\frac{x}{2}, \frac{v}{2}, t; \frac{\xi}{2}, \frac{v}{2}, 0\right) |f_0(\xi, v)|\, \mathrm{d}\xi\, \mathrm{d}v = H(x, t)$$

with $H(x, t) \in L^{1\mathrm{loc}}_{x,t}$. Therefore, $F_\delta(x, t) \to F(x, t)$ in $L^1_{xt,\mathrm{loc}}$. Since $E_\delta \to E$ in L^∞_{xt} weak * we can pass to the limit.

Passing to the limit in ($\mathrm{I1}_\delta$) and ($\mathrm{I2}_\delta$) we see that $\Gamma_E(x, v, t; \xi, v, \tau)$ is such that the solution of (P_E) is given by

$$(\mathrm{I1})\ \ f(x, v, t) = \int\int \Gamma_E(x, v, t; \xi, v, 0) f_0(\xi, v)\, \mathrm{d}\xi\, \mathrm{d}v$$

and Γ_E is a solution to the integral equation

$$\text{(I2)}\ \Gamma_E(x, v, t; \xi, \nu, \tau) = G(x, v, t; \xi, \nu, \tau) + \int_\tau^t \int\int \partial_{v'} G(x, v, t; \xi', v', s) E(s, \xi') \Gamma_E(\xi', v', s; \xi, \nu, \tau)\, d\xi'\, dv'\, ds$$

$$= G(x, v, t; \xi, \nu, \tau) - \int_\tau^t \int\int G(x, v, t; \xi', v', s) E(s, \xi') \partial_{v'} \Gamma_E(\xi', v', s; \xi, \nu, \tau)\, d\xi'\, dv'\, ds.$$

In the same way, (ii) and (iii) in Lemma 3 remain true in the limit. We know (see Reference 18) that the solution of (I2) satisfies (FS1) and (FS2).

2.2. *$E(x, t)$ decaying at infinity*

2.2.1. Estimate on the fundamental solution. For E as in section 2.1 we ignore the possibility of getting the decay for the solutions. The only bounds we have are

$$\|g(t)\|_{L^p_{xv}} \leqslant \|g_0\|_{L^p_{xv}}, \quad 1 \leqslant p \leqslant \infty.$$

However, if we assume some decay on E as t grows, we can bound the fundamental solution Γ_E by means of the fundamental solution of the problem with $E = 0$. Thanks to this bound, the decay estimates are immediate. More precisely, we have the following theorem:

Theorem 1. *Let us assume that the field E is such that*

(i) $E \in L^\infty_{x,t}$;

(ii) $\|E(t)\|_{L^\infty_x} \leqslant C_\alpha/((1+t)^{(1/2)+\alpha})$ *with* $\alpha \geqslant 0$ *and* C_α *small enough if* $\alpha = 0$.

Then, the fundamental solution Γ_E *for the problem* $\mathscr{P}_E$ *satisfies a global in time estimate of the format*

$$0 \leqslant \Gamma_E(x, v, t; \xi, \nu, \tau) \leqslant C(\|E\|_{L^\infty_{xt}}) G\left(\frac{x}{2}, \frac{v}{2}, t; \frac{\xi}{2}, \frac{\nu}{2}, \tau\right)$$

for $0 < t - \tau < \infty$, $\tau \geqslant 0$, $x, v, \xi, \nu \in \mathbb{R}^3$.

Remark 6. Condition (i) guarantees the existence of a fundamental solution satisfying the estimates and identifies in Corollary 1. Condition (ii) allows to get the global in time version of (FS1) removing the time dependence of the constant.

Proof. Step 1: C_α *small*. The procedure to construct the fundamental solution was as follows [17, 15]:

$$\Gamma(x, v, t; \xi, \nu, \tau) = G(x, v, t; \xi, \nu, \tau) + \sum_{n=0}^{\infty} (\Gamma_{n+1} - \Gamma_n)(x, v, t; \xi, \nu, \tau)$$

with

$$\Gamma_0(x,v,t;\xi,v,\tau)=G(x,v,t;\xi,v,\tau),$$
$$\Gamma_{n+1}(x,v,t;\xi,v,\tau)=G(x,v,t;\xi,v,\tau)$$
$$+\int_\tau^t\int\int\partial_{v'}G(x,v,t;\xi',v',s)E(s,\xi')\Gamma_n(\xi',v',s;\xi,v,\tau)\,d\xi'\,dv'\,ds.$$

We have the following estimates:

$$|(\Gamma_{n+1}-\Gamma_n)(x,v,t;\xi,v,\tau)|\leqslant(M_v2^6I_1C_\alpha)^{n+1}\,G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi}{2},\frac{v}{2},\tau\right)$$

and for $\alpha\in(0,\frac{1}{2})$

$$|(\Gamma_{n+1}-\Gamma_n)(x,v,t;\xi,v,\tau)|\leqslant(M_v2^6C_\alpha)^{n+1}I_1^nI_2t^{-\alpha}G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi}{2},\frac{v}{2},\tau\right),$$

where the constants M_v, I_1, I_2, C_α are given by

$$|\partial_vG(x,v,t;\xi,v,\tau)|\leqslant M_v(t-\tau)^{-1/2}\,G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi}{2},\frac{v}{2},\tau\right),$$

$$\int_0^t(t-s)^{-1/2}s^{-1/2}\,ds=I_1,$$

$$\int_0^t(t-s)^{-1/2}s^{-\alpha-1/2}\,ds=I_2t^{-\alpha},\quad \alpha\in(0,\tfrac{1}{2}),$$

$$\|E(t)\|_{L^\infty_x}\leqslant\frac{C_\alpha}{(1+t)^{1/2+\alpha}}.$$

The estimates are proved by induction. For $n=0$ we have

$$|(\Gamma_1-\Gamma_0)(x,v,t;\xi,v,\tau)|$$
$$=\left|\int_\tau^t\int\int\partial_{v'}G(x,v,t;\xi',v',s)E(s,\xi')G(\xi',v',s;\xi,v,\tau)\,d\xi'\,dv'\,ds\right|$$
$$\leqslant M_vC_\alpha\int_\tau^t(t-s)^{-1/2}(1+s)^{-(\alpha+1/2)}\,ds$$
$$\times\int\int G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi'}{2},\frac{v'}{2},s\right)G(\xi',v',s;\xi,v,\tau)\,d\xi'\,dv'$$
$$\leqslant M_vC_\alpha2^6\int_\tau^t(t-s)^{-1/2}(1+s)^{-(\alpha+1/2)}\,ds\,G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi}{2},\frac{v}{2},\tau\right).$$

We can bound this last quantity by either

$$M_vC_\alpha2^6I_1\,G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi}{2},\frac{v}{2},\tau\right)\quad\text{or}\quad M_v2^6C_\alpha I_2t^{-\alpha}G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi}{2},\frac{v}{2},\tau\right).$$

Assuming the bounds to hold for $n-1$, let us see that they hold for n. We have

$$\begin{aligned}
&|(\Gamma_{n+1}-\Gamma_n)(x,v,t;\xi,v,\tau)|\\
&\leqslant \left|\int_\tau^t\int\int \partial_{v'}G(x,v,t;\xi',v',s)E(s,\xi')(\Gamma_n-\Gamma_{n-1})(\xi',v',s;\xi,v,\tau)\,\mathrm{d}\xi'\,\mathrm{d}v'\,\mathrm{d}s\right|\\
&\leqslant M_vC_\alpha(M_v2^6I_1C_\alpha)^n\int_\tau^t(t-s)^{-1/2}(1+s)^{-(\alpha+1/2)}\,\mathrm{d}s\\
&\quad\times\int\int G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi'}{2},\frac{v'}{2},s\right)G\left(\frac{\xi'}{2},\frac{v'}{2},t;\frac{\xi}{2},\frac{v}{2}v\right)\mathrm{d}\xi'\,\mathrm{d}v'\\
&\leqslant (M_v2^6C_\alpha)^{n+1}I_1^n\int_\tau^t(t-s)^{-1/2}(1+s)^{-(\alpha+1/2)}\,\mathrm{d}s\,G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi}{2},\frac{v}{2},\tau\right).
\end{aligned}$$

We can bound this last quantity by either

$$(M_v2^6I_1C_\alpha)^{n+1}G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi}{2},\frac{v}{2},\tau\right)\ \text{or}\ (M_v2^6C_\alpha)^{n+1}I_1^nI_2t^{-\alpha}G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi}{2},\frac{v}{2},\tau\right).$$

The series $\sum_{n=0}^{\infty}(M_v2^6I_1C_\alpha)^{n+1}$ converges if $M_v2^6I_1C_\alpha<1$. This can be achieved if C_α is small enough. In conclusion

$$|\Gamma(x,v,t;\xi,v,\tau)|\leqslant\left(M+\sum_{n=0}^{\infty}(M_v2^6I_1C_\alpha)^{n+1}\right)G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi}{2},\frac{v}{2},\tau\right)$$

when C_α is small, where M is such that

$$\Gamma_E(x,v,t;\xi,v,\tau)\leqslant MG\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi}{2},\frac{v}{2},\tau\right).$$

Step 2: $\alpha>0$. When $\alpha>0$ we can write for $t\geqslant T$

$$\|E(x,t)\|_{L^\infty_x}\leqslant\frac{C_\alpha}{(1+t)^{\alpha+1/2}}\leqslant\frac{C_{\alpha/2}}{(1+t)^{(\alpha/2)+1/2}}$$

with $C_{\alpha/2}=C_\alpha/((1+T)^{\alpha+1/2})<1$ when T is large enough. Going back to the estimates in step 1, we can use this estimate on E in the integrals above when $\tau>T$, so that we get

$$|\Gamma(x,v,t;\xi,v,\tau)|\leqslant\left(M+\sum_{n=0}^{\infty}(M_v2^6I_1C_{\alpha/2})^{n+1}\right)G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi}{2},\frac{v}{2},\tau\right)$$

when $t>\tau>T$.

