Mathematical Models and Methods in Applied Sciences

LONG TIME ASYMPTOTICS FOR THE SEMICONDUCTOR VLASOV-POISSON-BOLTZMANN EQUATIONS

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In this paper we analyze the long time behavior of solutions to the one dimensional Vlasov-Poisson-Boltzmann (VPB) equations for semiconductors in unbounded domains when only one type of carriers (electrons) are considered. We prove that the distribution of electrons tends for large times to a steady state of the VPB equations with vanishing collision term and the same total charge as the initial data. In the proof of the main result, the conservation law of charge, the balance of energy and entropy inequalities are rigorously derived. An important argument in the proof is to use a Lyapunov-type functional related to these physical quantities.

1. Introduction

The Vlasov-Poisson-Boltzmann equations provide a kinetic description for transport phenomena in semiconductors. In the semiclassical kinetic model [20], the carriers are described by distribution functions $f(x,k,t)$ which express the probability of finding a carrier at time t in a position x with a wave vector k. Typically, two distribution functions are needed, one for electrons and another one for holes, resulting in a system of coupled transport equations with integral terms describing collisions with the lattice and recombination-generation of electrons and holes. We refer the reader to [20] for more details on the modelling.

In this paper we consider a reduced onedimensional model where only the electrons are taken into account. Setting all constants equal to unity, the equation for the electron distribution function reads:

$$\begin{cases} \partial_t f(x,k,t) + v(k)\partial_x f(x,k,t) - E(x,t)\partial_k f(x,k,t) = Q(f)(x,k,t) \\ f(x,k,0) = f^0(x,k) \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}$, $k \in \mathbb{R}$ and $t \in \mathbb{R}^+$. The electric field $E(x,t)$ is coupled to the distribution function $f(x,k,t)$ by the formula

$$E(x,t) = \int_{\mathbb{R}} H(x-y)\rho(y,t)\,dy \tag{1.2}$$

where

$$\rho(x,t) = n_D(x) - \int_{\mathbb{R}} f(x,k,t)\,dk \tag{1.3}$$

is the charge density and $n_D(x)$ is the given doping profile in the semiconductor. H is the Heaviside function, so that

$$\partial_x E(x,t) = n_D(x) - \int_{\mathbb{R}} f(x,k,t)\,dk \tag{1.4}$$

We consider solutions to (1.1) with zero total global charge. This property is conserved with time and allows to get L^2 bounds for the electric field. It is also essential to control the long time behavior of the solutions.

The speed of the electrons $v(k)$ is obtained from the (energy) band diagram $\varepsilon(k)$:

$$v(k) = \frac{d}{dk}\varepsilon(k). \tag{1.5}$$

The term $Q(f)(x,k,t)$ is a nonlinear integral operator which accounts for the collisions between the carriers and the crystal lattice. It has the structure:

$$\begin{aligned} Q(f)(x,k,t) = \int_{\mathbb{R}} &\{S(x,k',k)f(x,k',t)\,(1-f(x,k,t)) \\ &-S(x,k,k')f(x,k,t)\,(1-f(x,k',t))\}\;dk' \end{aligned} \tag{1.6}$$

where $S(x,k,k')$ is the scattering rate for an electron whose wave vector k changes to k' during interaction at the position x. The scattering rate satisfies:

$$S(x,k',k) = \exp\left[\varepsilon(k') - \varepsilon(k)\right] S(x,k,k') \tag{1.7}$$

for $x \in \mathbb{R}$ and it can be written as:

$$S(x,k,k') = \exp\left[-\varepsilon(k')\right] \Upsilon(x,k,k') \tag{1.8}$$

for some function Υ. Substituting in (1.7) we get $\Upsilon(x,k',k) = \Upsilon(x,k,k')$. Property (1.7) plays a key role when proving the H-theorem (see Theorem A.1). Property (1.8) is also used to investigate the vanishing of the collision term (Corollary 4.5) for limiting steady solutions.

The current density is given by

$$j(x,t) = -\int_{\mathbb{R}} v(k)f(x,k,t)\,dk \tag{1.9}$$

and the total energy

$$W(t) = T(t) + \frac{1}{2}\int_{\mathbb{R}} |E(x,t)|^2\,dx = T(t) + \frac{1}{2}\|E\|^2_{L^2(\mathbb{R})}(t), \tag{1.10}$$

where $T(t)$ is the kinetic energy:

$$T(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} \varepsilon(k) f(x,k,t)\, dx\, dk. \tag{1.11}$$

Some existence results for this reduced VPB system are known. Mustieles [16] proved existence and uniqueness of classical solutions. He obtained global solutions in dimensions 1 and 2 and local solutions in dimension 3. H.Andréasson extended this result in [1] proving the existence of a global classical solution in dimension 3.

Concerning the long time behavior, some results have been obtained by Fuchs in [11] for bounded domains in dimension 3. More work has been done for related kinetic models, in particular for Vlasov-Poisson (VP) equations with different collision terms. In [4], a Lyapunov technique is used to establish that solutions to Vlasov-Poisson-Fokker-Planck (VPFP) equations in bounded domains tend to Maxwellians determined by the boundary conditions. A general framework for the study of convergence towards stationary solutions in bounded domains based on relative entropy methods is established in [2] for several VP models. In [5] the VPFP equations in unbounded domains with an external potential are studied. Choosing appropiate external potentials, convergence to steady states is proved.

The main difficulty when studying the long time asymptotics of the Vlasov-Poisson-Boltzmann equations in unbounded domains is to find a way to bound the electric field and control the distribution function at infinity. This difficulty was overcome in [5] for VPFP thanks to the external potential. Working in dimension one, we are able to prove weak convergence to steady states for VPB without adding any external potential. In dimension 3 such a result cannot be expected for general initial data due to runaway and dispersion effects. In one dimension, the Coulomb potential created by a charge distribution is such that the force does not decay at infinity. The doping profile n_D is then responsible for a potential which turns out to be confining (see [7]).

We study the long time behavior of strong solutions to VPB with initial data having bounded total energy, vanishing total charge and decaying fast enough at infinity. For such solutions we prove rigorously conservation of charge, energy balance and entropy inequalities. We define a Lyapunov functional in terms of the total energy plus an entropy term, inspired in the "free energy" of the system. In proving that this functional is decreasing, it plays a key role the semiconductor version of the H-Theorem (see Theorem A.1). The Lyapunov functional provides uniform bounds on the kinetic energy and the electric field of the time translates $f_n(x,k,t) = f(x,k,t+nT)$. Passing to the limit in the sequence as n tends to ∞ we conclude that $f(x,k,t)$ converges weakly in L^1 to a stationary solution conserving the total charge and for which the collision term vanishes as t tends to infinity. We only obtain weak convergence in L^1, so, an additional difficulty is to establish that the limit function conserves the initial total charge. This is solved by using compactness of averaged quantities, following the ideas of [17,13,14,9].

The paper is organized as follows. In Section 2 we state the main result and present the proof. This proof uses some bounds and convergences whose proofs are detailed in the next sections. In Section 3 we investigate the Lyapunov functional and obtain some uniform bounds in time. In Section 4, we prove the convergence results and obtain some properties of the limit function (vanishing collision term, conservation of initial mass). In an Appendix we include some auxiliary regularity results and recall some known theorems needed in the proofs.

2. Main Result

Since we work in dimension 1, we rely on the procedure to construct classical solutions developed in [16]. For $\gamma > 0$ we define the Banach space (endowed with the natural norm)

$$\chi^\gamma = \Big\{\varphi : \mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_z \to \mathbb{R} : \varphi(x,y,z) \in L^\infty\left(\mathbb{R}_x \times \mathbb{R}_y, L^1(\mathbb{R}_z)\right)$$
$$\textit{and} \quad (1+\|z\|^2)^{\gamma/2}\varphi(x,y,z) \in L^\infty\left(\mathbb{R}_x \times \mathbb{R}_z, L^1(\mathbb{R}_y)\right)\Big\}.$$

To study the long time behavior we use the space χ^γ with $\gamma > 4$. We collect below the assumptions on the scattering rate, the velocity field, the doping profile and the initial data to be used in the paper.

H. 1 Assumptions on the scattering rate

$$S(x,k,k') \in L^\infty\left(\mathbb{R} \times \mathbb{R} \times \mathbb{R}\right) \tag{2.12}$$
$$S(x,k,k') > 0 \quad \text{and} \quad S(x,k,k') \in \chi^\gamma, \tag{2.13}$$
$$\varepsilon(k')S(x,k,k') \in L^\infty\left(\mathbb{R}_x \times \mathbb{R}_k, L^1\left(\mathbb{R}_{k'}\right)\right) \tag{2.14}$$
$$\|\nabla_{x,k,k'} S(x,k,k')\| \in \chi^\gamma, \tag{2.15}$$

H. 2 Assumptions on the velocity field

$$\exists C > 0, \delta > 1 \,\text{such that}\, \varepsilon(k) \geq C|k|^\delta \,\text{if}\, |k| \to \infty, \quad \varepsilon(k) \geq 0, \quad \varepsilon(k) = \varepsilon(-k) \tag{2.16}$$
$$\frac{d}{dk} v(k) \in L^\infty\left(\mathbb{R}\right) \tag{2.17}$$

$$\forall R < \infty \quad \exists C > 0, \beta > 0 \text{ such that if } \; e = (e', e'') \in \mathbb{R}^2 \text{ and}$$

$$H_e = \left\{k \in B_R : |e' + v(k)e''| \leq \alpha\right\}, \quad \textit{sup}\left\{|H_e| : \|e\| = 1\right\} \leq C\alpha^\beta \tag{2.18}$$

H. 3 Assumptions on the doping profile

$$n_D(x) \in W^{1,\infty}\left(\mathbb{R}\right) \tag{2.19}$$
$$\left(1+|x|^2\right)^{b/2} n_D(x) \in L^1\left(\mathbb{R}\right) \quad \text{with} \quad 0 < b < 1/2. \tag{2.20}$$

H. 4 Assumptions on the initial data

$$0 < f^0 < 1 \tag{2.21}$$
$$f^0 \in W^{1,1}\left(\mathbb{R} \times \mathbb{R}\right) \tag{2.22}$$
$$\left(1+|k|^2\right)^{\gamma/2}\left(f^0 + \left\|\nabla_{x,k} f^0\right\|\right) \in L^\infty\left(\mathbb{R} \times \mathbb{R}\right) \tag{2.23}$$

$$\left(1+|x|^2\right) f^0 \in L^\infty\left(\mathbb{R}\times\mathbb{R}\right) \tag{2.24}$$

$$\|f^0\|_{L^1(\mathbb{R}\times\mathbb{R})} = \int_{\mathbb{R}} n_D(x)dx \tag{2.25}$$

$$W(0) = T(0) + \frac{1}{2}\|E\|^2_{L^2(\mathbb{R})}(0) < \infty \tag{2.26}$$

We state a onedimensional version of the existence and uniqueness result in [16] taking into account the presence of the doping profile n_D:

