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Asymptotic profiles for convection–diffusion equations with variable diffusion

Gema Duro[a,*,1], Ana Carpio[b,2]

[a]Departamento de Análisis Económico: Economía Cuantitativa, Universidad Autónoma de Madrid, 28049 Madrid, Spain

[b]Departamento de Matematica Aplicada, Universidad Complutense, 28040 Madrid, Spain

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Abstract

We investigate the large time behavior of solutions of the convection-diffusion equation

$$u_t - \operatorname{div}(a(x)\nabla u) = d \cdot \nabla(|u|^{q-1}u) \;\; d \in \mathbb{R}^N, \;\text{ in } (0,\infty) \times \mathbb{R}^N$$

with integrable initial data $u_0(x)$. We take $a(x) = 1 + b(x) > 0$ with b smooth and decaying to zero fast enough as $x \to \infty$. When $q > 1 + \frac{1}{N}$, it is known that the solutions behave, in a first approximation, like the solutions of the head equation taking the same initial data as $t \to \infty$. We show here the influence of the nonlinear term and the variable diffusion in the large time behavior by obtaining the second term in the asymptotic development of solutions as $t \to \infty$.

Keywords: Convection–diffusion; Long-time behavior; Variable diffusion

1. Introduction and main results

In this paper, we study the large time behavior of solutions to convection–diffusion equations of the form

$$(\mathrm{P}) \quad \begin{aligned} &u_t - \operatorname{div}(a(x)\nabla u) = \boldsymbol{d} \cdot \nabla(|u|^{q-1}u) \text{ in } (0,\infty) \times \mathbb{R}^N \\ &u(0,x) = u_0(x) \end{aligned}$$

with integrable u_0.

* Corresponding author.

E-mail addresses: gema.duro@uam.es (G. Duro), Carpio@sunma4.mat.ucm.es (A. Carpio).

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This kind of equations are related to the study of the diffusion of pollution in fluids (see [4]). We are interested here in finding out how the variable diffusion and the nonlinearity affect the solution for large times.

Most of the previous work on these equations deals only with constant diffusion (see for instance [8], [9] and the references therein). However, when studying heterogeneous media we must work with variable diffusion.

For large times, a competition between the diffusive and convective effects is observed. When $a(x)=1$, the existence of a critical value for the power q, $q_N = 1 + 1/N$, has been established such that:

- If $q > q_N$, the diffusion is dominant and the solution behaves like a self-similar solution with mass $M = \int u_0$ of the heat equation.
- If $q = q_N$, diffusion and convection have the same strength and the solution behaves like a selfsimilar solution with mass M of the full equation.
- If $1 < q < q_N$, the convection effects are dominant and the solution behaves like a self-similar solution with mass M of a reduced equation where the dissipation in some direction vanishes.

More precise results have been obtained when working in weighted spaces in [12].

For general variable diffusion coefficients $a(x)$ it is not clear up to which point a similar description holds for the long time profiles. When $a(x) > 0$ is periodic, it is possible (see [5]) to prove an analogous result for $q \geq q_N$ replacing the heat operator in the limit problem by a homogenized evolution operator. When $a(x)=1+b(x)$ with $b(x)$ small enough at infinity, one can prove (see [5] and [7]) that, in a first approximation, the long time profiles remain the same as for $a = 1$. All these results follow from classical scaling techniques thanks to sharp estimates on the decay of the solutions.

Our purpose in this paper is to develop a technique to investigate the influence of variable diffusion coefficients and nonlinear convection terms on the solution profiles for large times. We are concerned here with the model problem (P) with $q > q_N$ and $a(x) = 1+b(x) > 0$, b small at infinity, so that a first approximation to the solution is given by the solution of the heat equation with the same initial datum. An analogous situation is observed in many other contexts where equations with a similar convection–diffusion structure are involved. For instance, for incompressible Navier–Stokes equations or Vlasov–Poisson–Fokker–Plank equations, it is also known that, in a first approximation, solutions with integrable initial data behave for a large time like the solutions of the underlying linear problem (the heat equation or the free VFP equation) taking the same initial data. Of course, one would like to know more terms in the asymptotic development as $t \to \infty$ in order to clarify the deviation of the solutions of the nonlinear problem from the solution of the linear problem. The study of the large time behavior of this style (see [3]) is important in discussing the asymptotic decay profiles of the small disturbances which unavoidably exist in all realistic situations and the stability of specific solutions.

In the study of problem (P) we face two types of difficulties. Those coming from the nonlinearity and those from a linear part with a variable coefficient. As said above, we take asymptotically constant diffusion of the form $a(x) = 1 + b(x)$ with $b(x) \in L^1(\mathbb{R}^N) \cap C^{1,\alpha}(\mathbb{R}^N)$ and $\|b(x)\|_\infty < 1$. Our analysis is valid as long as the semigroup generated by $u_t - div(a(x)\nabla u) = 0$ satisfies

$$\|\nabla T(t)(u_0)\|_p \le C\|u_0\|_s t^{-(N/2)(1/s-1/p)-1/2}$$

with $C = C(p,s)$ independent of u_0, a and t. This estimate holds when $b(x)$ (see [11]) satisfies

$$|b(x)| + (1+|x|^2)^{1/2}|\nabla b(x)| \le \frac{C}{(1+|x|^2)^{\delta/2}} \quad \forall x \in \mathbb{R}^N \tag{1}$$

for some positive constants C and δ. Whether the same bound can be obtained under weaker decay assumptions on $b(x)$ is a deep question in the theory of parabolic operators that we shall not address her. Note that classical semigroup theory yields bounds with C dependent on a and its derivatives. (see [1], [2] and [10]).

For the convenience of the reader we recall some known results on the initial value problem (P) (see [7]). For every $u_0 \in L^1(\mathbb{R}^N)$ there exists a unique solution $u \in C([0,\infty);\ L^1(\mathbb{R}^N))$ to (P). Integrating the equation over $\mathbb{R}^N$ we deduce that the mass is conserved, that is, $\int_{\mathbb{R}^N} u(t,x)\,dx = \int_{\mathbb{R}^N} u_0(x)\,dx = M$, $\forall t > 0$.

On the other hand, the following L^p-estimates hold:

$$\|u(t)\|_p \le C_p\|u_0\|_1 t^{-(N/2)(1-(1/p))}, \quad \forall t > 0, \tag{2}$$

$$\|\nabla u(t)\|_p \le C_p\|u_0\|_1 t^{-(N/2)(1-(1/p))-1/2}, \quad \forall t > 1 \tag{3}$$

for every $p \in [1,\infty]$. Owing to (1), the constant C_p does not depend on b or its derivatives.

Decay rates (2) and (3) do not depend on power q and are the same as for solutions of the linear problem. In fact, neither the variable diffusion, nor the nonlinearity appear in a first approximation of the long time profile. More precisely, the following result is known [7].

Theorem 1. *Assume that* $q > 1 + 1/N$. *Let be* $u_0 \in L^1(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ *with* $r > N_q/(N+2)$ *such that* $M = \int_{\mathbb{R}^N} u_0(x)\,dx$. *Then, the solution* $u = u(t,x)$ *of* (P) *satisfies*

$$t^{(N/2)(1-1/p)}\|u(t) - MG(t)\|_p \to 0 \quad \text{as } t \to \infty \tag{4}$$

for every $p \in [1,\infty]$, *where* G *is the heat kernel.*

Moreover, if $u_0 \in L^1(\mathbb{R}^N; 1+|x|) \cap L^\infty(\mathbb{R}^N)$, *then for every* $p \in [1,\infty]$,

$$\exists C_p > 0\colon \|u(t) - MG(t)\|_p \le C_p t^{-(N/2)(1-1/p)}\varepsilon(t) \quad \forall t \ge 1 \tag{5}$$

with

$$\varepsilon(t) = \begin{cases} t^{-1/2} & \text{when } q > 1 + \frac{2}{N}, \\ t^{-1/2}\log(t+2) & \text{when } q = 1 + \frac{2}{N}, \\ t^{-(N(q-1)-1)/2} & \text{when } 1 + \frac{1}{N} < q < 1 + \frac{2}{N}. \end{cases} \tag{6}$$

Since the difference $u(t) - MG(t)$ decays at a different speed depending on the value of q, we expect the nonlinearity to affect the second term in the development. We will see here that the decay estimates (5) with (6) are sharp. On the other hand, the importance of the perturbation $b(x)$ is greater in lower dimensions, particularly in dimension 1. We prove here that the variable diffusion appears in the second term of the development as $t \to \infty$ only when $q > 1 + 2/N$ or when $N = 1$ with $q = 3$. More precisely, we obtain the following results:

Theorem 2. *Let us assume that* $1 + 1/N < q < 1 + 2/N$, $N \geq 1$. *We take* $u_0 \in L^1(\mathbb{R}^N; 1 + |x|) \cap L^{N(q-1)}(\mathbb{R}^N)$, *with* $M = \int_{\mathbb{R}^N} u_0(x)\,\mathrm{d}x$.