For $0\leqslant\tau<t\leqslant3T$ we have

$$|\Gamma(x,v,t;\xi,v,\tau)|\leqslant C(\|E\|_{L^\infty(\mathbb{R}^3\times[0,2T])})\,G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi}{2},\frac{v}{2},\tau\right).$$

When $\tau \leqslant T$ and $t \geqslant 3T$ we may use the expression

$$\int_{\mathbb{R}^6} \Gamma_E(x, v, t; \xi', v', \tau')\Gamma_E(\xi', v', \tau'; \xi, v, \tau)\,d\xi'\,dv' = \Gamma_E(x, v, t; \xi, v, \tau)$$

with $x, v, \xi, v, \xi', v' \in \mathbb{R}^3$ and $t > \tau' > \tau \geqslant 0$. Taking $\tau' = 2T$ we get

$$\Gamma_E(x, v, t; \xi, v, \tau) \leqslant \left(M + \sum_{n=0}^{\infty} (M_v 2^6 I_1 C_{\alpha/2})^{n+1}\right) C(\|E\|_{L^\infty(\mathbb{R}^3 \times [0,2T])})$$

$$\times \iint G\left(\frac{x}{2}, \frac{v}{2}, t; \frac{\xi'}{2}, \frac{v'}{2}, \tau'\right) G\left(\frac{\xi'}{2}, \frac{v'}{2}, \tau'; \frac{\xi}{2}, \frac{v}{2}, \tau\right) d\xi'\,dv'$$

$$\leqslant \left(M + \sum_{n=0}^{\infty} (M_v 2^6 I_1 C_{\alpha/2})^{n+1}\right) C(\|E\|_{L^\infty(\mathbb{R}^3 \times [0,2T])})\, G\left(\frac{x}{2}, \frac{v}{2}, t; \frac{\xi}{2}, \frac{v}{2}, \tau\right),$$

where

$$\Gamma_E(x, v, t; \xi, v, \tau) \leqslant MG\left(\frac{x}{2}, \frac{v}{2}, t; \frac{\xi}{2}, \frac{v}{2}, \tau\right).$$

Therefore, we get a global estimate

$$|\Gamma(x, v, t; \xi, v, \tau)| \leqslant C(\|E\|_{L^\infty_{xt}})\, G\left(\frac{x}{2}, \frac{v}{2}, t; \frac{\xi}{2}, \frac{v}{2}, \tau\right)$$

for $0 \leqslant \tau < t$, when $\alpha > 0$.

2.2.2. Asymptotic behaviour of solutions. In view of section 2.2.1 we have the solution of $\mathscr{P}_E$

$$g(x, v, t) = \int \Gamma_E(x, v, t; \xi, v, 0) g_0(\xi, v)\,d\xi\,dv$$

satisfies the following decay estimates

$$\|g(t)\|_{L^p_{xv}} \leqslant \|g_0\|_{L^p_{xv}},$$

$$\|g(t)\|_{L^p_{xv}} \leqslant Ct^{-6(1/r - 1/p)}\|g_0\|_{L^r_{xv}}, \quad p \geqslant r,$$

$$\|\rho(g)(t)\|_{L^p_x} \leqslant Ct^{-(9/2)(1/r - 1/p)}\|g_0\|_{L^1_v(L^r_x)}, \quad p \geqslant r.$$

Expanding g_0 (see Reference 10) we have

$$g_0 = \sum_{\substack{|\alpha| \leqslant n \\ |\beta| \leqslant m}} \frac{(-1)^{|\alpha| + |\beta|}}{(|\alpha| + |\beta|)!}\left(\int x^\alpha v^\beta g_0\right) \partial_x^\alpha \partial_v^\beta \delta + \sum_{\substack{|\alpha| = n \\ |\beta| = m}} \partial_x^\alpha \partial_v^\beta F_{\alpha,\beta}$$

with $F_{\alpha,\beta} \in L^1_{xv}$, if $g_0(1+|x|^{n+1}|v|^{m+1}) \in L^1_{xv}$. For such g_0, we get an expansion

$$g(x,v,t) = \sum_{\substack{|\alpha|\leqslant n\\|\beta|\leqslant m}} \frac{(-1)^{|\alpha|+|\beta|}}{(|\alpha|+|\beta|)!}\int \Gamma_E(x,v,t;\xi,v,0)\left(\int x^\alpha v^\beta g_0\right)\partial_\xi^\alpha\partial_v^\beta \delta\,\mathrm{d}\xi\,\mathrm{d}v$$

$$+ \sum_{\substack{|\alpha|=n+1\\|\beta|=m+1}} \left(\int x^\alpha v^\beta g_0\right)\int \Gamma_E(x,v,t;\xi,v,0)\partial_\xi^\alpha\partial_v^\beta F_{\alpha,\beta}\,\mathrm{d}\xi\,\mathrm{d}v$$

$$= \sum_{\substack{|\alpha|\leqslant n\\|\beta|\leqslant m}} \frac{1}{(|\alpha|+|\beta|)!}\left(\int x^\alpha v^\beta g_0\right)\partial_\xi^\alpha\partial_v^\beta\Gamma_E(x,v,t;0,0,0)$$

$$+ \sum_{\substack{|\alpha|=n+1\\|\beta|=m+1}} (-1)^{|\alpha|+|\beta|}\int \partial_\xi^\alpha\partial_v^\beta\Gamma_E(x,v,t;\xi,v,0)F_{\alpha,\beta}\,\mathrm{d}\xi\,\mathrm{d}v$$

for $g_0(1+|x|^{n+1}|v|^{m+1}) \in L^1_{xv}$. When $E=0$, $\Gamma_E = G$ and we can estimate the decay rate of the different terms

$$\|\partial_\xi^\alpha\partial_v^\beta G(x,v,t;0,0,0)\|_{L^p_{xv}} \leqslant C\frac{\|G(x/2,v/2,t;0,0,0)\|_{L^p_{xv}}}{t^{(3|\alpha|/2)+|\beta|/2}}$$

$$\leqslant Ct^{-6(1-1/p)-(3|\alpha|/2)+|\beta|/2}.$$

Using the estimates obtained in Lemma 1, the decay norm of the remaining term, which we denote R_{mn}, is estimated as follows:

$$\|R_{mn}\|_{L^p_{xv}} \leqslant C\sum_{\substack{|\alpha|=n\\|\beta|=m}} \frac{\|\int G(x/2,v/2,t,\xi/2,v/2,\tau)F_{\alpha,\beta}(\xi,v)\,\mathrm{d}\xi\,\mathrm{d}v\|_{L^p_{xv}}}{t^{3(n+1)/2+(m+1)/2}}$$

$$\leqslant Ct^{-6(1-1/p)-(3(n+1))/2-(m+1)/2}.$$

When $E \neq 0$ is Lipschitz, we can use estimates (FS3) and (FS4) to bound the rest and we get

$$g(x,v,t) = \sum_{\substack{|\alpha|\leqslant 0\\|\beta|\leqslant 1}} \frac{1}{(|\alpha|+|\beta|)!}\left(\int x^\alpha v^\beta g_0\right)\partial_\xi^\alpha\partial_v^\beta\Gamma_E(x,v,t;0,0,0) + R_{01}$$

with

$$\|R_{01}\|_{L^p_{xv}} \leqslant Ct^{-6(1-1/p)-3/2-2/2}.$$

If E is only bounded we cannot estimate the rest in this way. Therefore, we have proved □

Theorem 2. (i) *For a solution g of (P_0) we have the expansion*

$$g(x,v,t) = \sum_{\substack{|\alpha|\leqslant n\\|\beta|\leqslant m}} \frac{1}{(|\alpha|+|\beta|)!}\left(\int x^\alpha v^\beta g_0\right)\partial_\xi^\alpha\partial_v^\beta\Gamma_E(x,v,t;0,0,0) + R_{nm},$$

$$\|R_{mn}\|_{L^p_{xv}} \leqslant Ct^{-6(1-1/p)-[3(n+1)]/2-(m+1)/2}$$

when $g_0(1+|x|^{n+1}|v|^{m+1}) \in L^1_{xv}$.