Theorem 2.1 *Let us assume that the scattering rate S, the velocity field v and the initial data $f^0(x,k)$ satisfy (2.13),(2.15),(2.17), (2.21),(2.22) and (2.23) for $\gamma > 1$. Let us assume that the doping profile satisfies $n_D \in L^1(\mathbb{R}) \cup W^{1,1}(\mathbb{R})$. Then, there exist a unique classical (and global) solution f for the VPB equation with $(x,k,t) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^+$*

$$\left\{ \begin{array}{l} \partial_t f(x,k,t) + v(k)\partial_x f(x,k,t) - E(x,t)\partial_k f(x,k,t) = Q(f)(x,k,t) \\ f(x,k,0) = f^0(x,k) \end{array} \right. \tag{2.27}$$

where the collision term $Q(f)$ is defined as in (1.6) and the electric field is coupled to the distribution function f as in (1.4) with the charge density defined by

$$\rho(x,t) = n_D(x) - \int_{\mathbb{R}} f(x,k,t)dk. \tag{2.28}$$

Moreover, the solution satisfies:

$$0 \le f \le 1 \tag{2.29}$$

$$f \in L^\infty_{loc}\left([0,\infty), W^{1,1}\left(\mathbb{R}\times\mathbb{R}\right)\right) \tag{2.30}$$

$$\left(1+\|k\|^2\right)^{\gamma/2}\left(f + \|\nabla_{x,k} f\|\right) \in L^\infty_{loc}\left([0,\infty), L^\infty\left(\mathbb{R}\times\mathbb{R}\right)\right) \tag{2.31}$$

$$E \in L^\infty_{loc}\left([0,\infty), W^{1,\infty}\left(\mathbb{R}\right)\right). \tag{2.32}$$

Note that our charge density (2.28) is not defined exactly as in [16] where the doping profile was absent. Nevertheless, it is enough to add the hypothesis on the doping, $n_D \in L^1(\mathbb{R}) \cup W^{1,1}(\mathbb{R})$ (see [6]) to prove Theorem 2.1.

We introduce the following auxiliary function $g(x)$

$$g(x) = \left\{ \begin{array}{ll} 1 & 0 \le |x| < 1 \\ |x|^{b^*} & 1 \le |x| \end{array} \right. \tag{2.33}$$

with $0 < b^* < b < 1/2$. This g verifies

$$0 < g(x) < \left(1+|x|^2\right)^{b/2} \tag{2.34}$$

$$\left(1+g(x)\right) exp\left[-g(x)\right] \in L^1\left(\mathbb{R}\right) \tag{2.35}$$

$$\frac{d}{dx}g(x) \in L^2\left(\mathbb{R}\right). \tag{2.36}$$

This function serves as a weight to control the behavior of solutions for large x. Such functions cannot be found in dimension greater than one. This adds a technical obstacle to the considerations made in the Introduction about the possibility of extending our results to higher dimensions.

Under the above assumptions one can prove that solutions to VPB tend to steady solutions for large times:

Theorem 2.2 - Long time behavior - *Let $f(x,k,t)$ be the solution of the VPB equations (1.1)-(1.6). We assume that the scattering rate S, the velocity field v and the doping profile n_D satisfy (2.12)-(2.20) and that the initial data satisfies (2.21)-(2.26) with $\gamma > 4$. The scattering rate satisfies also (1.7)-(1.8). Then, as $n \to \infty$*

$$f_n(x,k,t) \rightharpoonup \boldsymbol{f}(x,k) \qquad in \qquad L^1 \cap L^{\infty,*}(\mathbb{R}\times\mathbb{R}\times(0,T))$$

where $f_n(x,k,t) = f(x,k,t+nT)$ with $T > 0$.

The limit function $\boldsymbol{f}$ is the unique stationary solution of (1.1) satisfying

$$v(k)\partial_x \boldsymbol{f}(x,k) - \mathbb{E}(x)\partial_k \boldsymbol{f}(x,k) = Q(\boldsymbol{f})(x,k) = 0$$

with $\mathbb{E} \in L^2(\mathbb{R})$ such that $\frac{d}{dx}\mathbb{E}(x) = n_D(x) - \int_{\mathbb{R}} \boldsymbol{f}(x,k)\,dk$ and $\|\boldsymbol{f}\|_{L^1(\mathbb{R}\times\mathbb{R})} = \|f^0\|_{L^1(\mathbb{R}\times\mathbb{R})}$.

Remarks: The limit profile $\boldsymbol{f}$ is given by:

$$\boldsymbol{f}(x,k) = \frac{1}{1 + exp\left[\varepsilon(k) - \Phi(x)\right]} \qquad a.e.\ (x,k) \in \mathbb{R}\times\mathbb{R}$$

where Φ is a function verifying

$$\mathbb{E}(x) = -\frac{d}{dx}\Phi(x) \in L^2(\mathbb{R}) \qquad -\frac{d^2}{dx^2}\Phi(x) = n_D(x) - \int_{\mathbb{R}} \boldsymbol{f}(x,k)\,dk$$

and such that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{dx\,dk}{1 + exp\left[\varepsilon(k) - \Phi(x)\right]} = \|f^0\|_{L^1(\mathbb{R}\times\mathbb{R})}.$$

The uniqueness of the limit function implies that the convergence holds not only for the sequence of translates but also for the continuous translates $f(\cdot,\cdot,t+\tau)$, $t \in (0,T)$ as $\tau \to \infty$.

The sequence of translates $\{f_n(x,k,t)\}_{n\in\mathbb{N}\cup\{0\}}$ can also be defined as the sequence formed by the solutions of the VPB system with initial data $f_{n-1}(x,k,T)$. The sequence starts with $f_{-1}(x,k,T) = f^0(x,k)$. The conclusions (2.29)-(2.32) of Theorem 2.1, hold for each f_n and its associated E_n. The convergence $f_n(x,k,t) \rightharpoonup \boldsymbol{f}(x,k)$ in $L^{\infty,*}(\mathbb{R}\times\mathbb{R}\times(0,T))$ means $f_n(x,k,t) \overset{*}{\rightharpoonup} \boldsymbol{f}(x,k)$ in $L^{\infty}(\mathbb{R}\times\mathbb{R}\times(0,T))$. Although $\|\boldsymbol{f}\|_{L^1(\mathbb{R}\times\mathbb{R})} = \|f^0\|_{L^1(\mathbb{R}\times\mathbb{R})}$, we have not been able to prove strong convergence $f_n(x,k,t) \to \boldsymbol{f}(x,k)$ in $L^1(\mathbb{R}\times\mathbb{R}\times(0,T))$.

Proof: By Proposition 4.5 we extract from the sequence $\{f_n(x,k,t)\}_{n\in\mathbb{N}\cup\{0\}}$ a subsequence, denoted also by f_n, such that

$$f_n(x,k,t) \rightharpoonup \boldsymbol{f}(x,k,t) \qquad \text{in} \qquad L^1 \cap L^{\infty,*}(\mathbb{R}\times\mathbb{R}\times(0,T)). \tag{2.37}$$

The limit $\boldsymbol{f}$ satisfies (1.1)-(1.6) in the sense of distributions as a result of the following set of convergences:

a) $\partial_t f_n(x,k,t) + v(k)\partial_x f_n(x,k,t) \to \partial_t \mathcal{f}(x,k,t) + v(k)\partial_x \mathcal{f}(x,k,t)$ in $D'(\mathbb{R}\times\mathbb{R}\times(0,T))$.

Indeed, from (2.17) in **H. 2** it follows that $v(k) \in C^0(\mathbb{R})$. Using this and (2.37) we conclude for all $\varphi(x,k,t) \in C_0^1(\mathbb{R}\times\mathbb{R}\times(0,T))$ that

$$\int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} f_n(x,k,t)\left[\partial_t \varphi(x,k,t) + v(k)\partial_x \varphi(x,k,t)\right] dx\, dk\, dt$$

tends to

$$\int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} \mathcal{f}(x,k,t)\left[\partial_t \varphi(x,k,t) + v(k)\partial_x \varphi(x,k,t)\right] dx\, dk\, dt.$$

b) $E_n(x,t) \rightharpoonup \mathbb{E}(x,t)$ in $L^2(\mathbb{R}\times(0,T))$
since $\{E_n(x,k,t)\}_{n\in\mathbb{N}\cup\{0\}}$ is uniformly bounded in $L^2(\mathbb{R})$ (see Corollary 3.2).

c) $E_n(x,t)\partial_k f_n(x,k,t) \to \mathbb{E}(x,t)\partial_k \mathcal{f}(x,k,t)$ in $D'(\mathbb{R}\times\mathbb{R}\times(0,T))$.
This is proved in Proposition 4.6.

d) $Q(f_n)(x,k,t) \rightharpoonup Q(\mathcal{f})(x,k,t)$ in $L^p(\mathbb{R}\times\mathbb{R}\times(0,T))$ with $p>1$.
This is proved in Proposition 4.7.

e) $\rho_n(x,t) = n_D(x) - \int_{\mathbb{R}} f_n(x,k,t)\,dk \to \varrho(x,t) = n_D(x) - \int_{\mathbb{R}} \mathcal{f}(x,k,t)\,dk$ in $L^1_{loc}(\mathbb{R}\times(0,T))$.
Indeed, taking $\psi \equiv 1$ in Theorem A.4 we get

$$\int_{\mathbb{R}} f_n(x,k,t)\,dk \to \int_{\mathbb{R}} \mathcal{f}(x,k,t)\,dk \quad \text{in} \quad L^1_{loc}(\mathbb{R}\times(0,T)). \tag{2.38}$$

f) $\partial_x E_n(x,t) = \rho_n(x,t) \rightharpoonup \varrho(x,t) = \partial_x \mathbb{E}(x,t)$ in $L^2(\mathbb{R}\times(0,T))$
follows from a uniform bound in $L^2(\mathbb{R}\times(0,T))$ of $\partial_x E_n(x,t) = \rho_n(x,t)$. Such bound is obtained from inequality

$$\|\rho_n\|_{L^2(\mathbb{R})}(t) \le \|n_D\|_{L^2(\mathbb{R})} + \|\hat\rho_n\|_{L^2(\mathbb{R})}(t),$$

where $\hat\rho_n(x,t) = \int_{\mathbb{R}} f_n(x,k,t)\,dk$, using (2.19) and (2.20) in **H. 3** and Lemma A.5 with $p = (2\delta-1)/(\delta-1)$ and $\delta > 1$ (as required in (2.16) of **H. 2**). Since $E_n(x,t)$ tends weakly to $\mathbb{E}(x,t)$ in $L^2(\mathbb{R}\times(0,T))$ and

$$\partial_x E_n(x,t) \to \partial_x \mathbb{E}(x,t) \quad \text{in} \quad D'(\mathbb{R}\times(0,T))$$

the desired convergence follows.