Then, the solution u *of* (P) *satisfies*

$$t^{(N/2)(q-1/p)-1/2} \|u(t) - MG(t) - |M|^{q-1}MZ(t)\|_p \to 0 \quad \text{as } t \to \infty \tag{7}$$

for every $p \in [1, \infty]$, *where* $Z(x,t)$ *is the solution of*

$$(\mathrm{P}_z) \qquad \begin{aligned} & z_t - \Delta z = \boldsymbol{d} \cdot \nabla(G^q) \quad \text{in } (0,\infty) \times \mathbb{R}^N, \\ & z(0,x) = 0. \end{aligned}$$

Remark 1. We denote $L^q(\mathbb{R}^N; 1 + |x|) = \{\phi \in L^q(\mathbb{R}^N); \int_{\mathbb{R}^N} [|\phi|(1 + |x|)]^q \,\mathrm{d}x < \infty\}$.

Remark 2. The solution Z of (P_z) can be calculated in terms of the heat kernel. Indeed, $Z = \boldsymbol{d} \cdot \nabla g$ where

$$\begin{aligned} & g_t - \Delta g = G^q \quad \text{in } (0,\infty) \times \mathbb{R}^N, \\ & g(0) = 0. \end{aligned}$$

Theorem 3. *Let us assume that* $q = 1 + 2/N$. *The solution* u *of* (P) *satisfies*:

1. *If* $N \geq 2$ *and* $u_0 \in L^1(\mathbb{R}^N; 1 + |x|) \cap L^q(\mathbb{R}^N; 1 + |x|) \cap L^2(\mathbb{R}^N)$, *with mass* $M = \int_{\mathbb{R}^N} u_0(x)\,\mathrm{d}x$, *then*

$$\frac{t^{(N/2)(1-1/p)+1/2}}{\log t} \|u(t) - MG(t) - \alpha|M|^{q-1}M(\log t)\boldsymbol{d} \cdot \nabla G(t)\|_p \to 0, \tag{8}$$

2. *If* $N = 1$ *and* $u_0 \in L^1(\mathbb{R}; 1 + |x|) \cap L^3(\mathbb{R}; 1 + |x|)$

$$\frac{t^{(1/2)(1-1/p)+1/2}}{\log t} \|u(t) - MG(t) - (\mathscr{K}(t) + \alpha|M|^{q-1}Md)\log t G_x(t)\|_p \to 0 \tag{9}$$

with

$$\mathscr{K}(t) = \frac{1}{\log t} \int_0^t \int_{\mathbb{R}} b(y) u_x(s,y)\,\mathrm{d}s\,\mathrm{d}y$$

as $t \to \infty$, *for every* $p \in [1,\infty]$ *where* $\alpha = q^{-(N/2)}(4\pi)^{(N/2)(1-q)}$.

Remark 3. If we further assume that $u_0 = (v_0)_x$ for some $u_0 \in L^1(\mathbb{R})$ then (see Remark 10)

$$\mathscr{K}(t) = \lim_{t\to\infty} \left\{ \frac{1}{\log t} \int_0^t \int_{\mathbb{R}} b(y) u_x(s,y)\,\mathrm{d}s\,\mathrm{d}y \right\} = 0.$$

Therefore in this case we obtain that (8) holds for every $p \in [1, \infty]$ and for $N \geq 1$.

Theorem 4. *Assume that $q > 1 + 2/N$. Then the solution u of* (P) *satisfies*
1. *If $N \geq 2$ and $u_0 \in L^1(\mathbb{R}^N; 1 + |x|) \cap L^{N(q-1)}(\mathbb{R}^N)$ then*
$$t^{(N/2)(1-1/p)+1/2}\|u(t) - MG(t) - \boldsymbol{\beta}\nabla G(t)\|_p \to 0 \tag{10}$$
as $t \to \infty$, for every $p \in [1, \infty]$, with
$$\boldsymbol{\beta} = \boldsymbol{d}\int_0^\infty \int_{\mathbb{R}^N} |u|^{q-1}u(x,t)\,\mathrm{d}x\,\mathrm{d}t - \boldsymbol{m} + \int_0^\infty \int_{\mathbb{R}^N} \nabla u(x,t)b(x)\,\mathrm{d}x\,\mathrm{d}t, \tag{11}$$
where $\boldsymbol{m} = (m_1, \ldots, m_N)$ with $m_i = \int_{\mathbb{R}^N} u_0(x)x_i\,\mathrm{d}x$.
2. *If $N = 1$, $u_0 \in L^1(\mathbb{R}; 1 + |x|) \cap L^q(\mathbb{R})$ and $u_0 = (v_0)_x$ for some $u_0 \in L^1(\mathbb{R})$. Then* (10) *holds for every $p \in [1, \infty]$ with $\boldsymbol{\beta}$ as in* (11).

Remark 4. From (10) we conclude that, when $q > 1 + 2/N$, the L^p norm of the solutions of (P) will decay at most like $t^{-(N/2)(1-1/P)+1/2}$ as time tends to infinity, no matter how fast u_0 decays as $x \to \infty$. On the contrary, for $1 + 1/N < q \leq 1 + 2/N$, (7) and (8) give no restriction on the decay rate. Taking u_0 with zero mass and decaying fast enough as x tends to infinity, we could in principle obtain a faster decay in time for u.

Remark 5. The first moment $\boldsymbol{m}$ of the initial data u_0 appears is the second term of the solution of the linear heat equation and remains in the nonlinear problem if $q > 1 + 2/N$. The term $\int_0^\infty \int_{\mathbb{R}^N} \nabla u(x,t)b(x)\,\mathrm{d}x\,\mathrm{d}t = -\int_0^\infty \int_{\mathbb{R}^N} u(x,t)\nabla b(x)\,\mathrm{d}x\,\mathrm{d}t$ does not appear when b is constant (which agrees with the result in [12]).

We are able to obtain these results by analyzing the different terms appearing in the integral equation for u, which may be written in the following form:
$$u(t) = G(t) * u_0(x) + \int_0^t \partial_i G(t-s) * b\partial_i u(s)\,\mathrm{d}s$$
$$+ d_i \int_0^t \partial_i G(t-s) * |u|^{q-1}u(s)\,\mathrm{d}s, \tag{12}$$
where $G(x,t)$ is the heat kernel. The first term is the solution to the linear heat equation with initial datum u_0, whose development is known (see [6]):
$$t^{1/2}\|G(t) * u_0 - MG(t) + m_i\partial_i G(t)\|_1 \to 0 \quad \text{as } t \to \infty \tag{13}$$
with $m_i = \int_{\mathbb{R}^N} x_i u_0$, if $u_0 \in L^1(\mathbb{R}^N; 1 + |x|)$.