(ii) *For a solution g of* (P_E) *with a Lipschitz potential E we have the expansion*

$$g(x, v, t) = \sum_{\substack{|\alpha| \leqslant 0 \\ |\beta| \leqslant 1}} \frac{1}{(|\alpha| + |\beta|)!} \left(\int x^\alpha v^\beta g_0 \right) \partial_\xi^\alpha \partial_v^\beta \Gamma_E(x, v, t; 0, 0, 0) + R_{01},$$

$$\| R_{mn} \|_{L^p_{xv}} \leqslant C t^{-6(1-(1/p)) - (3/2) - 2/2}$$

when $g_0(1 + |x|^1 |v|^2) \in L^1_{xv}$.

Remark 7. The expansion in Reference 10 is proved when $g_0(1 + |(x, v)|^{k+1}) \in L^1_{xv}$ replacing the sums $\sum_{\substack{|\alpha| \leqslant n \\ |\beta| \leqslant m}}$ with $\sum_{|\alpha| + |\beta| \leqslant k}$ and $\sum_{\substack{|\alpha| = n+1 \\ |\beta| = m+1}}$ with $\sum_{|\alpha| + |\beta| \leqslant k+1}$. However, the result can be extended to cover our case.

2.2.3. Derivatives. We would like to get an estimate global in time for the derivatives of the fundamental solution. We have the following theorem:

Theorem 2. *Let us assume that the field E satisfies*

(i) $E \in L^\infty_{x,t}$;

(ii) $\| E(t) \|_{L^\infty_x} \leqslant C_\alpha / ((1 + t)^{(1/2) + \alpha})$ *with* $\alpha \geqslant 0$ *and* C_α *small enough if* $\alpha = 0$.

Then, the fundamental solution Γ_E *for the problem* $\mathscr{P}_E$ *satisfies an estimate global in time of the form*

$$|\nabla_v \Gamma_E(x, v, t; \xi, v, \tau)| \leqslant \frac{C(\| E \|_{L^\infty_{xt}})}{(t - \tau)^{1/2}} G\left(\frac{x}{2}, \frac{v}{2}, t; \frac{\xi}{2}, \frac{v}{2}, \tau \right)$$

for $0 < t - \tau < \infty$, $\tau \geqslant 0$, $x, v, \xi, v \in \mathbb{R}^3$.

Proof. The derivatives of Γ_E with respect to v solve

$$\nabla_v \Gamma_E(x, v, t; \xi, v, \tau) = \nabla_v G(x, v, t; \xi, v, \tau) - \int_\tau^t \int \int \nabla_v G(x, v, t; \xi', v', s) E(s, \xi') \nabla_{v'} \Gamma_E(\xi', v', s; \xi, v, \tau) \, d\xi' \, dv' \, ds.$$

On the other hand, we have

$$\nabla_v \Gamma_E(x, v, t; \xi, v, \tau) = \nabla_v G(x, v, t; \xi, v, \tau) + \sum_{n=0}^{\infty} (\nabla_v \Gamma_{n+1} - \nabla_v \Gamma_n)(x, v, t; \xi, v, \tau).$$

We shall prove that

$$|(\nabla_v \Gamma_{n+1} - \nabla_v \Gamma_n)(x, v, t; \xi, v, \tau)| \leqslant (M'' 2^6 C_\alpha I')^{n+1} (t - \tau)^{-1/2} G\left(\frac{x}{2}, \frac{v}{2}, t; \frac{\xi}{2}, \frac{v}{2}, \tau \right),$$

where $M'' = \max(M_v^2, M_{vv} M')$ with

$$|\nabla_v \nabla_v G(x, v, t; \xi, v, s)| \leqslant M_{vv} (t - s)^{-1} G\left(\frac{x}{2}, \frac{v}{2}, t; \frac{\xi}{2}, \frac{v}{2}, s \right)$$

and M_v as in section 2.2.1. The constants I' and M' will be fixed below. For $n=0$ we have

$$|(\nabla_v\Gamma_1-\nabla_v\Gamma_0)(x,v,t;\xi,v,\tau)|$$
$$\leqslant C_\alpha\left(\int_\tau^{(t+\tau)/2}\iint(1+(s-\tau))^{-(\alpha+1/2)}|\nabla_v\nabla_{v'}G(x,v,t;\xi',v',s)|G(\xi',v',s;\xi,v,\tau)\,ds\,d\xi'\,dv'\right.$$
$$\left.+\int_{(\tau+t)/2}^{t}(1+(s-\tau))^{-(\alpha+1/2)}M_v^2(s-\tau)^{-1/2}(t-s)^{-1/2}\,ds\,2^6G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi}{2},\frac{v}{2},\tau\right)\right).$$

Therefore,

$$|(\nabla_v\Gamma_1-\nabla_v\Gamma_0)(x,v,t;\xi,v,\tau)|\leqslant M_0 2^6C_\alpha I(t-\tau)^{-1/2}G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi}{2},\frac{v}{2},\tau\right)$$

with $M_0=\max(M_v,M_{vv},1)^2$ and $I=2^{1/2}\int_0^1 s^{1/2}\,ds=2^{3/2}/3$. Let us assume that

$$|(\nabla_v\Gamma_n-\nabla_v\Gamma_{n-1})(x,v,t;\xi,v,\tau)|\leqslant(2^6C_\alpha I')^nM_{n-1}(t-\tau)^{-1/2}G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi}{2},\frac{v}{2},\tau\right)$$

for $M_{n-1}=(\max(M_{vv},M_v,1))^{n+1}$. Then, we have

$$|(\nabla_v\Gamma_{n+1}-\nabla_v\Gamma_n)(x,v,t;\xi,v,\tau)|$$
$$\leqslant C_\alpha\left(\int_\tau^{(\tau+t)/2}\iint(1+(s-\tau))^{-(\alpha+1/2)}|\nabla_v\nabla_{v'}G(x,v,t;\xi',v',s)|\right.$$
$$\times|\Gamma_n-\Gamma_{n-1}|(\xi',v',s;\xi,v,\tau)\,ds\,d\xi'\,dv'$$
$$+\int_{(\tau+t)/2}^{t}\iint M_v(1+(s-\tau))^{-(\alpha+1/2)}(s-\tau)^{-1/2}(t-s)^{-1/2}G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi'}{2},\frac{v'}{2},s\right)$$
$$\left.\times\nabla_{v'}|\Gamma_n-\Gamma_{n-1}|(\xi',v',s;\xi,v,\tau)\,ds\,d\xi'\,dv'\right).$$

Using for $|\Gamma_n-\Gamma_{n-1}|$ the bounds in section 2.2.1 and for $|\nabla_v(\Gamma_n-\Gamma_{n-1})|$ the induction hypotheses, we get

$$|\nabla_v\Gamma_{n+1}-\nabla_v\Gamma_n)(x,v,t;\xi,v,\tau)|\leqslant C_\alpha 2^6M_n'I(t-\tau)^{-1/2}G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi}{2},\frac{v}{2},\tau\right)$$

with $M_n'=\max(M_{vv}(M_v2^6I_1C_\alpha)^n,\ M_v(2^6C_\alpha I)^nM_{n-1})$. We take $I'=\max(I,I_1)$ and $M_n=(\max(M_{vv},M_v,1))^{n+2}$. Then

$$|(\nabla_v\Gamma_{n+1}-\nabla_v\Gamma_n)(x,v,t;\xi,v,\tau)|\leqslant(C_\alpha 2^6I')^{n+1}M_nI(t-\tau)^{-1/2}G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi}{2},\frac{v}{2},\tau\right)$$

Therefore,

$$|\nabla_v\Gamma(x,v,t;\xi,v,\tau)|$$
$$\leqslant(M_v(t-\tau)^{-1/2}+\sum_{n=0}^{\infty}(C_\alpha 2^6I')^nM_n(t-\tau)^{-1/2})G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi}{2},\frac{v}{2},\tau\right).$$

The series above converges if C_α is small enough. We denote $M'_v = M_v + \sum_{n=0}^{\infty}(C_\alpha 2^6 I')^n$.

When $\alpha > 0$ we can write for $t \geqslant T$

$$\|E(x,t)\|_{L^\infty_x} \leqslant \frac{C_\alpha}{(1+t)^{\alpha+1/2}} \leqslant \frac{C_{\alpha/2}}{(1+t)^{\alpha/2+1/2}}$$

with $C_{\alpha/2} = C_\alpha/(1+T)^{\alpha+1/2} < 1$ when T is large enough. Going back to the previous estimates, we can use this estimate for E in the integrals above when $\tau \geqslant T$, so that we get

$$|\nabla_v \Gamma(x,v,t;\xi,v,\tau)| \leqslant M'_v(t-\tau)^{-1/2} G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi}{2},\frac{v}{2},\tau\right)$$

when $t > \tau \geqslant T$.