g) $\partial_t \rho_n(x,t) \to \partial_t \varrho(x,t) \quad$ in $\quad D'(\mathbb{R}\times(0,T))$
since, by (2.38),

$$\partial_t \int_{\mathbb{R}} f_n(x,k,t)\,dk \to \partial_t \int_{\mathbb{R}} \mathcal{f}(x,k,t)\,dk \quad \text{in} \quad D'(\mathbb{R}\times(0,T)).$$

h) $\partial_x j_n(x,t) \to \partial_x \mathcal{J}(x,t) \quad$ in $\quad D'(\mathbb{R}\times(0,T))$
with

$$j_n(x,t) = -\int_{\mathbb{R}} v(k) f_n(x,k,t) \quad \text{and} \quad \mathcal{J}(x,t) = -\int_{\mathbb{R}} v(k)\mathcal{f}(x,k,t)\,dk.$$

Indeed, taking $\psi(k) \equiv v(k)$ in Theorem A.4, we have

$$\int_{\mathbb{R}} v(k) f_n(x,k,t)\, dk \to \int_{\mathbb{R}} v(k) \boldsymbol{f}(x,k,t)\, dk \quad in \quad L^1_{loc}\left(\mathbb{R}\times(0,T)\right).$$

From the above set of convergences, we conclude that the limits $\boldsymbol{f}$ and $\mathbb{E}$ satisfy

$$\partial_t \boldsymbol{f}(x,k,t) + v(k)\partial_x \boldsymbol{f}(x,k,t) - \mathbb{E}(x,t)\partial_k \boldsymbol{f}(x,k,t) = Q(\boldsymbol{f})(x,k,t)$$

in the sense of distributions with

$$\partial_x \mathbb{E}(x,t) = n_D(x) - \int_{\mathbb{R}} \boldsymbol{f}(x,k,t)\, dk$$

$$\partial_t \int_{\mathbb{R}} \boldsymbol{f}(x,k,t)\, dk + \int_{\mathbb{R}} v(k)\partial_x \boldsymbol{f}(x,k,t)\, dk = 0. \tag{2.39}$$

The above equations determine the structure of $\boldsymbol{f}$. From Corollary 4.5, $Q(\boldsymbol{f}) = 0$ a.e. $(x,k,t) \in \mathbb{R}\times\mathbb{R}\times(0,T)$. Thus,

$$\partial_t \boldsymbol{f}(x,k,t) + v(k)\partial_x \boldsymbol{f}(x,k,t) - \mathbb{E}(x,t)\partial_k \boldsymbol{f}(x,k,t) = 0. \tag{2.40}$$

Now, Proposition 4.8 implies that for some function $u(x,t)$

$$\boldsymbol{f}(x,k,t) = \frac{1}{1 + u(x,t) exp\left[\varepsilon(k)\right]} \quad a.e.\ (x,k,t) \in \mathbb{R}\times\mathbb{R}\times(0,T).$$

Note that $\varepsilon(k)$ is an even function, and this implies that $v(k)$ is odd. Then,

$$\int_{\mathbb{R}} v(k)\partial_x \boldsymbol{f}(x,k,t)\, dk = -\int_{\mathbb{R}} v(k) \frac{exp\left[\varepsilon(k)\right]\partial_x u(x,t)}{\left(1 + u(x,t) exp\left[\varepsilon(k)\right]\right)^2}\, dk = 0.$$

By (2.39),

$$\partial_t \varrho(x,t) = -\partial_t \int_{\mathbb{R}} \boldsymbol{f}(x,k,t)\, dk = 0.$$

Hence, $\varrho(x,t) = \varrho(x)$ does not depend on time and $\mathbb{E}(x,t) = \mathbb{E}(x)$. Substituting in (2.40), we get

$$\partial_t u(x,t) + v(k)\left[\partial_x u(x,t) - u(x,t)\mathbb{E}(x)\right] = 0.$$

Therefore, $\partial_t u(x,t) = 0$ and $\partial_x u(x,t) - u(x,t)\mathbb{E}(x) = 0$. This implies $u(x,t) = u(x)$ and $\frac{1}{u(x)}\frac{d}{dx}u(x) = \mathbb{E}(x)$ a.e. $x \in \mathbb{R}$. We can write $ln\left(u(x)\right) = -\Phi(x)$ a.e. $x \in \mathbb{R}$ for some function $\Phi(x)$ such that $-\frac{d}{dx}\Phi(x) = \mathbb{E}(x)$. Thus,

$$\boldsymbol{f}(x,k) = \frac{1}{1 + exp\left[\varepsilon(k) - \Phi(x)\right]} \qquad a.e.\ (x,k) \in \mathbb{R}\times\mathbb{R}.$$

In Proposition 4.9 we prove $\|\boldsymbol{f}\|_{L^1(\mathbb{R}\times\mathbb{R})} = \left\|f^0\right\|_{L^1(\mathbb{R}\times\mathbb{R})}$. Therefore, Φ must satisfy

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{dx\, dk}{1 + exp\left[\varepsilon(k) - \Phi(x)\right]} = \|f^0\|_{L^1(\mathbb{R}\times\mathbb{R})}.$$

We prove in the next proposition (Proposition 2.1), the uniqueness of such Φ. As a result, $\boldsymbol{f}$ is also unique and the whole sequence f_n tends to $\boldsymbol{f}$. □

Proposition 2.1 - *The equation*

$$-\frac{d^2}{dx^2}\Phi(x) = n_D(x) - \int_{\mathbb{R}} \frac{dk}{1 + exp\,[\varepsilon(k) - \Phi(x)]}$$

with

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{dx\,dk}{1 + exp\,[\varepsilon(k) - \Phi(x)]} = \|f^0\|_{L^1(\mathbb{R}\times\mathbb{R})}$$

has a unique solution Φ *with* $\frac{d}{dx}\Phi \in L^2(\mathbb{R})$.

Proof: Assume that Φ_1 and Φ_2 are two solutions of

$$-\frac{d^2}{dx^2}\Phi(x) = n_D(x) - \int_{\mathbb{R}} \frac{dk}{1 + exp\,[\varepsilon(k) - \Phi(x)]}$$

satisfying $\frac{d}{dx}\Phi_1$, $\frac{d}{dx}\Phi_2 \in L^2(\mathbb{R})$ and

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{dx\,dk}{1 + exp\,[\varepsilon(k) - \Phi_1(x)]} = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{dx\,dk}{1 + exp\,[\varepsilon(k) - \Phi_2(x)]} = \|f^0\|_{L^1(\mathbb{R}\times\mathbb{R})}.$$

Let $U(x) = \Phi_1(x) - \Phi_2(x)$. Then, $\frac{d}{dx}U \in L^2(\mathbb{R})$ and

$$\frac{d^2}{dx^2}U(x) = \int_{\mathbb{R}} \frac{dk}{1 + exp\,[\varepsilon(k) - \Phi_1(x)]} - \int_{\mathbb{R}} \frac{dk}{1 + exp\,[\varepsilon(k) - \Phi_2(x)]}.$$

Let us assume first that $U(x) > 0$ for all $x \in \mathbb{R}$. Then

$$\int_{\mathbb{R}} \frac{dk}{1 + exp\,[\varepsilon(k) - \Phi_1(x)]} > \int_{\mathbb{R}} \frac{dk}{1 + exp\,[\varepsilon(k) - \Phi_2(x)]}$$

so that

$$\|f^0\|_{L^1(\mathbb{R}\times\mathbb{R})} = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{dx\,dk}{1 + exp\,[\varepsilon(k) - \Phi_1(x)]} >$$
$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{dx\,dk}{1 + exp\,[\varepsilon(k) - \Phi_2(x)]} = \|f^0\|_{L^1(\mathbb{R}\times\mathbb{R})},$$

which is impossible.

Let us assume now that there is a unique $x_0 \in \mathbb{R}$ such that $U(x_0) = 0$. To fix ideas, we take $U(x) < 0$ if $x < x_0$ and $U(x) > 0$ if $x_0 < x$. Thus, $\frac{d^2}{dx^2}U(x) < 0$ if $x < x_0$, $\frac{d^2}{dx^2}U(x) > 0$ if $x_0 < x$ and $\frac{d}{dx}U(x)$ is decreasing for $x < x_0$, $\frac{d}{dx}U(x)$ is increasing for $x_0 < x$. On the other hand, we know that

$$\int_{\mathbb{R}} \left(\frac{d}{dx}U(x)\right)^2 dx = \int_{-\infty}^{x^*} \left(\frac{d}{dx}U(x)\right)^2 dx + \int_{x^*}^{\infty} \left(\frac{d}{dx}U(x)\right)^2 dx$$

is convergent. This means that no $x^* \in \mathbb{R}$ exists such that $\frac{d}{dx}U(x^*) > 0$. On the contrary, assume that such x^* exists. Then, if $x^* < x_0$

$$\int_{-\infty}^{x^*} \left(\frac{d}{dx}U(x)\right)^2 dx > \left(\frac{d}{dx}U(x^*)\right)^2 \int_{-\infty}^{x^*} dx = \infty$$

and, similarly, if $x^* > x_0$ we get $\int_{x^*}^{\infty} \left(\frac{d}{dx}U(x)\right)^2 dx = \infty$. This is impossible, thus, $\frac{d}{dx}U(x) \leq 0 \; \forall x \in \mathbb{R}$ so that $U(x)$ is decreasing. This contradicts $U(x) < 0$ if $x < x_0$ and $U(x) > 0$ if $x_0 < x$. Therefore, U vanishes in more than one point.

Let x_0 and x_1 be such that $U(x_0) = U(x_1) = 0$. If

$$U(x_M) = \max\{U(x) : x_0 \leq x \leq x_1\} > 0,$$

we have (see [8,12]) $\frac{d^2}{dx^2}U(x_M) \leq 0$ because the maximum is attained at an interior point. However,

$$0 \geq \frac{d^2}{dx^2}U(x_M) = \int_{\mathbb{R}} \frac{dk}{1 + exp\left[\varepsilon(k) - \Phi_1(x_M)\right]} - \int_{\mathbb{R}} \frac{dk}{1 + exp\left[\varepsilon(k) - \Phi_2(x_M)\right]} > 0,$$

since $U(x_M) = \Phi_1(x_M) - \Phi_2(x_M) > 0$. Hence, $\max\{U(x) : x_0 \leq x \leq x_1\} = 0$. In an analogous way, we conclude that $U(x_m) = \min\{U(x) : x_0 \leq x \leq x_1\} = 0$. Hence, $U(x) = 0$ for all $x \in [x_0, x_1]$.