Therefore, we are left with the study of the asymptotic behavior of the remaining two integrals as $t \to \infty$, which is determined by the decay rates of $u(t)$ and $u(t) - MG(t)$. In view of the different decay rates for $u(t) - MG(t)$ we will have to distinguish three cases:
$$q \in \left(1 + \frac{1}{N}, 1 + \frac{2}{N}\right), \quad q = 1 + \frac{2}{N}, \quad q > 1 + \frac{2}{N}.$$
Also, we must distinguish the cases $N \geq 2$ and $N = 1$. The reason for this has to do with the convergence of the integral $\int_1^\infty \int_{\mathbb{R}^N} |\nabla u|(x,t)|b|(x)\,\mathrm{d}x\,\mathrm{d}t$. When $N \geq 2$, we

may bound this integral by $\int_1^\infty t^{-(N/2)-(1/2)}$. When $N=1$, this bound is useless since it does not converge. However, assuming that the initial datum is a derivative we can obtain the necessary bounds.

The rest of the paper is organized as follows. In Section 2, we study the second term of the asymptotic behavior when $N \geq 2$. In Section 3, we study the second term of asymptotic behavior when $N=1$ in the case $q \in (2,3]$. We also discuss our partial results for $q>3$.

2. Proof of the results in dimension $N \geq 2$

This section is devoted to the proof of Theorems 2–4 if $N \geq 2$. We study in detail the two integrals appearing in (12)

$$w_1(t,x) = \int_0^t \partial_i G(t-s) * b\partial_i u(s)\,\mathrm{d}s, \tag{14}$$

$$w_2(t,x) = \int_0^t \partial_i G(t-s) * |u|^{q-1} u(s)\,\mathrm{d}s. \tag{15}$$

2.1. Study of $w_1(t,x)$

We prove the following proposition.

Proposition 1. *Let w_1 given by* (14) *where u is a solution to* (P) *with $u_0 \in L^1(\mathbb{R}^N) \cap L^{N(q-1)}(\mathbb{R}^N)$ for $N \geq 2$. Then,*

(a) *There exists a constant $C>0$ such that*

$$t^{(N/2)(1-1/p)+1/2} \|w_1(t)\|_p \leq C \quad \forall t \geq 1,\ p \in [1,\infty]. \tag{16}$$

(b) *For every $p \in [1,\infty]$*

$$t^{(N/2)(1-1/p)+1/2} \left\| w_1(t) - \left(\int_0^\infty \int_{\mathbb{R}^N} \partial_i u(\sigma,y) b(y)\,\mathrm{d}y\,\mathrm{d}\sigma \right) \partial_i G(t) \right\|_p \to 0 \tag{17}$$

as $t \to \infty$.

Remark 6. It follows from Proposition 1 that

$$t^{(N/2)(q-(1/p))-1/2} \|w_1(t)\|_p \to 0 \quad \text{as } t \to \infty,\ \forall p \in [1,\infty] \tag{18}$$

for $1+1/N < q < 1+2/N$ and that

$$\frac{t^{(N/2)(1-(1/p))+1/2}}{\log(t)} \|w_1(t)\|_p \to 0 \quad \text{as } t \to \infty,\ \forall p \in [1,\infty] \tag{19}$$

if $q = 1+2/N$.

Remark 7. The proof of Proposition 1 is where the restriction over the dimension $N \geq 2$ is necessary.

Remark 8. The hypothesis $u_0 \in L^{N(q-1)}(\mathbb{R}^N)$ is needed to prove that $b\nabla u \in L^1(\mathbb{R}^N \times (0,\infty))$. In the particular case $q = 1 + 2/N$ this hypothesis is reduced $u_0 \in L^2(\mathbb{R}^N)$. More precisely, we have the following lemma, which we prove after Proposition 1.

Lemma 1. *Let $u = u(t,x)$ be the solution of* (P) *with $q > 1 + 1/N$ and $u_0 \in L^1(\mathbb{R}^N) \cap L^{N(q-1)}(\mathbb{R}^N)$. Then, there exists a constant $C = C(\|u_0\|_1, \|u_0\|_{N(q-1)})$ such that*

$$\|\nabla u(t)\|_1 \leq Ct^{-1/2}, \quad \forall t > 0. \tag{20}$$

Proof of Proposition 1. (a) We split the integral as follows:

$$\begin{aligned}
&\int_0^t \partial_i G(t-s) * b\partial_i u(s)\,\mathrm{d}s \\
&\quad = \int_0^{t/2} \partial_i G(t-s) * b\partial_i u(s)\,\mathrm{d}s + \int_{t/2}^t \partial_i G(t-s) * b\partial_i u(s)\,\mathrm{d}s \\
&\quad = I_1 + I_2.
\end{aligned}$$

Applying Minkowski's inequality

$$\begin{aligned}
\|I_1\|_p &\leq \int_0^{t/2} \left\| \int_{R^N} \partial_i G(t-s, x-y) \partial_i u(s,y) b(y)\,\mathrm{d}y \right\|_p \mathrm{d}s \\
&\leq C(t-s)^{-(N/2)(1-1/p)-1/2} \|\partial_i u(s,y) b(y)\|_1.
\end{aligned}$$

Then

$$\|I_1\|_p \leq Ct^{-(N/2)(1-1/p)-1/2} \int_0^{t/2} \int_{R^N} |\partial_i u|(s,y)|b|(y)\,\mathrm{d}s\,\mathrm{d}y.$$

Taking into account estimate (3) and Lemma 1 we obtain

$$\begin{aligned}
&\int_0^\infty \int_{R^N} |\partial_i u|(s,y)|b|(y)\,\mathrm{d}s\,\mathrm{d}y \\
&\quad \leq \int_0^1 \|\nabla u(s)\|_1 \|b\|_\infty\,\mathrm{d}s + \int_1^\infty \|\nabla u(s)\|_\infty \|b\|_1\,\mathrm{d}s \\
&\quad \leq \int_0^1 s^{-1/2}\,\mathrm{d}s + \int_1^\infty s^{-(N/2)-(1/2)}\,\mathrm{d}s.
\end{aligned}$$

Therefore, we have

$$\int_0^\infty \int_{R^N} |\partial_i u|(s,y)|b|(y)\,\mathrm{d}s\,\mathrm{d}y \leq C \tag{21}$$

if $N \geq 2$. Thus, $\|I_1\|_p \leq Ct^{(-N/2)(1-1/p)-1/2}$ if $N \geq 2$.

Now, by Young's inequality

$$\|I_2\|_p \le \left\| \int_{R^N} \partial_i G(t-s, x-y)\partial_i u(s,y)b(y)\,\mathrm{d}y \right\|_p$$

$$\le C(t-s)^{-(N/2)(1/r-1/p)-1/2}\|\partial_i u(s,y)b(y)\|_r$$

for every $r \le p$. Therefore,

$$\|\partial_i u(s,y)b(y)\|_r \le \|\nabla u\|_\infty \|b(y)\|_r \le C\|\nabla u\|_\infty,$$

where ∇u satisfies (3) for $p=\infty$. Choosing $r > Np/(p+N)$ to have $-N/2(1/r - 1/p) + 1/2 > 0$ we get

$$\|I_2\|_p \le C\int_{t/2}^t (t-s)^{-(N/2)(1/r-1/p)-1/2} s^{(-N-1)/2}\,\mathrm{d}s$$

$$\le Ct^{(-N-1)/2}\int_{t/2}^t (t-s)^{-(N/2)(1/r-1/p)-1/2}\,\mathrm{d}s$$

$$\le Ct^{(-(N/2))(1-1/p)-1/2+1/2(1-N/r)}, \quad \forall t \ge t_0 > 0.$$

Therefore, choosing $r \le N$ to have $1 - N/r \le 0$ we conclude that

$$\|I_2\|_p \le Ct^{-(N/2)(1-1/p)-1/2}, \quad \forall t \ge 1.$$

(b) We must prove that

$$t^{(N/2)(1-1/p)+1/2}\left\| \int_0^t \partial_i G(t-s) * b(y)\partial_i u(s)\,\mathrm{d}s - \beta_{1,i}\partial_i G(t) \right\|_p \to 0 \tag{22}$$