For $0 \leqslant \tau < t \leqslant 3T$ we have

$$|\nabla_v \Gamma(x,v,t;\xi,v,\tau)| \leqslant C(\|E\|_{L^\infty(\mathbb{R}^3\times[0,2T])})(t-\tau)^{-1/2} G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi}{2},\frac{v}{2},\tau\right).$$

When $\tau < T$ and $t \geqslant 3T$ we may use the expression

$$\int_{\mathbb{R}^6} \nabla_v \Gamma_E(x,v,t;\xi',v',\tau')\Gamma_E(\xi',v',\tau';\xi,v,\tau)\,\mathrm{d}\xi'\,\mathrm{d}v' = \nabla_v \Gamma_E(x,v,t;\xi,v,\tau)$$

with $x, v, \xi, v, \xi', v' \in \mathbb{R}^3$ and $t > \tau' > \tau \geqslant 0$. We get

$$\begin{aligned}|\nabla_v \Gamma_E(x,v,t;\xi',v,\tau)| &\leqslant M'_v(t-\tau')^{-1/2} C(\|E\|_{L^\infty(\mathbb{R}^3\times[0,\tau'])})\\ &\quad\times \iint G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi'}{2},\frac{v'}{2},\tau'\right) G\left(\frac{\xi'}{2},\frac{v'}{2},\tau',\frac{\xi}{2},\frac{v}{2},\tau\right)\mathrm{d}\xi'\,\mathrm{d}v'\\ &\leqslant M'_v(t-\tau')^{-1/2} C(\|E\|_{L^\infty(\mathbb{R}^3\times[0,r'])}) G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi}{2},\frac{v}{2},\tau\right).\end{aligned}$$

In the range of t we are dealing with $(t-\tau')^{-1/2} \leqslant (t-\tau)^{-1/2}$, so that we get a global estimate

$$|\nabla_v \Gamma(x,v,t;\xi,v,\tau)| \leqslant C(\|E\|_{L^\infty_{xt}})(t-\tau)^{-1/2} G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi}{2},\frac{v}{2},\tau\right)$$

for $0 \leqslant \tau < t$, when $\alpha > 0$.

2.2.4. Asymptotic behaviour of the derivatives. In view of section 2.2.3 we have that the derivative of the solution of $\mathscr{P}_E$ with respect to v

$$\nabla_v g(x,v,t) = \int \nabla_v \Gamma_E(x,v,t;\xi,v,0) g_0(\xi,v)\,\mathrm{d}\xi\,\mathrm{d}v$$

satisfies the following decay estimates:

$$\|\nabla_v g(t)\|_{L^p_{xv}} \leqslant C t^{-6(1/r-1/p)-1/2}\|g_0\|_{L^r_{xv}}, \quad p \geqslant r.$$

In the same way as in section 2.2.2 we get an expansion

$$\nabla_v g(x, v, t) = \sum_{\substack{|\alpha| \leqslant n \\ |\beta| \leqslant m}} \frac{(-1)^{|\alpha| + |\beta|}}{(|\alpha| + |\beta|)!} \left(\int x^\alpha v^\beta g_0 \right) \nabla_v \Gamma_E(x, v, t; \xi, \nu, 0) \partial_\xi^\alpha \partial_\nu^\beta \delta \, d\xi \, d\nu$$

$$+ \sum_{\substack{|\alpha| = n+1 \\ |\beta| = m+1}} \left(\int x^\alpha v^\beta g_0 \right) \nabla_v \Gamma_E(x, v, t; \xi, \nu, 0) \partial_\xi^\alpha \partial_\nu^\beta F_{\alpha,\beta} \, d\xi \, d\nu$$

$$= \sum_{\substack{|\alpha| \leqslant n \\ |\beta| \leqslant m}} \frac{1}{(|\alpha| + |\beta|)!} \left(\int x^\alpha v^\beta g_0 \right) \partial_\xi^\alpha \partial_\nu^\beta \nabla_v \Gamma_E(x, v, t; 0, 0, 0)$$

$$+ \sum_{\substack{|\alpha| = n+1 \\ |\beta| = m+1}} (-1)^{|\alpha| + |\beta|} \int \partial_\xi^\alpha \partial_\nu^\beta \nabla_v \Gamma_E(x, v, t; \xi, \nu, 0) F_{\alpha,\beta} \, d\xi \, d\nu$$

for $g_0(1 + |x|^{n+1} |v|^{m+1}) \in L^1_{xv}$, with $F_{\alpha,\beta} \in L^1_{xv}$. When $E = 0$, $\Gamma_E = G$ and we can estimate the decay rate of the different terms

$$\|\partial_\xi^\alpha \partial_\nu^\beta \nabla_v G(x, v, t; 0, 0, 0)\|_{L^p_{xv}} \leqslant C \frac{\|G(\frac{x}{2}, \frac{v}{2}, t; 0, 0, 0)\|_{L^p_{xv}}}{t^{3|\alpha|/2 + (|\beta| + 1)/2}}$$

$$\leqslant C t^{-6(1 - 1/p) - 3|\alpha|/2 - (|\beta| + 1)/2}.$$

The decay norm of the remaining term, which we denote R_{mn}, is estimated as follows:

$$\|R_{mn}\|_{L^p_{xv}} \leqslant C \sum_{\substack{|\alpha| = n \\ |\beta| = m}} \frac{\| \int G(x/2, v/2, t, \xi/2, \nu/2, \tau) F_{\alpha,\beta}(\xi, \nu) \, d\xi \, d\nu \|_{L^p_{xv}}}{t^{(3(n+1))/2 + (m+2)/2}}$$

$$\leqslant C t^{-6(1 - (1/p)) - 3(n+1)/2 - (m+2)/2}.$$

When $E \neq 0$ is Lipschitz we get this expansion but with $n = m = 0$.

3. Non-linear problem

3.1. *Long-time behaviour of the solutions*

Let f be a global solution of VPFP

$$f_t - \Delta_v f + v \nabla_x f + E(f) \nabla_v f = 0, \quad \text{in } \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+,$$

$$f(x, v, 0) = f_0(x, v), \quad \text{in } \mathbb{R}^3 \times \mathbb{R}^3,$$

where $E(\rho(f)) = \varepsilon (K *_x \rho(f))$ with $\rho(f) = \int_{\mathbb{R}^3} f(x, v, t) \, dv$, $\varepsilon = \pm 1$ (depending on whether the interaction between the particles is electrostatic or gravitational) and $K(x) = x / S_3 |x|^3$, S_3 being the area of the sphere in $\mathbb{R}^3$. We know that for initial data small enough and satisfying some suitable integrability conditions, global solutions are known to exist and to satisfy the associated integral equations

$$\text{(I1)}\quad f(x,v,t)=\int G(x,v,t;\xi,v,0)f_0(\xi,v)\,\mathrm{d}\xi\,\mathrm{d}v + \int_0^t \mathrm{d}s\int\int \nabla_v G(x,v,t;\xi,v,s)E(f)(\xi,s)f(\xi,v,s)\,\mathrm{d}\xi\,\mathrm{d}v,$$

$$\text{(I2)}\quad f(x,v,t)=\int\int \Gamma_E(x,v,t;\xi,v,0)f_0(\xi,v)\,\mathrm{d}\xi\,\mathrm{d}v,$$

where G is the fundamental solution of the problem with $E=0$, and Γ_E is the fundamental solution of the problem with potential $E(f)$.

Remark 8. For ease of notation we denote $E(\rho(f))=E(f)=E$.

We shall be interested in applying the results of the fundamental solutions obtained in section 2 to fields $E(\rho)$ with $\rho=\rho(f)$ and f a solution of (VPFP) corresponding to an initial datum f_0. Therefore, we must know for which data we get solutions of (VPFP) with Lipschitz or bounded potentials.

A first result in this direction is the following one: If

$$\text{(RW)}\ f_0\geqslant 0,\quad (1+v^2)^{\gamma/2}f_0\in(L^1\cap L^\infty)(\mathbb{R}^6),\quad (1+v^2)^{\gamma/2}\nabla_{(x,v)}f_0\in(L^1\cap L^\infty)(\mathbb{R}^6),\ \gamma>3$$

and we assume $E(\rho(f))\in L^\infty_{xt}$, then $\nabla_x E(\rho(f))$ belongs to L^∞_{xt} (see Reference 15). Therefore, when we have a global solution of (VPFP) and we know that the associated potential $E(f)$ is bounded, we can always guarantee that the potential is also Lipschitz, imposing more restrictions to the initial datum.

Let us recall some results of the global existence of solutions with bounded potentials.