We now set $x_0 = \min\{x \in R : U(x) = 0\}$ and $x_1 = \max\{x \in R : U(x) = 0\}$. Then, $U(x) < 0$; (resp. $U(x) > 0$) if $x < x_0$, $U(x) = 0$ if $x_0 \leq x \leq x_1$, $U(x) > 0$; (resp. $U(x) < 0$) if $x_1 < x$. Repeating the above arguments, we would obtain the existence of $x' \notin [x_0, x_1]$ such that $U(x') = 0$. This contradicts the definition of x_0 and x_1. In conclusion, $U(x) = 0$ for all $x \in \mathbb{R}$. □

3. Estimates

In this section we study more in detail the properties of classical solutions of the VPB equations for semiconductors, given by (1.1)-(1.6), when the hypotheses in theorem 2.2 hold. Some elementary bounds are stated in Theorem 2.1. Some other auxiliary regularity results are detailed in Appendix A. We are concerned here with obtaining the key uniform bounds, which follow from the existence of a Lyapunov functional bounded below. This functional is defined as:

$$A(t) = W(t) - \mathcal{S}(t) \tag{3.41}$$

where $W(t)$ is the total energy (defined in (1.10)) and $\mathcal{S}(t)$ the total entropy. The entropy can be expressed as:

$$\mathcal{S}(t) = -\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{s}(f(x,k,t))\, dx\, dk \tag{3.42}$$

with $\mathrm{s}(f) = f ln\,(f) + (1-f)\, ln\,(1-f)$, that is,

$$\mathrm{s}(f(x,k,t)) = f(x,k,t) ln\left(f(x,k,t)\right) + \left(1 - f(x,k,t)\right) ln\left(1 - f(x,k,t)\right). \tag{3.43}$$

The main result in this section is Proposition 3.4, where we establish that $A(t)$ is decreasing and bounded below. As corollaries we obtain some uniform bounds in time. Before proving Proposition 3.4, we need some preliminary results:

Proposition 3.2 - *$Q(f)$ is uniformly bounded in $L^\infty([0,\infty); L^p(\mathbb{R}\times\mathbb{R}))$ for all $p \geq 1$.*

Proof: We have

$$\|Q(f)\|_{L^1(\mathbb{R}\times\mathbb{R})}(t) \leq 2\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} S(x,k',k)f(x,k',t)\,(1-f(x,k,t))\,dx\,dk\,dk'$$

and $0 \leq 1-f(x,k,t) \leq 1$ (by (2.29)), so that:

$$\|Q(f)\|_{L^1(\mathbb{R}\times\mathbb{R})}(t) \leq 2\int_{\mathbb{R}}\int_{\mathbb{R}} \left(\int_{\mathbb{R}} S(x,k',k)\,dk\right) f(x,k',t)\,dx\,dk'.$$

From lemma A.1 we get

$$\|Q(f)\|_{L^1(\mathbb{R}\times\mathbb{R})}(t) \leq C_1(S)\|f^0\|_{L^1(\mathbb{R}\times\mathbb{R})}$$

as $S(x,k',k) \in L^\infty\left(\mathbb{R}_x \times \mathbb{R}_{k'}, L^1(\mathbb{R}_k)\right)$ by (2.13) in **H. 1**.

On the other hand, $0 \leq 1-f(x,k,t) \leq 1$ and $0 \leq f(x,k,t) \leq 1$ by (2.29), so that

$$\|Q(f)\|_{L^\infty(\mathbb{R}\times\mathbb{R})}(t) \leq \left\|\int_{\mathbb{R}} S(x,k',k)\,dk'\right\|_{L^\infty(\mathbb{R}\times\mathbb{R})} + \left\|\int_{\mathbb{R}} S(x,k,k')\,dk'\right\|_{L^\infty(\mathbb{R}\times\mathbb{R})}.$$

From condition (2.13) in **H. 1** we deduce $\|Q(f)\|_{L^\infty(\mathbb{R}\times\mathbb{R})}(t) \leq C_2(S)$. The constants depend only on S, so that the bounds are uniform in time. □

Proposition 3.3 - *The following identity holds:*

$$\frac{d}{dt}W(t) = \int_{\mathbb{R}}\int_{\mathbb{R}} \varepsilon(k)Q(f)(x,k,t)\,dx\,dk.$$

Proof: Multiplying (1.1) by $\varepsilon(k)$, integrating with respect to x and k and the integrating by parts we get:

$$\frac{d}{dt}T(t) - \int_{\mathbb{R}} j(x,t)E(x,t)\,dx = \int_{\mathbb{R}}\int_{\mathbb{R}} \varepsilon(k)Q(f)(x,k,t)\,dx\,dk$$

On the other hand, integrating (1.1) with respect to k we get:

$$\partial_t\left(\int_{\mathbb{R}} f(x,k,t)\,dk\right) + \int_{\mathbb{R}} v(k)\partial_x f(x,k,t)\,dk = \int_{\mathbb{R}} Q(f)(x,k,t)\,dk = 0.$$

From $\partial_t\rho(x,t) = -\partial_t\left(\int_{\mathbb{R}} f(x,k,t)\,dk\right)$ and the definition of current density (1.9), we obtain $\partial_t\rho(x,t) = -\partial_x j(x,t)$. By (1.2) and lemma A.3, $\partial_t E(x,t) = -j(x,t)$. Thus, $\int_{\mathbb{R}} j(x,t)E(x,t)\,dx = -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} |E(x,t)|^2\,dx$, and we conclude

$$\frac{d}{dt}W(t) = \frac{d}{dt}T(t) + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} |E(x,t)|^2\,dx = \int_{\mathbb{R}}\int_{\mathbb{R}} \varepsilon(k)Q(f)(x,k,t)\,dx\,dk. \quad \square$$

Corollary 3.1 - *We have*

$$W(t) = \int_{\mathbb{R}}\int_{\mathbb{R}} \varepsilon(k)f(x,k,t)\,dx\,dk + \frac{1}{2}\int_{\mathbb{R}} |E(x,t)|^2\,dx \in L^\infty_{loc}\left([0,\infty)\right).$$

Proof:

$$\frac{d}{dt}W(t) = \int_{\mathbb{R}}\int_{\mathbb{R}} \varepsilon(k)Q(f)(x,k,t)\,dx\,dk \leq$$

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \varepsilon(k)S(x,k',k)f(x,k',t)\left(1-f(x,k,t)\right)\,dx\,dk\,dk',$$

since $\varepsilon(k)S(x,k,k')f(x,k,t)\left(1-f(x,k',t)\right) \geq 0$. Taking into account condition (2.14) in **H. 1**, (2.29) and lemma A.1 we get:

$$\frac{d}{dt}W(t) \leq \int_{\mathbb{R}}\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \varepsilon(k)S(x,k',k)\,dk\right) f(x,k',t)\,dx\,dk' \leq$$

$$C\int_{\mathbb{R}}\int_{\mathbb{R}} f(x,k',t)\,dx\,dk' \leq C\|f^0\|_{L^1(\mathbb{R}\times\mathbb{R})} = C'.$$

Hence, $W(t) \leq W(0) + C't$ and the result follows. □

Lemma 3.1 - *If $n_D(x)$ is the doping profile and $g(x)$ the function defined in (2.33), it holds*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} g(x)f(x,k,t)\,dk\,dx \leq C + \left\|\frac{d}{dx}g\right\|_{L^2(\mathbb{R})} \|E\|_{L^2(\mathbb{R})}(t) \tag{3.44}$$

with $\int_{\mathbb{R}} g(x)n_D(x)\,dx = C \in \mathbb{R}$.

Proof: By definition of charge density (1.3), hypothesis (2.20) in **H. 3**, corollary 3.1 and lemma A.4,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} g(x)f(x,k,t)\,dk\,dx = \int_{\mathbb{R}} g(x)\left[n_D(x) - \rho(x,t)\right]dx =$$

$$\int_{\mathbb{R}} g(x)n_D(x)\,dx - \int_{\mathbb{R}} g(x)\rho(x,t)\,dx = C - \int_{\mathbb{R}} g(x)\partial_x E(x,t)\,dx =$$

$$C + \int_{\mathbb{R}} \left(\frac{d}{dx}g(x)\right) E(x,t)\,dx \leq C + \left\|\frac{d}{dx}g\right\|_{L^2(\mathbb{R})} \|E\|_{L^2(\mathbb{R})}(t).\ \square$$

Lemma 3.2 - *If $0 < f^0(x,k) < 1$ for all $x \in \mathbb{R}, k \in \mathbb{R}$, then $0 < f(x,k,t) < 1$ for $t > 0, x \in \mathbb{R}, k \in \mathbb{R}$.*

Proof: Proceeding as in [16], we define the operators:

$$D(f) = \int_{\mathbb{R}} S(x,k',k)f(x,k',t)\,dk', \quad B(f) = \int_{\mathbb{R}} S(x,k,k')\left(1-f(x,k',t)\right)\,dk'$$

and $\lambda(f) = D(f) + B(f)$. Then, $Q(f) = D(f) - \lambda(f)f$ and the solution of (1.1) satisfies the integral equation (see [19]):

$$f(x,k,t) = f^0(XK(0;x,k,t))J(0;x,k,t)exp\left[-\int_0^t \lambda(f)(XK(s;x,k,t),s)ds\right] +$$

$$\int_0^t D(f)(XK(s;x,k,t),s)J(s;x,k,t)exp\left[-\int_s^t \lambda(f)(XK(\sigma;x,k,t),\sigma)\,d\sigma\right]ds$$

where $XK(t;x,k,s) = (X(t;x,k,s), K(t;x,k,s))$ are the characteristics of the convective problem which solve

$$\frac{d}{dt}\left(X(t;x,k,s), K(t;x,k,s)\right) = \left(v(K(t;x,k,s)), -E(X(t;x,k,s),t)\right)$$

$$(X(s;x,k,s), K(s;x,k,s)) = (x,k),$$

and $J(t;x,k,s)$ is the jacobian of the change of variables:

$$J(t;x,k,s) = det\begin{pmatrix} \partial_x X(t;x,k,s) & \partial_k X(t;x,k,s) \\ \partial_x K(t;x,k,s) & \partial_k K(t;x,k,s). \end{pmatrix}.$$

Note that $J(t;x,k,s) = 1$ since $div_{x,k}(v(k), -E(x,t)) = 0$. Using $D(f) \geq 0$ we get

$$f(x,k,t) \geq f^0(XK(0;x,k,t))exp\left[-\int_0^t \lambda(f)(XK(s;x,k,t),s)\,ds\right] > 0.$$

Analogously, letting $\breve{f}(x,k,t) = 1 - f(x,k,t)$, we obtain (see [16]) that $\breve{f}(x,k,t) > 0$, that is, $f(x,k,t) < 1$. □

Now, we are ready to study the properties of $A(t)$:

Proposition 3.4 - *The functional $A(t)$ defined in (3.41) is bounded from below and decreasing.*

Proof: By lemma 3.2 we have $0 < f(x,k,t) < 1$ so that both $ln\,(f(x,k,t))$ and $ln\,(1 - f(x,k,t))$ are finite.

1. *$A(t)$ is bounded from below:* Let $\varepsilon(k)$ the band diagram associated to $v(k)$ and $g(x)$ the function defined in (2.33). For some positive constant z, we have

$$f(x,k,t)ln\,(f(x,k,t)) \geq -\left(1 + \frac{\varepsilon(k)+g(x)}{z}\right) f(x,k,t)$$

when $exp\left[-\left(1 + z^{-1}(\varepsilon(k)+g(x))\right)\right] \leq f(x,k,t) < e^{-1}$, since the logarithmic function is increasing.