as $t \to \infty$, with $\beta_{1,i} = (\int_0^\infty \int_{\mathbb{R}^N} \partial_i u(\sigma,y)b(y)\,\mathrm{d}y\,\mathrm{d}\sigma)$. Making the change of variables $x = zt^{1/2}$ we have

$$t^{(N/2)(1-1/p)+1/2}\left\| \int_0^t \int_{\mathbb{R}^N} \partial_i G(t-s, x-y)b(y)\partial_i u(s,y)\,\mathrm{d}s\,\mathrm{d}y - \beta_{1,i}\partial_i G(t,x) \right\|_{L_x^p(\mathbb{R}^N)}$$

$$= t^{(N/2)+1/2}\left\| \int_0^t \int_{\mathbb{R}^N} \partial_i G(t-s, zt^{1/2}-y)b(y)\partial_i u(s,y)\,\mathrm{d}s\,\mathrm{d}y \right.$$

$$\left. -\beta_{1,i}\partial_i G(t, zt^{1/2}) \right\|_{L_z^p(\mathbb{R}^N)}.$$

On the other hand, by the self-similar structure of $\partial_i G$ it follows that

$$\partial_i G(t-s, zt^{1/2}-y) = t^{(-N-1)/2}\partial_i G\left(1 - \frac{s}{t}, z - \frac{y}{\sqrt{t}}\right),$$

and that

$$\partial_i G(t, zt^{1/2}) = t^{(-N-1)/2}\partial_i G(1,z).$$

In the following we shall rename z by x. We remark that proving (22) is equivalent to proving

$$\left\| \int_0^t \int_{\mathbb{R}^N} \partial_i G\left(1-\frac{s}{t}, x-\frac{y}{\sqrt{t}}\right) \partial_i u(s,y) b(y) - \left(\int_0^\infty \int_{\mathbb{R}^N} \partial_i u(\sigma,y) b(y)\,\mathrm{d}y\,\mathrm{d}\sigma\right) \partial_i G(1) \right\|_p \to 0$$

as $t \to \infty$. We remark first that

$$\left(\int_0^\infty \int_{\mathbb{R}^N} \partial_i u(\sigma,y) b(y)\,\mathrm{d}y\,\mathrm{d}\sigma - \int_0^t \int_{\mathbb{R}^N} \partial_i u(\sigma,y) b(y)\,\mathrm{d}y\,\mathrm{d}\sigma\right) \partial_i G(1,x) \to 0$$

in L^p when $t \to \infty$ for every $1 \le p \le \infty$, since by (21) it follows that $b\nabla u \in L^1(\mathbb{R}^N \times (0,\infty))$ if $N \ge 2$. Therefore, it is enough to prove that

$$\left\| \int_0^t \int_{\mathbb{R}^N} \left(\partial_i G\left(1-\frac{s}{t}, x-\frac{y}{\sqrt{t}}\right) - \partial_i G(1,x)\right) \partial_i u(s,y) b(y)\,\mathrm{d}s\,\mathrm{d}y \right\|_p \to 0$$

as $t \to \infty$. To do so we split the integral as follows:

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^N} \left(\partial_i G\left(1-\frac{s}{t}, x-\frac{y}{\sqrt{t}}\right) - \partial_i G(1,x)\right) \partial_i u(s,y) b(y)\,\mathrm{d}s\,\mathrm{d}y \\
&= \int_0^{t\delta} \int_{|y|\le\delta\sqrt{t}} \left(\partial_i G\left(1-\frac{s}{t}, x-\frac{y}{\sqrt{t}}\right) - \partial_i G(1,x)\right) \partial_i u(s,y) b(y)\,\mathrm{d}s\,\mathrm{d}y \\
&\quad - \int_{t\delta}^t \int_{\mathbb{R}^N} \partial_i G(1,x) \partial_i u(s,y) b(y)\,\mathrm{d}s\,\mathrm{d}y \\
&\quad - \int_0^{t\delta} \int_{|y|>\delta\sqrt{t}} \partial_i G(1,x) \partial_i u(s,y) b(y)\,\mathrm{d}s\,\mathrm{d}y \\
&\quad + \int_{t\delta}^t \int_{\mathbb{R}^N} \partial_i G\left(1-\frac{s}{t}, x-\frac{y}{\sqrt{t}}\right) \partial_i u(s,y) b(y)\,\mathrm{d}s\,\mathrm{d}y \\
&\quad + \int_0^{t\delta} \int_{|y|>\delta\sqrt{t}} \partial_i G\left(1-\frac{s}{t}, x-\frac{y}{\sqrt{t}}\right) \partial_i u(s,y) b(y)\,\mathrm{d}s\,\mathrm{d}y \\
&= J_\delta^1 + J_\delta^2 + J_\delta^3 + J_\delta^4 + J_\delta^5,
\end{aligned}$$

with $0 < \delta < 1$.

Estimate of J_δ^1:

Applying Minkowski's inequality we have

$$\|J_\delta^1\|_{L_x^p} \le C\int_0^{t\delta}\int_{|y|<\delta\sqrt{t}}\left\|\partial_i G\left(1-\frac{s}{t},x-\frac{y}{\sqrt{t}}\right) - \partial_i G(1,x)\right\|_{L_x^p} |\nabla u|(s,y)|b|(y)\,\mathrm{d}s\,\mathrm{d}y.$$

Thanks to the continuity of translations in L^p and the continuity with respect to t, given $\varepsilon > 0$, we can choose $0 < \delta < 1$ such that

$$\mathrm{Sup}_{s\le t\delta,|y|\le\delta\sqrt{t}}\left\|\partial_i G\left(1-\frac{s}{t},x-\frac{y}{\sqrt{t}}\right) - \partial_i G(1,x)\right\|_{L_x^p} \le \varepsilon.$$

Therefore,

$$\|J_\delta^1\|_{L_x^p} \le C\varepsilon\int_0^{t\delta}\int_{|y|\le\sqrt{t}\delta}|\nabla u|(s,y)|b|(y)\,\mathrm{d}s\,\mathrm{d}y$$

$$\le C\varepsilon\int_0^{t\delta}\int_{\mathbb{R}^N}|\nabla u|(s,y)|b|(y)\,\mathrm{d}s\,\mathrm{d}y.$$

By (21), it follows that if $N\ge 2$, $\|J_\delta^1\|_{L_x^p}\le C\varepsilon$ where C is a constant independent of t.

Estimate of J_δ^2:

We have

$$\|J_\delta^2\|_{L_x^p} \le C\|\nabla G(1)\|_{L_x^p}\int_{t\delta}^t\int_{\mathbb{R}^N}|\nabla u|(s,y)|b|(y)\,\mathrm{d}s\,\mathrm{d}y \le C\int_{t\delta}^t\|\nabla u(s)\|_\infty\|b\|_1\,\mathrm{d}s.$$

Taking into account (3) and $b\in L^1(\mathbb{R}^N)$ it follows that

$$\|J_\delta^2\|_p \le C\int_{t\delta}^t s^{-(N/2)-1/2} \le C(t\delta)^{(1-N)/2}\to 0$$

as $t\to\infty$ if $N\ge 2$.

Estimate of J_δ^3:

We have

$$\|J_\delta^3\|_{L_x^p} \le \|\nabla G(1)\|_{L_x^p}\int_0^{t\delta}\int_{|y|\ge\delta\sqrt{t}}|\nabla u|(s,y)|b|(y)\,\mathrm{d}s\,\mathrm{d}y.$$

By (21) we have that $b\nabla u\in L^1(\mathbb{R}^N\times(0,\infty))$ if $N\ge 2$. Therefore, we can deduce that

$$\int_0^{t\delta}\int_{|y|\ge\delta\sqrt{t}}|\nabla u|(s,y)|b|(y)\,\mathrm{d}s\,\mathrm{d}y\to 0 \quad \text{as } t\to\infty.$$

Thus $\|J_\delta^3\|_{L_x^p}\to 0$ as $t\to\infty$.