In Reference 16 solutions to (VPFP) with $E(\rho(f))\in L^\infty_{xt}$ are constructed for small initial data in the class

$$f_0\in L^1\cap L^\infty,\quad 0\leqslant f_0\leqslant Ah(x)g(v).$$

In Reference 2 solutions to (VPFP) with $E(\rho(f))\in L^\infty_{xt}$ are constructed for initial data in the class

$$f_0\geqslant 0,\quad f_0\in(L^1\cap L^\infty)(\mathbb{R}^6),\quad |v|^m f_0\in L^1(\mathbb{R}^6)\ \text{for some}\ m>6.$$

By Reference 6 we can take $m>\frac{15}{4}$. By Reference 3 we can take $m=2$ if $\rho(f_0)\in L^\infty_x$.

In these cases the potential $E(f)$ is Lipschitz when we impose the condition (RW) on the initial data. All the results stated in Lemma 3 on the fundamental solutions for linearized (VPFP) problems with Lipschitz potentials E hold.

If we do not add restrictions on the data, the potential is bounded and Corollary 1 applies: the existence of fundamental solutions and the estimates (FS1) and (FS2) on the fundamental solution and its derivative with respect to v, as well as (ii) and (iii) in Lemma 3 are guaranteed without adding conditions on the data.

We have the following theorem:

Theorem 4. *Let us assume that*

(i) *f is the unique solution to* (VPFP) *taking f_0 as initial data satisfying* (I1);

(ii) *the fundamental solution Γ_E corresponding to $E = E(f)$ exists and satisfies*

$$0 \leqslant \Gamma_E(x, v, t; \xi, v, \tau) \leqslant C(\|E\|_{L^\infty_{xt}})\, G\left(\frac{x}{2}, \frac{v}{2}, t; \frac{\xi}{2}, \frac{v}{2}, \tau\right)$$

for $0 < t - \tau < \infty$, $x, v, \xi, v \in \mathbb{R}^3$

(iii) $f_0 \in L^1 \cap L^\infty(\mathbb{R}^6)$, $f_0 \in L^1_v(L^\infty_x)$.

Then, f behaves for large times like the solution of the problem $\mathscr{P}_0$ taking the initial data f_0, in the sense that

$$t^{6(1-1/p)} \|G(f_0)(t) - f(t)\|_{L^p_{xv}} \to 0 \quad \textit{as } t \to \infty,\ 1 \leqslant p \leqslant \infty.$$

Moreover, we have the decay rate

$$\|G(f_0)(t) - f(t)\|_{L^p_{xv}} \leqslant C t^{-6(1-1/p)-1/2}, \quad t > 0,\ 1 \leqslant p \leqslant \infty.$$

Corollary 2. *Theorem* 4 *applies to the solutions for* (VPFP) *constructed in*

(1) *Reference* 16 *for small initial data in the class*

$$f_0 \in L^1 \cap L^\infty, \quad 0 \leqslant f_0 \leqslant A h(x) g(v),\ h, g \in L^1 \cap L^\infty.$$

(2) *Reference* 2 *for initial data in the class*

$$f_0 \geqslant 0, \quad f_0 \in (L^1 \cap L^\infty)(\mathbb{R}^6), \quad f_0 \in L^1_v(L^\infty_x)\ |v|^m f_0 \in L^1(\mathbb{R}^6) \textit{ for some } m > 6$$

provided that the data are small enough. By Reference 6 *we can take $m > \frac{15}{4}$. By Reference* 3 *we can take $m = 2$ if $\rho(f_0) \in L^\infty_x$.*

Proof. We know by Theorem 1 that if $E \in L^\infty_{xv}$ and

$$\|E(t)\|_{L^\infty_x} \leqslant \frac{C_\alpha}{(1+t)^{\alpha+1/2}}$$

for $\alpha \geqslant 0$, with C_α small enough if $\alpha = 0$, then (ii) holds. Therefore, it suffices to apply Theorems 1 and 4 with $E = E(f)$. For the solutions constructed by Triolo, $\alpha = \frac{3}{2} > 0$. For the solutions constructed by Soler and Carrillo, $\alpha = 0$.

Remark 8. In view of the results in section 2.2.2, it folows from this theorem that

$$\|MG(t) - f(t)\|_{L^p_{xv}} \leqslant C t^{-6(1-1/p)-1/2}, \quad t > 0,\ 1 \leqslant p \leqslant \infty,$$

where M is the mass of the initial data, provided that $f_0(|x| + |v|^2) \in L^1_{xv}$. For f_0 only satisfying (ii), we can only say that

$$t^{6(1-1/p)} \|MG(t) - f(t)\|_{L^p_{xv}} \to 0 \quad \text{as } t \to \infty,\ 1 \leqslant p \leqslant \infty$$

holds.

Proof of Theorem 4. From (i) and (ii) f satisfies (I2) with

$$0 \leqslant \Gamma_E(x, v, t; \xi, v, \tau) \leqslant C(\|E\|_{L^\infty_{xt}})\, G\left(\frac{x}{2}, \frac{v}{2}, t; \frac{\xi}{2}, \frac{v}{2}, \tau\right)$$

for $0 < t - \tau < \infty$, $x, v, \xi, v \in \mathbb{R}^3$.

In those conditions, if $f_0 \in L^1 \cap L^\infty$ it follows that, for $p \geqslant r$

$$\|f(t)\|_{L^p_{xv}} \leqslant Ct^{-6(1/r-1/p)}\|f_0\|_{L^r_{xv}},$$
$$\|\rho(t)\|_{L^p_x} \leqslant Ct^{-9/2(1/r-1/p)}\|f_0\|_{L^1_v(L^r_x)},$$
$$\|E(t)\|_{L^\infty_x} \leqslant Ct^{-3/r}\|f_0\|^{1/3}_{L^1_{xv}}\|f_0\|^{2/3}_{L^1_v(L^r_x)},$$
$$\|\nabla_v f(t)\|_{L^p_{xv}} \leqslant Ct^{-6(1/r-1/p)-1/2}\|f_0\|_{L^r_{xv}},$$

thanks to the results for the linear equations with potential E obtained in section 2. Using the integral equation

$$f(x, v, t) - G(f_0)(t) = \int_0^{t/2}\iint \partial_v G(x, v, t; \xi, v, s)E(f)(\xi, s)f(\xi, v, s)\,d\xi\,dv\,ds - \int_{t/2}^{t}\iint G(x, v, t; \xi, v, s)E(f)(\xi, s)\partial_v f(\xi, v, s)\,d\xi\,dv\,ds,$$

so that

$$\begin{aligned}&\|f(x, v, t) - G(f_0)(t)\|_{L^p_{xv}}\\ &\leqslant C\left(\int_0^{t/2}\left\|\iint (t-s)^{-1/2}G\left(\frac{x}{2}, \frac{v}{2}, t; \frac{\xi}{2}, \frac{v}{2}, s\right)(1+s)^{-3}f(\xi, v, s)\,d\xi\,dv\right\|L^p_{xv}\,ds\right.\\ &\quad\left.+\int_{t/2}^{t}\left\|\iint G\left(\frac{x}{2}, \frac{v}{2}, t; \frac{\xi}{2}, \frac{v}{2}, s\right)(1+s)^{-3}\partial_v f(\xi, v, s)\,d\xi\,dv\right\|L^p_{xv}\,ds\right)\\ &\leqslant C\left(\int_0^{t/2}(t-s)^{-1/2-6(1/r-1/p)}(1+s)^{-3}(1+s)^{-6(1-1/r)}\,ds\right.\\ &\quad\left.+\int_{t/2}^{t}(t-s)^{-6(1/r-1/p)}(1+s)^{-3}s^{-1/2-6(1-1/r)}\right).\end{aligned}$$

Taking $r = 1$ in $(0, t/2)$ and $r = p$ in $(t/2, t)$ we get

$$\begin{aligned}&\|f(x, v, t) - G(f_0)(t)\|_{L^p_{xv}}\\ &\quad\leqslant C\left(t^{-1/2-6(1-1/p)}\int (1+s)^{-3}\,ds + (1+t)^{-3}t^{-1/2-6(1-1/p)}t\right),\end{aligned}$$

that is,

$$\|f(x, v, t) - G(f_0)(t)\|_{L^p_{xv}} \leqslant Ct^{-(1/2)-6(1-1/p)}.$$

Remark 10. The decay rate is the same as that obtained in Theorem 3 when $n = m = 0$.