On the other hand, $fln(f)$ is decreasing between 0 and e^{-1} and

$$f(x,k,t)ln\,(f(x,k,t)) \geq -\left(1 + \frac{\varepsilon(k)+g(x)}{z}\right) exp\left[-\left(1 + \frac{\varepsilon(k)+g(x)}{z}\right)\right]$$

if $0 < f(x,k,t) < exp\left[-\left(1 + z^{-1}(\varepsilon(k)+g(x))\right)\right]$.

Taking into account that $\mathrm{S}\,(f(x,k,t))$ verifies:

- $\mathrm{S}\,(f(x,k,t)) \geq 2f(x,k,t)ln\,(f(x,k,t))$ if $0 < f(x,k,t) < e^{-1}$
- $\mathrm{S}\,(f(x,k,t)) \geq -2f(x,k,t)$ if $e^{-1} \leq f(x,k,t) < 1$

we have

$$\mathrm{S}\,(f(x,k,t)) \geq -2\left(1+\frac{\varepsilon(k)+g(x)}{z}\right)\left\{f(x,k,t) + exp\left[-\left(1+\frac{\varepsilon(k)+g(x)}{z}\right)\right]\right\}.$$

Now, $\varepsilon(k)$ satisfies condition (2.16) in **H. 2** and $g(x)$ verifies condition (2.35), so that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left(1+\frac{\varepsilon(k)+g(x)}{z}\right) exp\left[-\left(1+\frac{\varepsilon(k)+g(x)}{z}\right)\right] dx\,dk \le C.$$

From lemma A.1 we get

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left(1+\frac{\varepsilon(k)+g(x)}{z}\right) f(x,k,t)\,dx\,dk =$$

$$\|f^0\|_{L^1(\mathbb{R}\times\mathbb{R})} + \frac{1}{z}\int_{\mathbb{R}}\int_{\mathbb{R}} \varepsilon(k) f(x,k,t)\,dx\,dk + \frac{1}{z}\int_{\mathbb{R}}\int_{\mathbb{R}} g(x) f(x,k,t)\,dx\,dk.$$

Using lemma 3.1 and the definition of kinetic energy (1.11), we obtain

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left(1+\frac{\varepsilon(k)+g(x)}{z}\right) f(x,k,t)\,dx\,dk \le C + \frac{1}{z}T(t) + \frac{C}{z}\|E\|_{L^2(\mathbb{R})}(t)$$

where C is a positive constant depending on f^0, n_D, ε and g. Therefore,

$$-\mathcal{S}(t) = \int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{s}\left(f(x,k,t)\right)\,dx\,dk \ge -4C - \frac{2}{z}T(t) - \frac{2C}{z}\|E\|_{L^2(\mathbb{R})}(t).$$

Hence,

$$A(t) = W(t) - \mathcal{S}(t) \ge \left(1-\frac{2}{z}\right)T(t) + \left(\frac{1}{2}-\frac{2C}{z}\right)\|E\|^2_{L^2(\mathbb{R})}(t) - 5C. \tag{3.45}$$

Choosing in an appropiate way the constant z and taking into account that both $T(t)$ and $\|E\|^2_{L^2(\mathbb{R})}(t)$ are positive we conclude that $A(t) \ge -5C$.

2. *$A(t)$ is decreasing:* For $\mathrm{s} \in C^1((0,1))$ we have

$$\partial_t \mathrm{s}(f) + v(k)\partial_x \mathrm{s}(f) - E(x,t)\partial_k \mathrm{s}(f) = \mathrm{s}'(f)\,Q(f)$$

where $\mathrm{s}'(f(x,k,t)) = ln\left(\frac{f(x,k,t)}{1-f(x,k,t)}\right)$. From

$$\int_{\mathbb{R}}\int_{\mathbb{R}} v(k)\partial_x \mathrm{s}\left(f(x,k,t)\right)\,dx\,dk = 0 \quad \text{and} \quad \int_{\mathbb{R}}\int_{\mathbb{R}} E(x,t)\partial_k \mathrm{s}\left(f(x,k,t)\right)\,dx\,dk = 0$$

we conclude that

$$-\frac{d}{dt}\mathcal{S}(t) = \int_{\mathbb{R}}\int_{\mathbb{R}} \partial_t \mathrm{s}\left(f(x,k,t)\right)\,dx\,dk = \int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{s}'\left(f(x,k,t)\right)Q(f)(x,k,t)\,dx\,dk.$$

Thus,

$$-\frac{d}{dt}\mathcal{S}(t) = \int_{\mathbb{R}}\int_{\mathbb{R}} Q(f)(x,k,t) ln\left(\frac{f(x,k,t)}{1-f(x,k,t)}\right) dx\,dk.$$

Combining the later equality and Proposition 3.3 we obtain

$$\frac{d}{dt}A(t) = \int_{\mathbb{R}}\int_{\mathbb{R}} Q(f)(x,k,t) ln\left(exp\left[\varepsilon(k)\right]\frac{f(x,k,t)}{1-f(x,k,t)}\right) dx\,dk. \tag{3.46}$$

The H-theorem (see Theorem A.1) implies $\frac{d}{dt}A(t) \le 0$, that is, A is decreasing. □

As corollaries we get the following bounds:

Corollary 3.2 - *There exist constants C_ε, C_E and C_g, independent of time, such that*

$$\int_{\rm I\!R}\int_{\rm I\!R} \varepsilon(k) f(x,k,t)\, dx\, dk \le C_\varepsilon,$$

$$\int_{\rm I\!R} |E(x,t)|^2\, dx \le C_E,$$

$$\int_{\rm I\!R}\int_{\rm I\!R} g(x) f(x,k,t)\, dx\, dk \le C_g.$$

Proof: Since $A(t)$ is decreasing and $-\mathcal{S}(t)$ is negative, we get using (3.45)

$$W(0) \ge A(0) \ge C_1 \int_{\rm I\!R}\int_{\rm I\!R} \varepsilon(k) f(x,k,t)\, dx\, dk \;+\; C_2 \int_{\rm I\!R} |E(x,t)|^2\; dx \;-\; C_3.$$

where C_1, C_2 and C_3 are positive constants which do not depend on time. Taking $C_\varepsilon = (W(0)+C_3)C_1^{-1}$ and $C_E = (W(0)+C_3)C_2^{-1}$, we obtain:

$$\int_{\rm I\!R}\int_{\rm I\!R} \varepsilon(k) f(x,k,t)\, dx\, dk \le C_\varepsilon, \qquad \int_{\rm I\!R} |E(x,t)|^2\, dx \le C_E.$$

On the other hand, from lemma 3.1 we get

$$\int_{\rm I\!R}\int_{\rm I\!R} g(x) f(x,k,t)\, dx\, dk \le C + \left\| \frac{d}{dx} g \right\|_{L^2({\rm I\!R})} \|E\|_{L^2({\rm I\!R})}(t) \le C_5 + C_6 \|E\|_{L^2({\rm I\!R})}(t)$$

and the bounds on the electric field imply $\int_{\rm I\!R}\int_{\rm I\!R} g(x) f(x,k,t)\, dx\, dk \le C_g$. □

Corollary 3.3 - *It holds*

$$0 \le -\int_0^\infty \int_{\rm I\!R}\int_{\rm I\!R} Q(f)(x,k,t) ln \left(exp\,[\varepsilon(k)]\, \frac{f(x,k,t)}{1-f(x,k,t)} \right) dx\, dk\, dt < \infty.$$

Proof: By the H-Theorem (theorem A.1) and (3.46), we get

$$0 \le -\int_0^\infty \int_{\rm I\!R}\int_{\rm I\!R} Q(f)(x,k,t) ln \left(exp\,[\varepsilon(k)]\, \frac{f(x,k,t)}{1-f(x,k,t)} \right) dx\, dk\, dt =$$

$$-\int_0^\infty \frac{d}{dt} A(t)\, dt = A(0) - lim_{T\to\infty} A(T) < \infty,$$

since A is bounded from below. □

4. Convergences

In this section we prove all the convergences required in the proof of Theorem 2.2 in section 2. We also prove some properties of the the limit function: $Q(\boldsymbol{f}) = 0$ and $\|\boldsymbol{f}\|_1 = \|f_0\|_1$.

We keep the notations and assumptions of the previous sections. Let f_n be the translated functions defined by $f_n(x,k,t) = f(x,k,t+nT)$, where f is the solution of VPB considered in Theorem 2.2. There is a subsequence, which for simplicity we denote also f_n, satisfying the following set of convergences:

Proposition 4.5 -

$$f_n(x,k,t) \rightharpoonup \mathit{f\!\!\!f}(x,k,t) \qquad in \qquad L^1 \cap L^{\infty,*}(\mathbb{R}\times\mathbb{R}\times(0,T)).$$

Proof: By (2.29), $0 \le f_n \le 1$ for all $n \in \mathbb{N}\cup\{0\}$ so that $\{f_n(x,k,t)\}_{n\in\mathbb{N}\cup\{0\}}$ is uniformly bounded in $L^\infty(\mathbb{R}\times\mathbb{R}\times(0,T))$.

Now, using the Dunford-Pettis Theorem (theorem A.2) we conclude that the sequence $\{f_n(x,k,t)\}_{n\in\mathbb{N}\cup\{0\}}$ is weakly compact in $L^1(\mathbb{R}\times\mathbb{R}\times(0,T))$. Let us check that the hypotheses hold

1. Since $\|f_n\|_{L^1(\mathbb{R}\times\mathbb{R})}(t) = \|f^0\|_{L^1(\mathbb{R}\times\mathbb{R})}$, we get from lemma A.1

$$\|f_n\|_{L^1(\mathbb{R}\times\mathbb{R}\times(0,T))} \le T\|f^0\|_{L^1(\mathbb{R}\times\mathbb{R})}.$$

2. Given $A \subset \mathbb{R}\times\mathbb{R}\times(0,T)$, the bound $0 \le f_n \le 1$ implies:

$$\iiint_A |f_n(x,k,t)|\,dx\,dk\,dt = \iiint_A f_n(x,k,t)dx\,dk\,dt \le |A|.$$

3. For some $M > 0$ and $t_K \in (0,T)$, we define the compact set

$$K = \{(x,k,t) \in \mathbb{R}\times\mathbb{R}\times(0,T) : 1 + g(x) + \varepsilon(k) \le M,\ t = t_K\}$$

where $g(x) \to \infty$ if $|x| \to \infty$ and $\varepsilon(k) \to \infty$ if $|k| \to \infty$. In this way, denoting $K^c = (\mathbb{R}\times\mathbb{R}\times(0,T)) - K$, we get using corollary 3.2

$$\iiint_{K^c} f_n(x,k,t)dx\,dk\,dt = \iiint_{K^c} \frac{1+g(x)+\varepsilon(k)}{1+g(x)+\varepsilon(k)} f_n(x,k,t)dx\,dk\,dt \le$$

$$\frac{1}{M}\iiint_{K^c}(1+g(x)+\varepsilon(k))f_n(x,k,t)dx\,dk\,dt \le$$

$$\frac{1}{M}\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}}(1+g(x)+\varepsilon(k))f_n(x,k,t)dx\,dk\,dt \le \frac{T}{M}\left[\|f^0\|_{L^1(\mathbb{R}\times\mathbb{R})} + C_g + C_\varepsilon\right]. \ \square$$