Estimate of J_δ^4:

We have

$$\|J_\delta^4\|_{L_x^p} \le \int_{t\delta}^{t} \left\| \int_{\mathbb{R}^N} \partial_i G\left(1-\frac{s}{t}, x-\frac{y}{\sqrt{t}}\right) \partial_i u(s,y) b(y)\,\mathrm{d}y \right\|_{L_x^p} \mathrm{d}s.$$

Thus,

$$\begin{aligned}
&\left\| \int_{\mathbb{R}^N} \partial_i G\left(1-\frac{s}{t}, x-\frac{y}{\sqrt{t}}\right) \partial_i u(s,y) b(y)\,\mathrm{d}y \right\|_{L_x^p} \mathrm{d}s \\
&\quad = \left\| \int_{\mathbb{R}^N} \partial_i G\left(1-\frac{s}{t}, x-y\right) \partial_i u(s,y\sqrt{t}) b(y\sqrt{t}) t^{(N/2)}\,\mathrm{d}y \right\|_p \\
&\quad \le C\left(1-\frac{s}{t}\right)^{(-N/2)(1/r-1/p)-1/2} \|\partial_i u(s,y\sqrt{t}) b(y\sqrt{t}) t^{(N/2)}\|_r
\end{aligned}$$

for every $r \le p$. Since

$$\begin{aligned}
\|\partial_i u(s,y\sqrt{t}) b(y(\sqrt{t}) t^{(N/2)}\|_r &= t^{(N/2)-(N/2)r} \|\partial_i u(s,y) b(y)\|_r \\
&\le t^{(N/2)-(N/2)r} \|\nabla u\|_\infty \|b(y)\|_r \\
&\le C t^{(N/2)-(N/2)r} \|\nabla u\|_\infty
\end{aligned}$$

it follows that

$$\begin{aligned}
\|J_\delta^4\|_p &\le C t^{(N/2)-(N/2)r} \int_{t\delta}^{t} \left(1-\frac{s}{t}\right)^{(-N/2)(1/r-1/p)-1/2} \|\nabla u(s,y)\|_\infty \,\mathrm{d}s \\
&\le C t^{1/2-(N/2)r} \int_{\delta}^{1} (1-t)^{(-N/2)(1/r-1/p)-1/2}\,\mathrm{d}t \le C t^{1/2-(N/2)r}
\end{aligned}$$

taking $N > r > Np/(p+N)$ in order to have $-N/2(1/r-1/p)+1/2 > 0$. If $1-N/r < 0$, we conclude that $\|J_\delta^4\|_p$ tends to zero when $t \to \infty$ and δ is fixed.

Estimate of J_δ^5:

As in estimate J_δ^4 but taking $r = 1$ we get

$$\begin{aligned}
\|J_\delta^5\|_{L_x^p} &\le \int_0^{t\delta} \left\| \int_{|y|\ge\delta\sqrt{t}} \partial_i G\left(1-\frac{s}{t}, x-\frac{y}{\sqrt{t}}\right) \partial u(s,y) b(y)\,\mathrm{d}y \right\|_{L_x^p} \mathrm{d}s \\
&\le C \int_0^{t\delta} \left(1-\frac{s}{t}\right)^{(-N/2)(1-1/p)-1/2} \|\nabla u(s,y) b(y)\|_{L^1(|y|\ge\delta\sqrt{t})} \,\mathrm{d}s \\
&\le C \int_0^{t\delta} \int_{|y|\ge\delta\sqrt{t}} |\nabla u|(s,y)|b|(y)\,\mathrm{d}y\,\mathrm{d}s \to 0
\end{aligned}$$

as $t \to \infty$, since $b\nabla u \in L^1(\mathbb{R}^N \times (0,\infty))$ if $N \ge 2$.

The proof of Proposition 1 is now completed. □

Proof of Lemma 1. The rescaled functions $u_\lambda(t,x) = \lambda^N u(\lambda^2 t, \lambda x)$ solve

$$\begin{aligned} &u_{\lambda,t} - \operatorname{div}(a(\lambda x)\nabla u_\lambda) = \lambda^{N(1-q)+1} d \cdot \nabla(|u_\lambda|^{q-1}u_\lambda) \quad \text{in } (0,\infty)\times\mathbb{R}^N, \\ &u_\lambda(0,x) = u_{0,\lambda}(x) = \lambda^N u_0(\lambda x) \end{aligned} \tag{23}$$

and, therefore, for $s > 0$ fixed, they satisfy the integeral equation

$$u_\lambda(t+s) = T_\lambda(t)[u_\lambda(s)] + \lambda^{N(1-q)+1}\int_0^t T_\lambda(t-\tau) d \cdot [\nabla(F(u_\lambda(\tau+s)))]\,\mathrm{d}\tau,$$

where $F(u_\lambda) = |u_\lambda|^{q-1}u_\lambda$ and $T_\lambda(t)$ is the semigroup generated by the linear diffusion equation with coefficient $a_\lambda(x) = a(\lambda x)$.

Taking gradients in the integral equation we obtain

$$\begin{aligned} \nabla u_\lambda(t+s) &= \nabla(T_\lambda(t)[u_\lambda(s)]) \\ &\quad + \lambda^{N(1-q)+1}\int_0^t \nabla(T_\lambda(t-s) d \cdot [\nabla(F(u_\lambda(\tau+s)))])\,\mathrm{d}\tau. \end{aligned} \tag{24}$$

On the other hand, from the estimates in [11] we have that (see Proposition 1 of [7])

$$\|\nabla(T_\lambda(t)[v])\|_p \le C\|v\|_s t^{(-N/2)(1/s-1/p)-1/2}, \quad \forall t > 0 \tag{25}$$

with $C > 0$ independent of λ.

Taking $L^p(\mathbb{R}^N)$-norms in (24) we get,

$$\begin{aligned} \|\nabla u_\lambda(t+s)\|_1 &\le C\|u_0\|_1 t^{-1/2} + |d|C\lambda^{N(1-q)+1}q \\ &\quad \times \int_0^t (t-\tau)^{-1/2}\|u_\lambda(s+\tau)\|_\infty^{q-1}\|\nabla u_\lambda(s+\tau)\|_1\,\mathrm{d}\tau \end{aligned}$$

for every $\lambda > 0$ and $t > 0$. By (2) we deduce that

$$\|u_\lambda(t)\|_\infty \le Ct^{-(N/2)p}\lambda^{N(1-1/p)}\|u_0\|_p \quad \forall t \ge 0,\ \lambda > 0$$

with $C > 0$ independent of λ. Therefore

$$\|u_\lambda(\tau+s)\|_\infty \le Cs^{-(N/2)p}\lambda^{N(1-1/p)}\|u_0\|_p \quad \forall \tau \ge 0,\ \lambda > 0.$$

Taking $p = N(q-1)$ we obtain

$$\begin{aligned} \|\nabla u_\lambda(t+s)\|_1 &\le C\|u_0\|_1 t^{-1/2} + |d|C\lambda^{N(1-q)+1}q \\ &\quad \times \int_0^t (t-\tau)^{-1/2}\|u_\lambda(s+\tau)\|_\infty^{q-1}\|\nabla u_\lambda(s+\tau)\|_1\,\mathrm{d}\tau \\ &\le C\|u_0\|_1 t^{-1/2} + |d|Cq\|u_0\|_{N(q-1)}^{q-1}s^{-1/2} \\ &\quad \times \int_0^t (t-\tau)^{-1/2}\|\nabla u_\lambda(s+\tau)\|_1\,\mathrm{d}\tau \end{aligned}$$

for every $t > 0$ and $\lambda > 0$.

Applying Gronwall's Lemma we deduce for $t = s$

$$\|\nabla u_\lambda(2s)\|_1 \le C_s \quad \forall \lambda > 0 \text{ if } q > 1 + \frac{1}{N},$$

which is equivalent to (20). □

2.2. Study of $w_2(t,x)$

We prove here the following proposition:

Proposition 2. *Let w_2 be given by* (38) *where u is a solution to* (P) *with initial datum u_0. Then*

(a) *If $q \in (1 + 1/N, 1 + 2/N)$ and $u_0 \in L^1(\mathbb{R}^N)$,*

$$t^{(N/2)(q-1/p)-1/2}\|w_2(t) - |M|^{q-1}Mz(t)\|_p \to 0, \quad \text{as } t \to \infty \tag{26}$$

for every $p \in [1,\infty]$, where $z(t,x)$ is a solution of (P_z).