3.2. *Long-time behaviour of the derivatives*

We have the following theorem:

Theorem 5. *Let us assume that*

(i) *f is the unique solution to* (VPFP) *taking f_0 as initial data satisfying* (I1);

(ii) *the fundamental solution Γ_E corresponding to $E = E(f)$ exists and satisfies*

$$0 \leqslant \Gamma_E(x, v, t; \xi, \nu, \tau) \leqslant C(\|E\|_{L^\infty_{xt}}) G\left(\frac{x}{2}, \frac{v}{2}, t; \frac{\xi}{2}, \frac{\nu}{2}, \tau\right),$$

$$|\nabla_v \Gamma_E(x, v, t; \xi, \nu, \tau) \leqslant C(\|E\|_{L^\infty_{xt}}) G\left(\frac{x}{2}, \frac{v}{2}, t; \frac{\xi}{2}, \frac{\nu}{2}, \tau\right)$$

for $0 < t - \tau < \infty$, $x, v, \xi, \nu \in \mathbb{R}^3$;

(iii) $f_0 \in L^1 \cap L^\infty(\mathbb{R}^6)$, $f_0 \in L^1_v(L^\infty_x)$.

Then

$$t^{6(1-(1/p))+1/2} \|\nabla_v G(f_0)(t) - \nabla_v f(t)\|_{L^p_{xv}} \to 0 \quad \text{as } t \to \infty,\ 1 \leqslant p \leqslant \infty.$$

Moreover, we have the decay rate

$$\|\nabla_v G(f_0)(t) - \nabla_v f(t)\|_{L^p_{xv}} \leqslant C t^{-6(1-(1/p))-1}, \quad t > 0,\ 1 \leqslant p \leqslant \infty.$$

Corollary 3. *Theorem* 5 *applies to the solutions for* (VPFP) *constructed in*

(1) *Reference* 16 *for small initial data in the class*

$$f_0 \in L^1 \cap L^\infty, \quad 0 \leqslant f_0 \leqslant A h(x) g(v),\ h, g \in L^1 \in L^\infty;$$

(2) *Reference* 2 *for initial data in the class*

$$f_0 \geqslant 0, \quad f_0 \in (L^1 \cap L^\infty)(\mathbb{R}^6), \quad f_0 \in L^1_v(L^\infty_x),\ |v|^m f_0 \in L^1(\mathbb{R}^6) \text{ for some } m > 6$$

provided that the data are small enough. By Reference 6 *we can take $m > \frac{15}{4}$. By Reference* 3 *we can take $m = 2$ if $\rho(f_0) \in L^\infty_x$.*

Proof. The same as in Corollary 2 but using Theorem 2.

Remark 11. In view of the results in section 2.2.2, it follows from this theorem that

$$\|M\nabla_v G(t) - \nabla_v f(t)\|_{L^p_{xv}} \leqslant C t^{-6(1-1/p)-1}, \quad t > 0,\ 1 \leqslant p \leqslant \infty,$$

where M is the mass of the initial data, provided that $f_0(|x| + |v|^2) \in L^1_{xv}$. For f_0 only satisfying (ii), we can only say that

$$t^{6(1-1/p)+1} \|M\nabla_v G(t) - \nabla_v f(t)\|_{L^p_{xv}} \to 0 \quad \text{as } t \to \infty,\ 1 \leqslant p \leqslant \infty$$

holds.

Proof. From (i) and (ii) f satisfies (I2) with

$$0 \leqslant \Gamma_E(x, v, t; \xi, v, \tau) \leqslant C(\|E\|_{L^\infty_{xt}})\, G\left(\frac{x}{2}, \frac{v}{2}, t; \frac{\xi}{2}, \frac{v}{2}, \tau\right),$$

$$|\nabla_v \Gamma_E(x, v, t; \xi, v, \tau)| \leqslant C(\|E\|_{L^\infty_{xt}})\, G\left(\frac{x}{2}, \frac{v}{2}, t; \frac{\xi}{2}, \frac{v}{2}, \tau\right).$$

Under those conditions, if $f_0 \in L^1 \cap L^\infty$ it follows that

$$\|\nabla_v f(t)\|_{L^p_{xv}} \leqslant C t^{-6(1/r-1/p)-1/2} \|f_0\|_{L^r_{xv}}, \quad p \geqslant r,$$

thanks to the results for the linear equations with potential E obtained in section 2.

Using the integral equation

$$|\nabla_v f(x, v, t) - \nabla_v G(f_0)(t)| \leqslant C \left| \int_{t/2}^{t} \iint (t-s)^{-1/2} \nabla_v G\left(\frac{x}{2}, \frac{v}{2}, t; \frac{\xi}{2}, \frac{v}{2}, s\right) \right.$$
$$\left. \times \frac{|\nabla_v f(\xi, v, s)|}{(1+s)^3} \mathrm{d}s \int_0^{t/2} \iint (t-s)^{-1/2} \nabla_v \nabla_v G\left(\frac{x}{2}, \frac{v}{2}, t; \frac{\xi}{2}, \frac{v}{2}, s\right) \frac{|f(\xi, v, s)|}{(1+s)^3} \mathrm{d}s \right|,$$

so that

$$\|\nabla_v f(x, v, t) - \nabla_v G(f_0)(t)\|_{L^p_{xv}}$$
$$\leqslant C \int_0^{t/2} (t-s)^{-1-6(1/r-1/p)} (1+s)^{-3} (1+s)^{-6(1-1/r)} \mathrm{d}s$$
$$+ C \int_{t/2}^{t} (t-s)^{-(1/2)+6((1/r)-1/p)} (1+s)^{-3} s^{-6(1-(1/r))-1/2} \|f_0\|_{L^1_{xv}} \mathrm{d}s.$$

Choosing $r = p$ in the interval $(t/2, t)$ and $r = 1$ in $(0, t/2)$ we obtain

$$\|\nabla_v f(x, v, t) - \nabla_v G(f_0)(t)\|_{L^p_{xv}} \leqslant C t^{-1-6(1-1/p)}.$$

3.3. *Second term*

We know that the first term in the asymptotic development of a solution f of (VPFP) with small data $f_0 \in L^1 \cap L^\infty(\mathbb{R}^6)$ is $MG(x, v, t; 0, 0, 0)$, $M = \iint f_0(x, v)\, \mathrm{d}x\, \mathrm{d}v$.

In order to find a more precise development we are going to study each of the terms appearing in the integral equation

$$\text{(I1)}\ f(x, v, t) = \int G(x, v, t; \xi, v, 0) f_0(\xi, v)\, \mathrm{d}\xi\, \mathrm{d}v$$
$$+ \int_0^t \mathrm{d}s \iint \nabla_v G(x, v, t; \xi, v, s) E(f)(\xi, s) f(\xi, v, s)\, \mathrm{d}\xi\, \mathrm{d}v.$$

As far as the first term is concerned, we know (section 2.2.2) that

$$\int G(x, v, t; \xi, v, 0) f_0(\xi, v)\, \mathrm{d}x\, \mathrm{d}v = G(f_0)$$
$$= MG(x, v, t; 0, 0, 0) + m_i \partial_{v_i} G(x, v, t; 0, 0, 0) + R(x, v, t),$$

where $m_i = \iint v_i f_0(x, v)\,dx\,dv$ and the remaining term R satisfies

$$\|R\|_{L^p_{xv}} \leqslant Ct^{-1-6(1-1/p)}$$

when $v^2 f_0 \in L^1_{xv}$, $xf_0 \in L^1_{xv}$.

To study the asymptotic behaviour of the second integral, denoted $w(x, v, t)$, that is,

$$w(x, v, t) = \int_0^t ds \iint \nabla_v G(x, v, t; \xi, v, s) E(f)(\xi, s) f(\xi, v, s)\,d\xi\,dv,$$

we are going to use a scaling technique. By rescaling, we see that the functions $w_\lambda(x, v, t) = \lambda^{12} w(\lambda^3 x, \lambda v, \lambda^2 t)$ satisfy

$$w_\lambda(x, v, t) = \lambda^{-5} \int_0^t \iint \partial_\xi G(x, v, t; \xi, v, s) E_\lambda(\xi, s) f_\lambda(\xi, v, s)\,d\xi\,dv\,ds$$

with $E_\lambda(\xi, s) = \lambda^6 E(\lambda^3 \xi, \lambda^2 s)$, $f_\lambda(\xi, v, s) = \lambda^{12} f(\lambda^3 \xi, \lambda v, \lambda^2 s)$, that is,

$$w_\lambda(x, v, 1) = \lambda^{-1} \int_0^{\lambda^2} \iint \partial_\xi G\left(x, v, 1; \frac{\xi}{\lambda^3}, \frac{v}{\lambda}, \frac{s}{\lambda^2}\right) E(\xi, s) f(\xi, v, s)\,d\xi\,dv\,ds.$$

We remark that

$$\|w_\lambda(1)\|_{L^p_{xv}} = \lambda^{12(1-1/p)} \|w(\lambda^2)\|_{L^p_{xv}}.$$