Proposition 4.6 -

$$E_n(x,t)\partial_k f_n(x,k,t) \to \mathbb{E}(x,t)\partial_k \mathit{f\!\!\!f}(x,k,t) \qquad in \qquad D'(\mathbb{R}\times\mathbb{R}\times(0,T)).$$

Proof: We prove the convergence for test functions $\varphi(x,k,t) = \varphi_1(x,t)\varphi_2(k)$, where $\varphi_1(x,t) \in C_0^\infty(\mathbb{R}\times(0,T))$ and $\varphi_2(k) \in C_0^\infty(\mathbb{R})$. By density, it extends to test functions in $C_0^\infty(\mathbb{R}\times\mathbb{R}\times(0,T))$. We have

$$\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi_1(x,t)\varphi_2(k)E_n(x,t)\partial_k f_n(x,k,t)\,dx\,dk\,dt =$$

$$\int_0^T\int_{\mathbb{R}} \varphi_1(x,t)E_n(x,t)\left[\int_{\mathbb{R}} f_n(x,k,t)\partial_k\varphi_2(k)\,dk\right]dx\,dt.$$

By theorem A.4, for any ball B_{R_x}

$$\int_{\mathbb{R}} f_n(x,k,t)\partial_k\varphi_2(k)\,dk \to \int_{\mathbb{R}} \boldsymbol{f}(x,k,t)\partial_k\varphi_2(k)\,dk \quad \text{in} \quad L^1\left(B_{R_x}\times(0,T)\right).$$

Using $0 \le f_n \le 1$ we get $\left|\int_{\mathbb{R}} f_n(x,k,t)\partial_k\varphi_2(k)\,dk\right|^2 \le C(\varphi_2)$, where $C(\varphi_2)$ is a constant depending only on φ_2. Thus, we have

$$\int_{\mathbb{R}} f_n(x,k,t)\partial_k\varphi_2(k)\,dk \to \int_{\mathbb{R}} \boldsymbol{f}(x,k,t)\partial_k\varphi_2(k)\,dk \quad \text{in} \quad L^2\left(B_{R_x}\times(0,T)\right).$$

We conclude using $E_n(x,t) \rightharpoonup \mathbb{E}(x,t)$ in $L^2(\mathbb{R}\times(0,T))$. □

Lemma 4.3 -

$$f_n(x,k',t)\left(1-f_n(x,k,t)\right) \rightharpoonup \boldsymbol{f}(x,k',t)\left(1-\boldsymbol{f}(x,k,t)\right)$$

in $L^p(\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times(0,T))$ *for all* $p>1$.

Proof: For any function $\varphi(x,k,k',t)=\varphi_1(x,t)\varphi_2(k)\varphi_3(k')$ with $\varphi_3(k')\in C_0^\infty(\mathbb{R})$, $\varphi_2(k)\in C_0^\infty(\mathbb{R})$ and $\varphi_1(x,t)\in C_0^\infty(\mathbb{R}\times(0,T))$, we have

$$\int_0^T\!\!\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(x,k,k',t)f_n(x,k',t)\left(1-f_n(x,k,t)\right)dx\,dk\,dk'\,dt -$$

$$\int_0^T\!\!\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(x,k,k',t)\boldsymbol{f}(x,k',t)\left(1-\boldsymbol{f}(x,k,t)\right)dx\,dk\,dk'\,dt =$$

$$\int_0^T\!\!\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(x,k,k',t)\left(f_n(x,k',t)-\boldsymbol{f}(x,k',t)\right)dx\,dk\,dk'\,dt +$$

$$\int_0^T\!\!\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(x,k,k',t)(\boldsymbol{f}(x,k',t)\boldsymbol{f}(x,k,t)-f_n(x,k',t)f_n(x,k,t))\,dx\,dk\,dk'\,dt.$$

Both terms tend to zero when $n\to\infty$ thanks to Proposition 4.5 and the bounds $0\le\boldsymbol{f}\le1$ and $0\le f_n\le1$ for all n. The proof is concluded by a density argument. □

Corollary 4.4 -

$$exp\left[-\varepsilon(k)\right]f_n(x,k',t)\left(1-f_n(x,k,t)\right) \rightharpoonup exp\left[-\varepsilon(k)\right]\boldsymbol{f}(x,k',t)\left(1-\boldsymbol{f}(x,k,t)\right)$$

in $L^p(\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times(0,T))$ *for all* $p>1$.

Proof: It is straightforward using the previous lemma, since $exp\left[-\varepsilon(k)\right]\in L^\infty(\mathbb{R})$ by hypothesis (2.16) in **H. 2**. □

Proposition 4.7 -

$$Q(f_n)(x,k,t) \rightharpoonup Q(\boldsymbol{f})(x,k,t) \qquad in \qquad L^p(\mathbb{R}\times\mathbb{R}\times(0,T)) \qquad \forall p>1.$$

Proof: For all $\varphi(x,k,t)\in C_0^\infty(\mathbb{R}\times\mathbb{R}\times(0,T))$, identity

$$\int_{\mathbb{R}} Q(f)(x,k,t)\varphi(x,k,t)\,dk =$$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} [\varphi(x,k,t)-\varphi(x,k',t)]\, S(x,k',k)\, [f(x,k',t)\,(1-f(x,k,t))]\; dk\, dk'$$

holds with $f = \boldsymbol{f}$ and $f = f_n$. By (2.12) and (2.13),

$$\tilde{\varphi}(x,k,k',t) = [\varphi(x,k,t)-\varphi(x,k',t)]\, S(x,k',k) \in L^p\left(\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times(0,T)\right)$$

for $p \geq 1$. Thus,

$$\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}} [Q(f_n)(x,k,t)-Q(\boldsymbol{f})(x,k,t)]\,\varphi(x,k,t)\, dx\, dk\, dt =$$

$$\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \tilde{\varphi}(x,k,k',t)\, [f_n(x,k',t)\,(1-f_n(x,k,t)) -$$

$$\boldsymbol{f}(x,k',t)\,(1-\boldsymbol{f}(x,k,t))]\, dx\, dk\, dk'\, dt$$

which tends to zero by lemma 4.3 as $n \to \infty$. □

Lemma 4.4 - *Let g be the function defined in (2.33). We have*

$$\sqrt{g(x)}f_n(x,k,t) \rightharpoonup \sqrt{g(x)}\boldsymbol{f}(x,k,t) \qquad in \qquad L^1\left(\mathbb{R}\times\mathbb{R}\times(0,T)\right).$$

Proof: Using the Dunford-Pettis Theorem (theorem A.2) with corollary 3.2 we obtain weak compactness of $\left\{\sqrt{g(x)}f_n(x,k,t)\right\}_{n\in\mathbb{N}\cup\{0\}}$ in $L^1\left(\mathbb{R}\times\mathbb{R}\times(0,T)\right)$. Then, there exists a function F such that

$$\sqrt{g(x)}f_n(x,k,t) \rightharpoonup F(x,k,t) \qquad in \qquad L^1\left(\mathbb{R}\times\mathbb{R}\times(0,T)\right).$$

Now, $\sqrt{g(x)}\varphi(x,k,t) \in L^\infty\left(\mathbb{R}\times\mathbb{R}\times(0,T)\right)$ if $\varphi(x,k,t) \in C_0^\infty\left(\mathbb{R}\times\mathbb{R}\times(0,T)\right)$. Using Proposition 4.5, we get

$$\sqrt{g(x)}f_n(x,k,t) \to \sqrt{g(x)}\boldsymbol{f}(x,k,t) \qquad in \qquad D'\left(\mathbb{R}\times\mathbb{R}\times(0,T)\right).$$

Thus, $F(x,k,t) = \sqrt{g(x)}\boldsymbol{f}(x,k,t)$. □

As a result of the previous convergences, we obtain some properties of the limit function: the collision term vanishes and the total charge is the same as the total charge of the initial data.

Proposition 4.8 -

$$exp\,[-\varepsilon(k)]\,\boldsymbol{f}(x,k',t)\,(1-\boldsymbol{f}(x,k,t)) - exp\,[-\varepsilon(k')]\,\boldsymbol{f}(x,k,t)\,(1-\boldsymbol{f}(x,k',t)) = 0$$

a.e. $(x,k,k',t) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times(0,T)$.

Proof: By Corollary 3.3,

$$I_n = -\int_0^T\int_{\mathbb{R}}\int_{\mathbb{R}} Q(f_n)(x,k,t) ln\left(exp\,[\varepsilon(k)]\,\frac{f_n(x,k,t)}{1-f_n(x,k,t)}\right)\, dk\, dx\, dt \to 0$$

when $n \to \infty$. Let $\mathrm{A}_n(x,k',k,t) = exp\,[-\varepsilon(k)]\, f_n(x,k',t)\,(1 - f_n(x,k,t))$. Using (1.8), the definition of A_n and the symmetry of Υ with respect to k, k' in (1.6) we get:

$$- \int_{\mathrm{IR}} Q(f_n)(x,k,t) ln \left(exp\,[\varepsilon(k)]\, \frac{f_n(x,k,t)}{1 - f_n(x,k,t)} \right) dk =$$

$$\frac{1}{2} \int_{\mathrm{IR}} \int_{\mathrm{IR}} \Upsilon(x,k',k) \left\{ \mathrm{A}_n(x,k',k,t) - \mathrm{A}_n(x,k,k',t) \right\}$$
$$\left\{ ln\,(\mathrm{A}_n(x,k',k,t)) - ln\,(\mathrm{A}_n(x,k,k',t)) \right\} dk'\, dk.$$

Thus,

$$I_n = \int_0^T \int_{\mathrm{IR}} \int_{\mathrm{IR}} \int_{\mathrm{IR}} \Upsilon(x,k',k) \left\{ \mathrm{A}_n(x,k',k,t) - \mathrm{A}_n(x,k,k',t) \right\}$$
$$\left\{ ln\,(\mathrm{A}_n(x,k',k,t)) - ln\,(\mathrm{A}_n(x,k,k',t)) \right\} dk'\, dk\, dx\, dt \to 0$$

when $n \to \infty$. By corollary 4.4,

$$\mathrm{A}_n(x,k',k,t) \rightharpoonup \mathrm{A}(x,k',k,t) = exp\,[-\varepsilon(k)]\, \mathbf{f}(x,k',t)\,(1 - \mathbf{f}(x,k,t))$$

in $L^p\,(\mathrm{IR}\times\mathrm{IR}\times\mathrm{IR}\times(0,T))$ for all $p > 1$. Taking into account that $\Upsilon(x,k',k) > 0$ and $\{\mathrm{A}_n(x,k',k,t) - \mathrm{A}_n(x,k,k',t)\}\,\{ln\,(\mathrm{A}_n(x,k',k,t)) - ln\,(\mathrm{A}_n(x,k,k',t))\}$ is convex, we get

$$\left\{ \mathrm{A}(x,k',k,t) - \mathrm{A}(x,k,k',t) \right\} \left\{ ln\,(\mathrm{A}(x,k',k,t)) - ln\,(\mathrm{A}(x,k,k',t)) \right\} = 0$$

a.e. $(x,k,k',t) \in \mathrm{IR}\times\mathrm{IR}\times\mathrm{IR}\times(0,T)$ (see [3]). Thus,

$$\mathrm{A}(x,k',k,t) - \mathrm{A}(x,k,k',t) = 0 \quad a.e.\ (x,k,k',t) \in \mathrm{IR}\times\mathrm{IR}\times\mathrm{IR}\times(0,T). \ \square$$

Corollary 4.5 -

$$Q(\mathbf{f})(x,k,t) = 0 \qquad a.e.\ (x,k,t) \in \mathrm{IR}\times\mathrm{IR}\times(0,T).$$

Proof: By (1.8) and the symmetry of Υ,

$$Q(\mathbf{f})(x,k,t) = \int_{\mathrm{IR}} \Upsilon(x,k',k) \left\{ exp\,[-\varepsilon(k)]\, \mathbf{f}(x,k',t)\,(1 - \mathbf{f}(x,k,t)) \right.$$
$$\left. - exp\,[-\varepsilon(k')]\, \mathbf{f}(x,k,t)\,(1 - \mathbf{f}(x,k',t)) \right\} dk'.$$

We conclude using Proposition 4.8. $\square$

Remark: Using the above results, in the proof of Theorem 2.2 in Section 2 we established that $\mathbf{f}$ does not depend on t. In the next proposition, we drop the variable t.