(b) *If $q = 1 + 2/N$ and $u_0(x) \in L^1(\mathbb{R}^N; 1 + |x|) \cap L^q(\mathbb{R}^N; 1 + |x|)$,*

$$\frac{t^{(N/2)(1-1/p)+1/2}}{\log(t)}\|w_2(t) - \alpha|M|^{q-1}M\log(t)d \cdot \nabla G(t)\|_p \to 0 \quad \text{as } t \to \infty \tag{27}$$

for every $p \in [1,\infty]$, where $\alpha = q^{(-N/2)}(4\pi)^{(N/2)(1-q)}$.

(c) *If $q > 1 + 2/N$ and $u_0 \in L^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$,*

$$t^{(N/2)(1-1/p)+1/2}\left\|w_2(t) - d\left(\int_0^\infty \int_{\mathbb{R}^N} \|u\|^{q-1}u(t,x)\,\mathrm{d}x\,\mathrm{d}t\right) \cdot \nabla G(t)\right\|_p \to 0 \tag{28}$$

as $t \to \infty$, for every $p \in [1,\infty]$.

Remark 9. Proposition 2 is also satisfied in the dimension $N = 1$.

Proof of Proposition 2. (a) We introduce the scaling $w_\lambda(t,x) = \lambda^N w_2(\lambda^2 t, \lambda x)$. Note that w_λ satisfies

$$(\mathrm{P}_\lambda) \qquad \begin{aligned} &w_{\lambda,t} - \Delta w_\lambda = \lambda^{N(1-q)+1} d \cdot \nabla(|u_\lambda|^{q-1}u_\lambda) \quad \text{in } (0,\infty) \times \mathbb{R}^N, \\ &w_\lambda(0,x) = 0 \end{aligned}$$

and that we can rewrite the integral expression for w_λ,

$$w_\lambda(t) = \lambda^{N(1-q)+1} d_i \int_0^t \partial_i G(t-s) * |u_\lambda|^{q-1}u_\lambda(s)\,\mathrm{d}s.$$

We see that proving (26) is equivalent to proving

$$\lambda^{-N(1-q)-1}w_\lambda(1) \to d_i \int_0^1 \partial_i G(1-s) * |MG|^{q-1}(MG)(s)\,\mathrm{d}s,$$

in $L^p(\mathbb{R}^N)$ when $\lambda \to \infty$. We have

$$\left\| \int_0^1 \partial_i G(1-s) * \{|u_\lambda|^{q-1} u_\lambda(s) - |MG|^{q-1}(MG)(s)\} \, ds \right\|_p$$

$$\leq \int_0^{1/2} \|\nabla G(1-s)\|_p \||u_\lambda|^{q-1} u_\lambda(s) - |MG|^{q-1}(MG)(s)\|_1 \, ds$$

$$+ \int_{1/2}^1 \|\nabla G(1-s)\|_1 \||u_\lambda|^{q-1} u_\lambda(s) - |MG|^{q-1}(MG)(s)\|_p \, ds$$

$$= I_\lambda^1 + I_\lambda^2.$$

We see that $I_\lambda^1 \to 0$ as $\lambda \to \infty$. Taking into account (2) and using classical estimates on the heat kernel we have

$$\|\nabla G(1-s)\|_p \||u_\lambda|^{q-1} u_\lambda(s) - |MG|^{q-1}(MG)(s)\|_1$$

$$\leq C_p (1-s)^{-(N/2)(1-1/p)-1/2} S^{-(N/2)(q-1)} \in L^1(0, \tfrac{1}{2}), \quad \forall \lambda > 0, \tag{29}$$

since $N/2(q-1) < 1$. On other hand, from [8] we have

$$u_\lambda(s) \to MG(s) \quad \text{in } L^p(\mathbb{R}^N) \quad \text{as } \lambda \to \infty \tag{30}$$

for every $p \in [1, \infty]$ and for every $s \in (0, \frac{1}{2})$. Combining (29) and (30) and applying the dominated convergence theorem we deduce $I_\lambda^1 \to 0$ as $\lambda \to \infty$.

To prove that $I_\lambda^2 \to 0$ as $\lambda \to \infty$ we proceed in a similar way as in estimate I_λ^1, taking into account that

$$\|\nabla G(1-s)\|_1 \||u_\lambda|^{q-1} u_\lambda(s) - |MG|^{q-1}(MG)(s)\|_p$$

$$\leq C_p (1-s)^{-1/2} s^{-(N/2)(q-1/p)} \in L^1(\tfrac{1}{2}, 1), \quad \forall \lambda > 0.$$

(b) The proof is analogous to the proof of section (b) of Proposition 1. Making the change of variable $x = zt^{1/2}$ and taking into account the self-similar structure $\partial_i G$ we remark that

$$\frac{t^{(N/2)(1-(1/p))+1/2}}{\log(t)} \left\| d_i \int_0^t \partial_i G(t-s) * |u|^{q-1} u(s) \, ds \right.$$

$$\left. - \alpha |M|^{q-1} M (\log t) d_i \partial_i G(t) \right\|_p \to 0,$$

when $t \to \infty$ it is equivalent to

$$\left\| \int_0^t \int_{\mathbb{R}^N} \partial_i G\left(1 - \frac{s}{t}, x - \frac{y}{\sqrt{t}}\right) \frac{|u|^{q-1} u(s,y)}{\log t} \, ds \, dy - \partial_i G(1,x) \alpha |M|^{q-1} M \right\|_p \to 0, \tag{31}$$

when $t \to \infty$. On the other hand, assuming that

$$\lim_{t\to\infty}\left\{\frac{1}{\log t}\int_0^t\int_{\mathbb{R}^N}|u|^{q-1}u\right\} = |M|^{q-1}M\int G(1)^q = |M|^{q-1}M\alpha, \tag{32}$$

proving (31) is equivalent to proving that

$$\left\|\int_0^t\int_{\mathbb{R}^N}\partial_i G\left(1-\frac{s}{t}, x-\frac{y}{\sqrt{t}}\right)\frac{|u|^{q-1}u(s,y)}{\log t}\,\mathrm{d}s\,\mathrm{d}y \right.$$
$$\left. -\partial_i G(1,x)\lim_{t\to\infty}\left\{\frac{1}{\log t}\int_0^t\int_{\mathbb{R}^N}|u|^{q-1}u\right\}\right\|_p$$

tends to zero as $t \to \infty$.

Therefore, it is enough to prove that

$$\int_0^t\int_{\mathbb{R}^N}\partial_i G\left(1-\frac{s}{t}, x-\frac{y}{\sqrt{t}}\right)\frac{|u|^{q-1}u(s,y)}{\log t}\,\mathrm{d}s\,\mathrm{d}y$$
$$-\partial_i G(1,x)\int_0^t\int_{\mathbb{R}^N}\frac{|u|^{q-1}u(s,y)}{\log t}\,\mathrm{d}s\,\mathrm{d}y$$

and

$$\partial_i G(1,x)\int_0^t\int_{\mathbb{R}^N}\frac{|u|^{q-1}u(s,y)}{\log t}\,\mathrm{d}s\,\mathrm{d}y - \partial_i G(1,x)\lim_{t\to\infty}\left\{\frac{1}{\log t}\int_0^t\int_{\mathbb{R}^N}|u|^{q-1}u(s,y)\right\}$$

tend to zero in L_x^p when $t \to \infty$. If (32) holds then the last convergence is clear. We must only prove the first convergence and (32).