Thus, if we want to make precise the asymptotic behaviour of w when $t \to \infty$ it suffices to find a function g such that

$$\|w_\lambda(1) - g_\lambda(1)\|_{L^p_{xv}} \delta(\lambda) \to 0 \quad \text{as } \lambda \to \infty$$

for some $\delta(\lambda)$ tending to ∞ as $\lambda \to \infty$, where $g_\lambda(x. v, t) = \lambda^{12} g(\lambda^3 x, \lambda v, \lambda^2 t)$. That implies

$$\|w(t) - g(t)\|_{L^p_{xv}} t^{1-1/p} \delta(t^{1/2}) \to 0 \quad \textit{as } t \to \infty.$$

It is easy to prove that $\|w_\lambda\|_{L^p_{xv}}$ is bounded by $C\lambda^{-1}$. Therefore, the same kind of bound must hold for g_λ but the difference $\|g_\lambda(t) - w_\lambda(t)\|_q$ must go to zero faster. Under certain conditions it is possible to take $g(x, v, t) = K_i \partial_{v_i} G(x, v, t; 0, 0, 0)$ with K_i to be precised below and $\delta(t) = t^{1/2}$. More precisely, we prove the following:

Proposition 1. *Let us assume that*

(i) f *is the unique solution to* (VPFP) *taking* f_0 *as initial data satisfying* (I1):
(ii) *the fundamental solution* Γ_E *corresponding to* $E = E(f)$ *exists and satisfies*

$$0 \leqslant \Gamma_E(x, v, t; \xi, v, \tau) \leqslant C(\|E\|_{L^\infty_{xt}}) G\left(\frac{x}{2}, \frac{v}{2}, t; \frac{\xi}{2}, \frac{v}{2}, \tau\right),$$

$$|\nabla_v \Gamma_E(x, v, t; \xi, v, \tau)| \leqslant C(\|E\|_{L^\infty_{xt}}) G\left(\frac{x}{2}, \frac{v}{2}, t; \frac{\xi}{2}, \frac{v}{2}, \tau\right)$$

for $0 < t - \tau < \infty$, $x, v, \xi, v \in \mathbb{R}^3$;
(iii) $f_0 \in L^1 \cap L^\infty(\mathbb{R}^6)$, $f_0 \in L^1_v(L^\infty_x)$.

Then, the integral w satisfies

$$t^{1+6(1-(1/p))}\left\| w(t)-\left(\int_0^\infty\int_{\mathbb{R}^6}E(f)f\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}s\right)\nabla_\xi G(x,v,t;0,0,0)\right\|_{L^p_{xv}}\to 0$$

as $t\to\infty$.

Proof. In view of the identity

$$w_\lambda(x,v,1)=\lambda^{-1}\int_0^{\lambda^2}\iint \partial_\xi G\left(x,v,1;\frac{\xi}{\lambda^3},\frac{v}{\lambda},\frac{s}{\lambda^2}\right)E(\xi,s)f(\xi,v,s)\,\mathrm{d}\xi\,\mathrm{d}v\,\mathrm{d}s,$$

we expect

$$\lambda(w_\lambda(1)-g_\lambda(1))=\int_0^{\lambda^2}\iint\partial_\xi G\left(x,v,1;\frac{\xi}{\lambda^3},\frac{v}{\lambda},\frac{s}{\lambda^2}\right)E(\xi,s)f(\xi,v,s)\,\mathrm{d}\xi\,\mathrm{d}v\,\mathrm{d}s$$
$$-\left(\int_0^\infty\iint E(\xi,s)f(\xi,v,s)\,\mathrm{d}\xi\,\mathrm{d}v\right)\mathrm{d}s\,\partial_\xi G(x,v,1;0,0,0)$$

to tend to 0 as $\lambda\to\infty$. Taking into account that

$$\left(\int_0^{\lambda^2}\iint E(\xi,s)f(\xi,v,s)\,\mathrm{d}\xi\,\mathrm{d}v\,\mathrm{d}s\right.$$
$$\left.-\int_0^\infty\iint E(\xi,s)f(\xi,v,s)\,\mathrm{d}\xi\,\mathrm{d}v\,\mathrm{d}s\right)\partial_\xi G(x,v,1;0,0,0)$$

tends to zero in any L^p_{xv} as $\lambda\to\infty$, we must only prove that

$$I_\lambda=\int_0^{\lambda^2}\iint\left(\partial_\xi G\left(x,v,1;\frac{\xi}{\lambda^3},\frac{v}{\lambda},\frac{s}{\lambda^2}\right)\right.$$
$$\left.-\partial_\xi G(x,v,1;0,0,0)\right)E(\xi,s)f(\xi,v,s)\,\mathrm{d}\xi\,\mathrm{d}v\,\mathrm{d}s$$

goes to zero in L^p_{xv} as $\lambda\to\infty$. We split the integral as follows:

$$I_\lambda=I^1_{\delta,\lambda}+I^2_{\delta,\lambda}+J^2_{\delta,\lambda}+I^3_{\delta,\lambda}+J^3_{\delta,\lambda}+I^4_{\delta,\lambda}+I^5_{\delta,\lambda},$$

where

$$I^1_{\delta,\lambda}=\int_0^{\lambda^2\delta}\int_{\substack{|\xi|\leqslant\lambda^3\delta\\|v|\leqslant\lambda\delta}}\left(\partial_\xi G\left(x,v,1;\frac{\xi}{\lambda^3},\frac{v}{\lambda},\frac{s}{\lambda^2}\right)\right.$$
$$\left.-\partial_\xi G(x,v,1;0,0,0)\right)E(\xi,s)f(\xi,v,s)\,\mathrm{d}\xi\,\mathrm{d}v\,\mathrm{d}s,$$

$$I^2_{\delta,\lambda}=\int_{\lambda^2\delta}^{\lambda^2}\int_{\mathbb{R}^6}\partial_\xi G(x,v,1;0,0,0)E(\xi,s)f(\xi,v,s)\,\mathrm{d}\xi\,\mathrm{d}v\,\mathrm{d}s,$$

$$J^2_{\delta,\lambda}=\int_{\lambda^2\delta}^{\lambda^2}\int_{\mathbb{R}^6}\partial_\xi G\left(x,v,1;\frac{\xi}{\lambda^3},\frac{v}{\lambda},\frac{s}{\lambda^2}\right)E(\xi,s)f(\xi,v,s)\,\mathrm{d}\xi\,\mathrm{d}v\,\mathrm{d}s,$$

$$I^3_{\delta,\lambda} = \int_0^{\lambda^2\delta} \int_{\substack{|\xi| > \lambda^3\delta \\ |v| > \lambda\delta}} \partial_\xi G(x, v, 1; 0, 0, 0) E(\xi, s) f(\xi, v, s)\, d\xi\, dv\, ds,$$

$$J^3_{\delta,\lambda} = \int_0^{\lambda^2\delta} \int_{\substack{|\xi| > \lambda^3\delta \\ |v| > \lambda\delta}} \partial_\xi G\left(x, v, 1; \frac{\xi}{\lambda^3}, \frac{v}{\lambda}, \frac{s}{\lambda^2}\right) E(\xi, s) f(\xi, v, s)\, d\xi\, dv\, ds,$$

$$I^4_{\delta,\lambda} = \int_0^{\lambda^2\delta} \int_{\substack{|\xi| \leqslant \lambda^3\delta \\ |v| > \lambda\delta}} \left(\partial_\xi G\left(x, v, 1; \frac{\xi}{\lambda^3}, \frac{v}{\lambda}, \frac{s}{\lambda^2}\right)\right.$$

$$\left. - \partial_\xi G(x, v, 1; 0, 0, 0)\right) E(\xi, s) f(\xi, v, s)\, d\xi\, dv\, ds,$$

$$I^5_{\delta,\lambda} = \int_0^{\lambda^2\delta} \int_{\substack{|\xi| > \lambda^3\delta \\ |v| \leqslant \lambda\delta}} \left(\partial_\xi G\left(x, v, 1; \frac{\xi}{\lambda^3}, \frac{v}{\lambda}, \frac{s}{\lambda^2}\right)\right.$$

$$\left. - \partial_\xi G(x, v, 1; 0, 0, 0)\right) E(\xi, s) f(\xi, v, s)\, d\xi\, dv\, ds.$$

Given $\varepsilon > 0$, taking δ small enough we get

$$\|I^2_{\delta,\lambda}\|_p \leqslant C \int_0^{\lambda^2\delta} \int_{\substack{|\xi| \leqslant \lambda^3\delta \\ |v| \leqslant \lambda\delta}} \left\| \left(\partial_\xi G\left(x, v, 1; \frac{\xi}{\lambda^3}, \frac{v}{\lambda}, \frac{s}{\lambda^2}\right) - \partial_\xi G(x, v, 1; 0, 0, 0)\right) \right\|_p$$

$$\times |E(\xi, s) f(\xi, v, s)|\, d\xi\, dv\, ds \leqslant C\varepsilon \int_0^\infty \int_{\mathbb{R}^6} |E(\xi, s) f(\xi, v, s)|\, d\xi\, dv\, ds.$$