Proposition 4.9 -

$$\|\mathbf{f}\|_{L^1(\mathrm{IR}\times\mathrm{IR})} = \|f^0\|_{L^1(\mathrm{IR}\times\mathrm{IR})}.$$

Proof:

$$\frac{T}{2}\left| \|f^0\|_{L^1(\mathbb{R}\times\mathbb{R})} - \|\boldsymbol{f}\|_{L^1(\mathbb{R}\times\mathbb{R})}\right| = \int_{\frac{T}{4}}^{\frac{3T}{4}} \left| \|f^0\|_{L^1(\mathbb{R}\times\mathbb{R})} - \|\boldsymbol{f}\|_{L^1(\mathbb{R}\times\mathbb{R})}\right| dt =$$

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \left| \|f_n\|_{L^1(\mathbb{R}\times\mathbb{R})}(t) - \|\boldsymbol{f}\|_{L^1(\mathbb{R}\times\mathbb{R})}\right| dt = \int_{\frac{T}{4}}^{\frac{3T}{4}} \left| \int_{\mathbb{R}}\int_{\mathbb{R}} (f_n(x,k,t) - \boldsymbol{f}(x,k)) dx dk \right| dt$$

$$\leq \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} (f_n(x,k,t) - \boldsymbol{f}(x,k))\, dk \right| dx\, dt =$$

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{G_1} \left| \int_{\mathbb{R}} (f_n(x,k,t) - \boldsymbol{f}(x,k)) dk \right| dx dt + \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{G_2} \left| \int_{\mathbb{R}} (f_n(x,k,t) - \boldsymbol{f}(x,k)) dk \right| dx dt$$

with $G_1 = \{x \in \mathbb{R} : \sqrt{g(x)} \leq R_\epsilon\}$ and $G_2 = \{x \in \mathbb{R} : \sqrt{g(x)} > R_\epsilon\}$. By (2.38), for all $\epsilon > 0$ there is n_0 such that if $n > n_0$

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{G_1} \left| \int_{\mathbb{R}} (f_n(x,k,t) - \boldsymbol{f}(x,k))\, dk \right| dx\, dt < \frac{\epsilon T}{4}.$$

On the other hand,

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{G_2} \left| \int_{\mathbb{R}} (f_n(x,k,t) - \boldsymbol{f}(x,k))\, dk \right| dx\, dt \leq$$

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{G_2} \left| \int_{\mathbb{R}} f_n(x,k,t) dk \right| dx\, dt + \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{G_2} \left| \int_{\mathbb{R}} \boldsymbol{f}(x,k) dk \right| dx\, dt =$$

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{\mathbb{R}} \int_{G_2} \frac{\sqrt{g(x)}}{\sqrt{g(x)}} f_n(x,k,t) dx\, dk\, dt + \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{\mathbb{R}} \int_{G_2} \frac{\sqrt{g(x)}}{\sqrt{g(x)}} \boldsymbol{f}(x,k) dx\, dk\, dt \leq$$

$$\frac{1}{R_\epsilon} \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{\mathbb{R}} \int_{G_2} \sqrt{g(x)} f_n(x,k,t) dx\, dk\, dt + \frac{1}{R_\epsilon} \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{\mathbb{R}} \int_{G_2} \sqrt{g(x)} \boldsymbol{f}(x,k) dx\, dk\, dt \leq$$

$$\frac{1}{R_\epsilon} \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} \sqrt{g(x)} f_n(x,k,t) dx\, dk\, dt + \frac{1}{R_\epsilon} \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} \sqrt{g(x)} \boldsymbol{f}(x,k) dx\, dk\, dt.$$

By lemma 4.4, we get $\frac{T}{2}\left| \|f^0\|_{L^1(\mathbb{R}\times\mathbb{R})} - \|\boldsymbol{f}\|_{L^1(\mathbb{R}\times\mathbb{R})}\right| < \frac{\epsilon T}{4} + 2\frac{C_g T}{R_\epsilon}$. Taking $R_\epsilon = 8C_g/\epsilon$, we get $\left| \|f^0\|_{L^1(\mathbb{R}\times\mathbb{R})} - \|\boldsymbol{f}\|_{L^1(\mathbb{R}\times\mathbb{R})}\right| < \frac{2}{T}\left[\frac{\epsilon T}{4} + \frac{\epsilon T}{4}\right] = \epsilon$.□

Appendix

Proposition A.1 - *Let us assume that the hypotheses in Theorem 2.1 hold. Then, the solution of the VPB equations for semiconductors (1.1)-(1.6) satifies*

$$\left(1 + \|x\|^2\right)^{\delta_x/2} f \in L^\infty_{loc}\left([0, T^*), L^\infty\left(\mathbb{R}^d \times \mathbb{R}^d\right)\right)$$

when $\left(1 + \|x\|^2\right)^{\delta_x/2} f^0 \in L^\infty\left(\mathbb{R}^d \times \mathbb{R}^d\right)$, *for* $0 \leq \delta_x \leq \gamma/\delta_v$, *provided there is* $C > 0$ *such that* $C\left(1 + \|k\|^2\right)^{\delta_v/2} \geq \|v(k)\|$ *with* $\gamma \geq \delta_v > 0$.

Proof: Similar to regularity results in [16].

Lemma A.1 - *Classical solutions of VPB satisfy* $\|f\|_{L^1(\mathbb{R}\times\mathbb{R})}(t) = \|f^0\|_{L^1(\mathbb{R}\times\mathbb{R})}$ *for all* $t \in [0,\infty)$.

Proof: Integrating (1.1) with respect to x and k we get

$$\frac{d}{dt}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} f(x,k,t)\,dx\,dk\right) = \int_{\mathbb{R}}\left[\int_{\mathbb{R}} Q(f)(x,k,t)\,dk\right]dx$$

and $\int_{\mathbb{R}} Q(f)(x,k,t)\,dk = 0$ gives $\frac{d}{dt}\|f\|_{L^1(\mathbb{R}\times\mathbb{R})}(t) = 0$.

Lemma A.2 - *If* $0 < b < 1/2$ *is such that condition (2.20) in* **H. 3**, *holds*

$$\left(1+|x|^2\right)^{b/2}\rho \in L^\infty_{loc}\left([0,\infty), L^1(\mathbb{R})\right).$$

Proof: By (1.3),

$$\int_{\mathbb{R}}\left(1+|x|^2\right)^{b/2}|\rho(x,t)|\,dx \le \int_{\mathbb{R}}\left(1+|x|^2\right)^{b/2}\left(|n_D(x,t)| + f(x,k,t)\right)dk\,dx.$$

Now, since

$$\left(1+|x|^2\right)^{b/2} f(x,k,t) = \frac{\left(1+|x|^2\right)^{3/4}\left(1+|k|^2\right)^{\gamma/8}}{\left(1+|x|^2\right)^{(3-2b)/4}\left(1+|k|^2\right)^{\gamma/8}} f(x,k,t) \le$$

$$\left[\left(1+|x|^2\right) + \left(1+|k|^2\right)^{\gamma/2}\right]\frac{f(x,k,t)}{\left(1+|x|^2\right)^{(3-2b)/4}\left(1+|k|^2\right)^{\gamma/8}},$$

we get using (2.20) of **H. 3** and (2.31)

$$\int_{\mathbb{R}}\left|\left(1+|x|^2\right)^{b/2}\rho(x,t)\right|dx \le$$

$$C\left[1 + \left\|\left(1+|k|^2\right)^{\gamma/2} f\right\|_{L^\infty(\mathbb{R}\times\mathbb{R})}(t) + \left\|\left(1+|x|^2\right) f\right\|_{L^\infty(\mathbb{R}\times\mathbb{R})}(t)\right],$$

since $3-2b > 2$ and $\gamma > 4$.

Lemma A.3 - *We have* $\left(1+|x|^2\right)^{1/4} j \in L^\infty_{loc}\left([0,\infty), L^1(\mathbb{R})\right)$, *so that* $j(x,t) \to 0$ *when* $|x| \to \infty$.