To prove (32) it is sufficient to prove that

$$I \equiv s\int_{\mathbb{R}^N}(|u|^{q-1}u(s,y) - |M|^{q-1}MG^q(s,y))\,\mathrm{d}y \to 0 \quad \text{as } s \to \infty. \tag{33}$$

We first observe that

$$||u|^{q-1}u(s,y) - |M|^{q-1}MG^q(s,y)| \le C[|u|^{q-1} + |MG|^{q-1}]|u(s,y) - MG(s,y)|.$$

Therefore,

$$\||u|^{q-1}u(s) - |M|^{q-1}MG^q(s)\|_1 \le C[\|u\|_\infty^{q-1} + \|MG\|_\infty^{q-1}]\|u(s) - MG(s)\|_1.$$

Since $q-1 = 2/N$ we have $\|u\|_\infty^{q-1} \le Cs^{-1}$ and $\|G\|_\infty^{q-1} \le Cs^{-1}$ so that

$$s\||u|^{q-1}u(s) - |M|^{q-1}MG^q(s)\|_1 \le C\|u(s) - MG(s)\|_1.$$

Using (7) for $p = 1$, we conclude that (33) holds. We proceed now to prove that

$$\int_0^t\int_{\mathbb{R}^N}\partial_i G\left(1-\frac{s}{t}, x-\frac{y}{\sqrt{t}}\right)\frac{|u|^{q-1}u(s,y)}{\log t}\,\mathrm{d}s\,\mathrm{d}y$$
$$-\partial_i G(1,x)\int_0^t\int_{\mathbb{R}^N}\frac{|u|^{q-1}u(s,y)}{\log t}\,\mathrm{d}s\,\mathrm{d}y$$

tends to zero in L_x^p when $t \to \infty$. We decompose the integral as follows:

$$\int_0^t \int_{\mathbb{R}^N} \left(\partial_i G\left(1-\frac{s}{t}, x-\frac{y}{\sqrt{t}}\right) - \partial_i G(1,x)\right) \frac{|u|^{q-1}u(s,y)}{\log t}\, \mathrm{d}s\, \mathrm{d}y$$

$$= J_\delta^1 + J_\delta^2 + J_\delta^3 + J_\delta^4 + J_\delta^5,$$

where each J_δ^i is the same integral as in section (b) of Proposition 1, but replacing $\partial_i u(s,y)b(y)$ by $|u|^{q-1}u(s,y)/\log s$.

Estimate of J_δ^1:

From (2) we deduce that

$$\|u(t)\|_p \le C_p(t+1)^{(-N/2)(1-1/p)}, \quad \forall t>0, \tag{34}$$

when $u_0 \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ since then $u \in C([0,\infty); L^p(\mathbb{R}^N))$.

Repeating the same argument as in J_δ^1 of section (b) of Proposition 1 and applying (34) for $q = 1+2/N$ it follows that

$$\|J_\delta^1\|_{L_x^p} \le \varepsilon \int_0^{t\delta} \int_{|y|\le\delta\sqrt{t}} \frac{|u|^q(s,y)}{\log t}\, \mathrm{d}s\, \mathrm{d}y \le \varepsilon \int_0^{t\delta} \frac{(s+1)^{-1}}{\log t}\, \mathrm{d}s \le \varepsilon C,$$

uniformly with respect to t for $t \ge t_0 > 0$.

Estimate of J_δ^2:

We have

$$\|J_\delta^2\|_{L_x^p} \le C\|\nabla G(1)\|_{L_x^p} \int_{t\delta}^t \int_{\mathbb{R}^N} \frac{|u|^q(s,y)}{\log t}\, \mathrm{d}s\, \mathrm{d}y$$

$$\le \|\nabla G(1)\|_{L_x^p} \int_{t\delta}^t \frac{(s+1)^{-1}}{\log t}\, \mathrm{d}s \le C\frac{\log((1+t)/(1+t\delta))}{\log t}.$$

Therefore, it tends to zero when $t\to\infty$ and δ is fixed.

Estimate of J_δ^4:

In the same way as for estimate J_δ^4, from Proposition 1 we have

$$\|J_\delta^4\|_{L_x^p} \le \frac{C}{\log t} \int_{t\delta}^t \left\| \int_{\mathbb{R}^N} \partial_i G\left(1-\frac{s}{t}, x-\frac{y}{\sqrt{t}}\right) |u|^{q-1}u(s,y)\, \mathrm{d}y \right\|_{L_x^p} \mathrm{d}s$$

$$\le \frac{Ct^{(N/2)(1-1/r)}}{\log t} \int_{t\delta}^t \left(1-\frac{s}{t}\right)^{-(N/2)(1/r-1/p)-1/2} s^{-(Nq/2)(1-1/qr)}$$

if $p \ge r$. Since $q = 1+2/N$, we deduce

$$\|J_\delta^4\|_{L_x^p} \le \frac{Ct^{-1}}{\log t} \int_{t\delta}^t \left(1-\frac{s}{t}\right)^{(-N/2)(1/r-1/p)-1/2} \mathrm{d}s$$

$$\le \frac{C}{\log t} \int_\delta^1 (1-t)^{(-N/2)(1/r-1/p)-1/2}\, \mathrm{d}t \le \frac{C}{\log t}$$

if $p \ge r > Np/(p+N)$.

Therefore, we conclude that $\|J_\delta^4\|_p \le C/\log t \to 0$ when $t\to\infty$ for δ fixed.
Estimate of J_δ^3:
We have

$$\|J_\delta^3\|_{L_x^p} \le \|\nabla G(1)\|_{L_x^p} \int_0^{t\delta}\int_{|y|\ge\delta\sqrt{t}} \frac{|u|^q(s,y)}{\log t}\,\mathrm{d}s\,\mathrm{d}y.$$

and we must prove

$$\int_0^{t\delta}\int_{|y|\ge\delta\sqrt{t}} \frac{|u|^q(s,y)}{\log t} \to 0 \quad \text{as } t\to\infty \tag{35}$$

for δ fixed. Since we are integrating in $|y|/\sqrt{t}\ge\delta$ we have

$$\int_0^{t\delta}\int_{|y|\ge\delta\sqrt{t}} \frac{|u|^q(s,y)}{\log t} \le C\int_0^{t\delta}\int_{|y|\ge\delta\sqrt{t}} \frac{|u(s,y)-G*u_0(s,y)|^q}{\log t}$$
$$+C\int_0^{t\delta}\int_{|y|\ge\delta\sqrt{t}} \frac{|y|^q G*u_0(s,y)|^q}{t^{q/2}\log t} \equiv J_1^3 + J_2^3.$$

Therefore to prove (35) it is sufficient to prove that J_1^3 and J_2^3 tend to zero when $t\to\infty$. For this we need the following Lemmae, which we prove later.

Lemma 2. *Take* $q=1+2/N$ *and* $u_0(x)\in L^1(\mathbb{R}^N;1+|x|)$. *Then*,

$$\||y|[G*u_0](t,y)\|_q^q \le C(t^{(q/2)-1}+t^{-1}) \le Ct^{(q/2)-1}, \quad \forall t\ge 1. \tag{36}$$

Lemma 3. *Let* $u(t,x)$ *be the solution of* (P). *Then*, *if* $u_0\in L^1(\mathbb{R}^N;1+|x|)\cap L^p(\mathbb{R}^N)$

$$(t+1)^{(N/2)(1-1/p)+\varepsilon}\|u(t)-G*u_0(t)\|_p \le C, \quad \forall t\ge 0 \tag{37}$$

for some $\varepsilon>0$.

Thanks to Lemma 2 for J_2^3 we have

$$\int_0^{t\delta}\int_{|y|\ge\delta\sqrt{t}} \frac{|y|^q|G*u_0|^q(s,y)}{t^{q/2}\log t} \le C\frac{(t\delta)^{q/2}}{t^{q/2}\log t} \le \frac{C}{\log t}.$$

Applying Lemma 3 with $p=q=1+\frac{2}{N}$, to J_1^3, we get that:

$$\int_0^{t\delta}\int_{|y|\ge\delta\sqrt{t}} \frac{|u(s,y)-G*u_0(s,y)|^q}{\log t} \le \frac{C}{\log t}\int_0^{t\delta}(1+t)^{-1-\varepsilon} \le \frac{C}{\log t}.$$

Therefore, for δ fixed, we obtain $\|J_\delta^3\|_{L_x^p}\to 0$ as $t\to\infty$.