Now, for δ fixed

$$\|I^2_{\delta,\lambda}\|_p \leqslant \|\partial_\xi G(x, v, 1; 0, 0, 0)\|_p \int_{\lambda^2\delta}^{\lambda^2} \int_{\mathbb{R}^6} |E(\xi, s) f(\xi, v, s)|\, d\xi\, dv\, ds,$$

$$\|J^2_{\delta,\lambda}\|_p \leqslant C \int_{\lambda^2\delta}^{\lambda^2} \left(1 - \frac{s}{\lambda^2}\right)^{-1/2} \int_{\mathbb{R}^6} G\left(\frac{x}{2}, \frac{v}{2}, 1; \frac{\xi}{2\lambda^3}, \frac{v}{2\lambda}, \frac{s}{\lambda^2}\right) E(\xi, s) f(\xi, v, s)\, d\xi\, dv\, ds$$

$$\leqslant C \int_\delta^1 (1 - s)^{-1/2} \int_{\mathbb{R}^6} G\left(\frac{x}{2}, \frac{v}{2}; 1; \xi, v, s\right) |E(2\lambda^3\xi, \lambda^2 s) f(2\lambda^3\xi, 2\lambda v, \lambda^2 s)|\, \lambda^{14}\, d\xi\, dv\, ds$$

$$\leqslant C \int_\delta^1 (1 - s)^{-1/2} \frac{\lambda^2}{(1 + \lambda^2 s)^3} \frac{\lambda^{12(1-1/p)}}{(1 + \lambda^2\delta)^{12(1-1/p)}} \leqslant \frac{C}{(1 + \lambda^2\delta)^2},$$

$$\|I^3_{\delta,\lambda}\|_p \leqslant C \|\partial_\xi G(x, v, 1; 0, 0, 0)\|_p \int_0^{\lambda^2\delta} \int_{\substack{|\xi| > \lambda^3\delta \\ |v| > \lambda\delta}} |E(\xi, s) f(\xi, v, s)|\, d\xi\, dv\, ds,$$

$$\|J^3_{\delta,\lambda}\|_p \leqslant C \int_0^{\lambda^2\delta} \left(1 - \frac{s}{\lambda^2}\right)^{-1/2} \int_{\mathbb{R}^6} G\left(\frac{x}{2}, \frac{v}{2}, 1; \frac{\xi}{2\lambda^3}, \frac{v}{2\lambda}, \frac{s}{\lambda^2}\right) E(\xi, s) f(\xi, v, s)\, d\xi\, dv\, ds$$

$$\leqslant C \int_0^{\lambda^2\delta} \left(1 - \frac{s}{\lambda^2}\right)^{-1/2} \int_{\mathbb{R}^6} G\left(\frac{x}{2}, \frac{v}{2}, 1; \xi, v, \frac{s}{\lambda^2}\right) |E(2\lambda^3\xi, s) f(2\lambda^3\xi, 2\lambda v, s)|\, \lambda^{12}\, d\xi\, dv\, ds$$

$$\leqslant C\int_0^{\lambda^2\delta}\left(1-\frac{s}{\lambda^2}\right)^{-(1/2)-6(1-1/p)}\int_{\substack{|\xi|>\lambda^3\delta/2\\|v|>\lambda\delta/2}}|E(\xi,s)f(\xi,v,s)|\,\mathrm{d}\xi\,\mathrm{d}v\,\mathrm{d}s,$$

$$\leqslant C\int_0^{\lambda^2\delta}\int_{\substack{|\xi|>\lambda^3\delta/2\\|v|>\lambda\delta/2}}|E(\xi,s)f(\xi,v,s)|\,\mathrm{d}\xi\,\mathrm{d}v\,\mathrm{d}s,$$

$$\|I^4_{\delta,\lambda}\|_p\leqslant C\int_0^{\lambda^2\delta}\int_{\substack{|\xi|\leqslant\lambda^3\delta\\|v|>\lambda\delta}}\left(\partial_\xi G\left(x,v,1;\frac{\xi}{\lambda^3},\frac{v}{\lambda},\frac{s}{\lambda^2}\right)-\partial_\xi G(x,v,1;0,0,0)\right)$$

$$\times|E(\xi,s)f(\xi,v,s)|\,\mathrm{d}\xi\,\mathrm{d}v\,\mathrm{d}s$$

$$\leqslant C(\|\partial_\xi G(x,v,1;0,0,0)\|_p\int_0^{\lambda^2\delta}\int_{\substack{|\xi|\leqslant\lambda^3\delta\\|v|>\lambda\delta}}|E(\xi,s)f(\xi,v,s)|\,\mathrm{d}\xi\,\mathrm{d}v\,\mathrm{d}s$$

$$+\int_0^{\lambda^2\delta}\left(1-\frac{s}{\lambda^2}\right)^{-(1/2)-6(1-1/p)}\int_{\substack{|\xi|\leqslant\lambda^3\delta/2\\|v|>\lambda\delta/2}}|E(\xi,s)f(\xi,v,s)|\,\mathrm{d}\xi\,\mathrm{d}v\,\mathrm{d}s$$

$$\leqslant C\int_0^{\lambda^2\delta}\int_{\substack{|\xi|\leqslant\lambda^3\delta/2\\|v|>\lambda\delta/2}}|E(\xi,s)f(\xi,v,s)|\,\mathrm{d}\xi\,\mathrm{d}v\,\mathrm{d}s,$$

$$\|I^5_{\delta,\lambda}\|_p\leqslant C\int_0^{\lambda^2\delta}\int_{\substack{|\xi|>\lambda^3\delta/2\\|v|>\lambda\delta/2}}|E(\xi,s)f(\xi,v,s)|\,\mathrm{d}\xi\,\mathrm{d}v\,\mathrm{d}s.$$

All these integrals tend to 0 as $\lambda\to\infty$, since $\int_0^\infty\int_{\mathbb{R}^6}|E(\xi,s)f(\xi,v,s)|\,\mathrm{d}\xi\,\mathrm{d}v\,\mathrm{d}s$ is finite. Going back to the original variables we get

$$t^{1+6(1-1/p)}\left\|w(t)-\left(\int_0^\infty\int_{\mathbb{R}^6}E(f)f\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}s\right)\nabla_\xi G(x,v,t;0,0,0)\right\|_{L^p_{xv}}\to 0$$

$$\text{as } t\to\infty.$$

In conclusion, we have proved the following result:

Theorem 6. *Let us assume that*

(i) f *is the unique solution to* (VPFP) *taking* f_0 *as initial data satisfying* (I1);

(ii) *the fundamental solution* Γ_E *corresponding to* $E=E(f)$ *exists and satisfies*

$$0\leqslant\Gamma_E(x,v,t;\xi,v,\tau)\leqslant C(\|E\|_{L^\infty_{xt}})G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi}{2},\frac{v}{2},\tau\right),$$

$$|\nabla_v\Gamma_E(x,v,t;\xi,v,\tau)\leqslant C(\|E\|_{L^\infty_{xt}})G\left(\frac{x}{2},\frac{v}{2},t;\frac{\xi}{2},\frac{v}{2},\tau\right)$$

for $0<t-\tau<\infty$, x, v, ξ, $v\in\mathbb{R}^3$;

(iii) $f_0\in L^1\cap L^\infty(\mathbb{R}^6)$, $f_0\in L^1_v(L^\infty_x)$.

Then, the solution to (VPFP) *taking* f_0 *as initial data satisfies*

$$t^{1+6(1-1/p)}\left\| f(t) - MG(x,v,t) - \left(m_v + \int_0^\infty \int_{\mathbb{R}^6} E(f) f \,\mathrm{d}\xi \,\mathrm{d}v \,\mathrm{d}s\right)\nabla_\xi G(x,v,t)\right\|_{L^p_{xv}} \to 0$$

as $t \to \infty$

where

$$M = \int f_0(\xi, v)\,\mathrm{d}v\,\mathrm{d}\xi, \qquad m_v = \int v f_0(\xi, v)\,\mathrm{d}v\,\mathrm{d}\xi$$

and $G(x,v,t) = G(x,v,t;0,0,0)$.

Corollary 4. *Theorem* 5 *applies to the solutions for* (VPFP) *constructed in*

(1) *Reference* 16 *for small initial data in the class*

$$f_0 \in L^1 \cap L^\infty, \quad 0 \leqslant f_0 \leqslant Ah(x)g(v), \; h, g \in L^1 \in L^\infty;$$

(2) *Reference* 2 *for initial data in the class*

$$f_0 \geqslant 0, \quad f_0 \in (L^1 \cap L^\infty)(\mathbb{R}^6), \quad f_0 \in L^1_v(L^\infty_x), \; |v|^m f_0 \in L^1(\mathbb{R}^6) \text{ for some } m > 6$$

provided that the data are small enough. By Reference 6 *we can take* $m > \frac{15}{4}$. *By Reference* 3 *we can take* $m = 2$ *if* $\rho(f_0) \in L^\infty_x$.

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