Proof: By (2.17) of **H. 2**, there is C such that $|v(k)| \le C\left(1+|k|^2\right)^{1/2}$,

$$\left(1+|x|^2\right)^{1/4}|j(x,t)| \le \left(1+|x|^2\right)^{1/4}\int_{\mathbb{R}}|v(k)|\,f(x,k,t)\,dk \le$$

$$C\int_{\mathbb{R}}\left(1+|x|^2\right)^{1/4}\frac{\left(1+|k|^2\right)^{3/2}}{\left(1+|k|^2\right)} f(x,k,t)\,dk \le$$

$$C\left[\left\|\left(1+|x|^2\right) f\right\|_{L^\infty(\mathbb{R}\times\mathbb{R})}(t) + \left\|\left(1+|k|^2\right)^{\gamma/2} f\right\|_{L^\infty(\mathbb{R}\times\mathbb{R})}(t)\right]\int_{\mathbb{R}}\frac{1}{\left(1+|k|^2\right)}\,dk.$$

Lemma A.4 - *If $0 < b < 1/2$ is such that (2.20) of* **H. 3** *holds, then*

$$\left(1+|x|^2\right)^{b/2} E \in L^\infty_{loc}\left([0,\infty), L^\infty(\mathbb{R})\right).$$

Proof: By (1.2), if $x < 0$

$$\left(1+|x|^2\right)^{b/2} |E(x,t)| \le \left(1+|x|^2\right)^{b/2} \int_{\mathbb{R}} H(x-y)\, |\rho(y,t)|\, dy =$$

$$\left(1+|x|^2\right)^{b/2} \int_{-\infty}^{x} \frac{\left(1+|y|^2\right)^{b/2}}{\left(1+|y|^2\right)^{b/2}} |\rho(y,t)|\, dy \le \int_{-\infty}^{x} \left(1+|y|^2\right)^{b/2} |\rho(y,t)|\, dy.$$

By lemma A.2,

$$\left(1+|x|^2\right)^{b/2} |E(x,t)| \to 0 \qquad \text{if} \qquad x \to -\infty. \tag{A.1}$$

On the other hand,

$$\partial_x\Big(\left(1+|x|^2\right)^{b/2} E(x,t)\Big) = bx\left(1+|x|^2\right)^{(b-2)/2} E(x,t) + \left(1+|x|^2\right)^{b/2}\rho(x,t).$$

Taking into account (A.1) and corollary 3.2,

$$\left(1+|x|^2\right)^{b/2} |E(x,t)| \le \int_{-\infty}^{x} \left| by\left(1+|y|^2\right)^{(b-2)/2} E(y,t) + \left(1+|y|^2\right)^{b/2}\rho(y,t)\right| dy$$

$$\le b\, \|E\|_{L^2(\mathbb{R})}(t) \left[\int_{\mathbb{R}} \left(1+|y|^2\right)^{(b-1)}\, dy\right]^{1/2} + \int_{\mathbb{R}} \left(1+|y|^2\right)^{b/2} |\rho(y,t)|\, dy.$$

This is bounded due to $b < 1/2$ and lemma A.2.

Lemma A.5 (see [17]) *Let $\varepsilon(k)$ be such that $\varepsilon(k) \ge C|k|^\delta$ if $|k| \to \infty$ for some constants $C > 0$ and $\delta > 0$. Given a function $f \in L^p(\mathbb{R}\times\mathbb{R})$ such that $\varepsilon(k) f \in L^1(\mathbb{R}\times\mathbb{R})$, we have*

$$\hat{\rho}(x) = \int_{\mathbb{R}} f(x,k)\, dk \in L^r(\mathbb{R}) \quad \textit{with} \quad r = \frac{p-1+\delta p}{p-1+\delta}$$

and for $q = (1-p^{-1})^{-1}$

$$\|\hat{\rho}\|_{L^r(\mathbb{R})} \le \|f\|_{L^p(\mathbb{R}\times\mathbb{R})}^{\frac{\delta q}{1+\delta q}} \|\varepsilon(k) f\|_{L^1(\mathbb{R}\times\mathbb{R})}^{\frac{1}{1+\delta q}}.$$

Theorem A.1 (H-theorem [18]) *For every increasing function $\psi : \mathbb{R}^+ \to \mathbb{R}$*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} Q(f)(x,k,t)\psi\left(exp\left[\varepsilon(k)\right] \frac{f(x,k,t)}{1-f(x,k,t)}\right) dx\, dk \le 0.$$

Proof:

Set $M(x,k,t) = exp\left[\varepsilon(k)\right] \frac{f(x,k,t)}{1-f(x,k,t)}$. Using (1.7) we get

$$\int_{\mathbb{R}}\int_{\mathbb{R}} Q(f)(x,k,t)\psi\left(exp\left[\varepsilon(k)\right] \frac{f(x,k,t)}{1-f(x,k,t)}\right) dx\, dk =$$

$$\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} exp\left[-\varepsilon(k')\right] S(x,k',k)\left(1-f(x,k',t)\right)\left(1-f(x,k,t)\right)$$
$$\left\{M(x,k',t)-M(x,k,t)\right\}\left\{\psi\left(M(x,k,t)\right)-\psi\left(M(x,k',t)\right)\right\}\,dx\,dk\,dk'.$$

Taking into account that ψ is increasing, we get

$$\left\{M(x,k',t)-M(x,k,t)\right\}\left\{\psi\left(M(x,k,t)\right)-\psi\left(M(x,k',t)\right)\right\}\leq 0.$$

On the other hand, since $S(x,k',k)>0$ and $0\leq f(x,k,t)\leq 1$, inequality

$$exp\left[-\varepsilon(k')\right] S(x,k',k)\left(1-f(x,k',t)\right)\left(1-f(x,k,t)\right)\geq 0$$

follows. Thus,

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} exp\left[-\varepsilon(k')\right] S(x,k',k)\left(1-f(x,k',t)\right)\left(1-f(x,k,t)\right)$$
$$\left\{M(x,k',t)-M(x,k,t)\right\}\left\{\psi\left(M(x,k,t)\right)-\psi\left(M(x,k',t)\right)\right\} dxdkdk'\leq 0.$$

Theorem A.2 (Dunford-Pettis, see [10]) *Let T be a separable and locally compact space. A subset P in $L^1(T)$ is weakly relatively compact if and only if the following conditions hold:*

i) Boundedness: $\sup\left\{\int_T |f|\,d\mu : f\in P\right\}<\infty$

ii) $\forall\varepsilon>0\ \exists\delta>0$ *s.t. if* $|A|\leq\delta$, $A\subset T$, *then* $\sup\left\{\int_A |f|\,d\mu : f\in P\right\}\leq\varepsilon$

iii) $\forall\varepsilon>0\ \exists K$compact $\subset T$ *such that* $\sup\left\{\int_{T-K}|f|\,d\mu : f\in P\right\}\leq\varepsilon.$

Theorem A.3 (see [9], [14], [15], [17]) *Let $r(x,k,t)$ be bounded in $L^2(\mathbb{R}_x\times B_R\times\mathbb{R}_t)$ $(\forall R<\infty)$. Let $f(x,k,t)$ and $h(x,k,t)$ be bounded in $L^2(\mathbb{R}_x\times\mathbb{R}_k\times\mathbb{R}_t)$. If*

$$\partial_t f(x,k,t)+v(k)\partial_x f(x,k,t)=h(x,k,t)+\partial_k r(x,k,t)$$

and the speed $v(k)$ satisfies the hypotheses (2.17) and (2.18) of **H. 2***, then*

$$\int_{\mathbb{R}} f(x,k,t)\psi(k)\,dk \quad \textit{is bounded in} \quad H^s(\mathbb{R}_x\times\mathbb{R}_t)$$

for all $\psi\in C_0^\infty(\mathbb{R}_k)$ with $s=\beta/(4+\beta)$.

Theorem A.4 (see [17]) *If (2.16) of* **H. 2** *holds, for all $\psi\in C^0(\mathbb{R}_k)$ such that $|k|^{-\delta}\,\psi(k)\to 0$ when $|k|\to\infty$ we have*

$$\int_{\mathbb{R}} f_n(x,k,t)\psi(k)\,dk\to\int_{\mathbb{R}} \mathcal{f}(x,k,t)\psi(k)\,dk \quad \textit{in} \quad L^1\left(B_{R_x}\times(0,T)\right)\ \ \forall R_x<\infty.$$

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1. H.Andréasson. *Global Existence of Smooth Solutions in Three Dimensions for the Vlasov-Poisson-Boltzmann Equation Arising in Semiconductor Physics*, *Nonlinear Anal. Theory Methods Appl.*, **28**, 7 (1997) pp.1193-1211.

2. N. Ben Abdallah, J. Dolbeaut. *Entropies Relatives pour le Système de Vlasov-Poisson dans des Domaines Bornés,C.R.A.S. Paris Sér. I Mathématiques*, **320** (2000),pp.867-872.
3. H.Brezis. *Analyse Fonctionnelle*, (Masson, Paris, 1983).
4. L.L.Bonilla, J.A.Carrillo, J.Soler. *Asymptotic Behavior of an Initial-Boundary Value Problem for the Vlasov-Poisson-Fokker-Planck System*, *SIAM J. Appl. Math.*, **5**, 5 (1997), pp.1343-1372.
5. F.Bouchut, J.Dolbeault. *On Long Time Asymptotics of The Vlasov-Fokker-Planck Equation and of The Vlasov-Poisson-Fokker-Planck System with Coulombic and Newtonian Potentials*, *Differential and Integral Equations*, **8**, 3 (1995), pp.487-514.
6. E.Cebrián. *Existencia de Soluciones y Simulación Numérica para el Sistema de Ecuaciones de Vlasov-Poisson-Boltzmann del Modelo Cinético de Semiconductores*, (Tesina, U.C.M.,1994).
7. J.Dolbeaut. *Free Energy and Solutions of the Vlasov-Poisson-Fokker-Planck System: External Potential and Confinement (large time behavior and steady states)*, *J. Math. Pures et Appliquées*, **78** (1999), pp.121-157.
8. K.Dressler. *Steady States in Plasma Physics - The Vlasov-Fokker-Planck Equation*, *Mathematical Methods in the Applied Sciences*, **12** (1990), pp.471-487.
9. R.J.DiPerna, P.L.Lions. *Global Weak Solutions of Vlasov-Maxwell Systems*, *Comm. Pure. Appl. Math.*, **XLII** (1989) pp.729-757.
10. R.E.Edwards. *Functional Analysis. Theory and Applications*, (Hold, Rinehart and Winston. New York, 1965).
11. F.Fuchs. *Analyse Mathématique et Numérique de quelques Modèles du Transport des Charges dans les Semiconducteurs*, (Thèse de Doctorat, Universite de Nice, 1995).
12. D.Gogny, P.L.Lions. *Sur les États d'Équilibre pour les Densités Électroniques dans les Plasmas*, *RAIRO, Mod. Math. Anal. Numér.*, **23** (1989), pp.137-153.
13. F.Golse, P.L.Lions, B.Perthame, R.Sentis *'Regularity of the moments of the solution of a transport equation'. J. on Funct. Anal., 88, pp.110-125, 1988.*
14. F.Golse, F.Poupaud. *Limite Fluide des Équations de Boltzmann des Semiconducteurs pour une Statistique de Fermi-Dirac*, *J. on Asympt. Anal.*, **6** (1992), pp.135-160.
15. F.J.Mustieles. *L'Equation de Boltzmann des Semiconducteurs. Etude Mathematique et Simulation Numérique*, (Thése de Doctorat de L'Ecole Polytechnique. Paris, 1990).
16. F.J.Mustieles. *Global Existence of Solutions for the Nonlinear Boltzmann Equation of Semiconductor Physics*, *Revista Matemática iberoamericana*, **6**, 1-2 (1990), pp.43-59.
17. F.J.Mustieles. *Global Existence of Weak Solutions for a System of Non-linear Boltzmann Equation in Semiconductor Physics. Mathematical Methods in the Applied Sciences*, **14** (1991), pp.139-153.
18. F.Poupaud. *On a System of Nonlinear Boltzmann Equations of Semiconductor Physics. SIAM J. Appl. Math.*, **50**, 6 (1990), pp.1593-1606.
19. P.A.Raviart. *Lecture notes in Mathematics*, **1127**, *F. Brezzi ed.*, (Springer Verlag, Berlin 1985).
20. C.M.Snowden. *Introduction to semiconductor device modelling*, (World Scientific, Singapure, 1986).