Estimate of J_δ^5:

We have

$$\|J_\delta^5\|_{L_x^p} \le \int_0^{t\delta} \left\| \int_{|y|\ge\delta\sqrt{t}} \partial_i G\left(1-\frac{s}{t}, x-\frac{y}{\sqrt{t}}\right) \frac{|u|^{q-1}u(s,y)}{\log t} dy \right\|_{L_x^p} \mathrm{d}s$$

$$\le C\int_0^{t\delta} \left(1-\frac{s}{t}\right)^{(-N/2)(1-1/p)-1/2} \left\| \frac{|u|^{q-1}u(s,y)}{\log t} \right\|_{L^1(|y|\ge\delta\sqrt{t})} \mathrm{d}s$$

$$\le C\int_0^{t\delta} \int_{|y|\ge\delta\sqrt{t}} \frac{|u|^q u(s,y)}{\log t} \,\mathrm{d}y\,\mathrm{d}s \to 0,$$

as $t\to\infty$ by (35). This concludes the proof of section (b). □

(c) It is analogous to the proof of section (b) of Proposition 1.

Proof of Lemma 2. We have

$$\||y|[G*u_0](t,y)\|_q^q = \int_{\mathbb{R}^N} |y|^q \left| \int_{\mathbb{R}^N} G(t,y-x)u_0(x)\,\mathrm{d}x \right|^q \mathrm{d}y$$

$$\le C\int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} |y-x| G(t,y-x)u_0(x)\,\mathrm{d}x \right|^q \mathrm{d}y$$

$$+C\int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} G(t,y-x)|x|u_0(x)\,\mathrm{d}x \right|^q \mathrm{d}y$$

$$\le \||\cdot|G(t,\cdot)\|_q^q \|u_0\|_1^q + Ct^{-1}\||x|u_0\|_1^q \le C\||\cdot|G(t,\cdot)\|_q^q + Ct^{-1}$$

if $u_0, |x|u_0 \in L^1(\mathbb{R}^N)$.

Since $\||\cdot|G(t,\cdot)\|_q^q \le Ct^{(-Nq/2)(1-1/q)+q/2}$, we obtain for $q=1+2/N$ that $\||\cdot|G(t,\cdot)\|_{(N+2)/N}^{(N+2)/N} \le Ct^{((N+2)/2N)-1}$. Therefore, (36) holds.

Proof of Lemma 3. Taking into acount

$$\|u(t)-G*u_0(t)\|_p \le \|u(t)-MG(t)\|_p + \|G*u_0(t)-MG(t)\|_p$$

by (5), we deduce that Lemma 3 is satisfied. □

3. Proof of the results in dimension $N=1$

This section is devoted to the proofs of Theorems 2, 3, 4 if $N=1$. Our results on the second term of asymptotic behavior in the case $q>3$ are partial in the sense that the hypothesis on u_0 are too strong. Again, the behavior of u follows from the behavior of the three terms in the integral equation (12). The only difference with the case $N\ge 2$

arises in the study of

$$w_1(t,x) = \int_0^t \int_{\mathbb{R}} G_x(t-s, x-y) b(y) u_x(s,y) \, dy \, ds. \tag{38}$$

Next, we develop this integral when $N = 1$.

3.1. *Case $q \leq 3$*

In this section we prove Theorems 2, 3 for $q \leq 3$ and $N = 1$. As said above, we only need to study the integral term w_1.

Lemma 4. *If $u_0 \in L^1(\mathbb{R}) \cap L^{q-1}(\mathbb{R})$ there exists a constant $C > 0$ such that*

$$\|w_1(t)\|_p \leq C t^{-(1/2)(1-(1/p))-1/2} \log t \tag{39}$$

for every $p \in [1, \infty]$.

Proposition 3. *Let us assume that $u_0 \in L^1(\mathbb{R}) \cap L^{q-1}(\mathbb{R})$. Then,*
(a) *if $q = 3$ and $u_0 \in L^3(\mathbb{R})$ then, for every $p \in [1, \infty]$,*

$$\frac{t^{(1/2)(1-1/p)+1/2}}{\log(t)} \|w_1(t) - \mathscr{K}(t) \log(t) G_x(t)\|_p \to 0 \quad as \quad t \to \infty, \tag{40}$$

(b) *if $2 < q < 3$, for every $p \in [1, \infty]$*

$$t^{(1/2)(q-1/p)-1/2} \|w_1(t)\|_p \to 0 \quad as \quad t \to \infty. \tag{41}$$

Proof of Lemma 4. We decompose the integral as follows:

$$\int_0^t G_x(t-s) * b u_x(s) \, ds$$
$$= \int_0^{t/2} G_x(t-s) * b u_x(s) \, ds + \int_{t/2}^t G_x(t-s) * b u_x(s) \, ds = I_1 + I_2.$$

We have

$$\|I_1\|_p \leq \int_0^{t/2} \left\| \int_{\mathbb{R}} G_x(t-s, x-y) u_x(s,y) b(y) \, dy \right\|_p ds.$$

On the other hand,

$$\left\| \int_{\mathbb{R}} G_x(t-s, x-y) u_x(s,y) b(y) \, dy \right\|_p \leq C (t-s)^{-(1/2)(1-1/p)-1/2} \|u_x(s,y) b(y)\|_1.$$

Thus,

$$\|I_1\|_p \le C\int_0^{t/2}(t-s)^{-(1/2)(1-1/p)-1/2}\|u_x(s,y)b(y)\|_1\,\mathrm{d}s$$

$$\le Ct^{-(1/2)(1-1/p)-1/2}\int_0^{t/2}\int_{\mathbb{R}}|u_x|(s,y)|b|(y)\,\mathrm{d}s\,\mathrm{d}y.$$

Since u_x satisfies estimates (3) and (20), if we fix $t_0\in(0,t/2)$ it follows that

$$\int_0^{t/2}\int_{\mathbb{R}}|u_x|(s,y)|b|(y)\,\mathrm{d}s\,\mathrm{d}y \le \int_0^{t_0}\|u_x(s)\|_1\|b\|_\infty ds + \int_{t_0}^{t/2}\|u_x(s)\|_\infty\|b\|_1$$

$$\le C_1 + C_2\log t \tag{42}$$

$C_1 = C(t_0)$ and C_2 being positive constants. Therefore, $\|I_1\|_p \le Ct^{-(1/2)(1-1/p)-1/2}\log t$.

To prove that $\|I_2\|_p \le Ct^{-(1/2)(1-1/p)-1/2}\log t$ we repeat the same argument as in estimate I_2 of section (a) in Proposition 1. □

Proof of Proposition 3. (a) We want to prove that

$$\frac{t^{(1/2)(1-1/p)+1/2}}{\log(t)}\|w(t)-\mathscr{K}(t)\log(t)G_x(t)\|_p \to 0 \quad \text{as } t\to\infty, \tag{43}$$

where

$$\mathscr{K}(t) = \frac{1}{\log t}\int_0^t\int_{\mathbb{R}} b(y)u_x(s,y)\,\mathrm{d}s\,\mathrm{d}y.$$

Making the change of variable $x = zt^{1/2}$ and taking into account the self-similar structure of we see that (43) is equivalent to proving

$$\left\|\int_0^t\int_{\mathbb{R}}\left(G_x\left(1-\frac{s}{t},x-\frac{y}{\sqrt{t}}\right)-G_x(1,x)\right)\frac{u_x(s,y)b(y)}{\log t}\,\mathrm{d}s\,\mathrm{d}y\right\|_p \to 0,$$

as $t\to\infty$. We decompose the integral as follows:

$$\int_0^t\int_{\mathbb{R}}\left(G_x\left(1-\frac{s}{t},x-\frac{y}{\sqrt{t}}\right)-G_x(1,x)\right)\frac{u_x(s,y)b(y)}{\log t}\,\mathrm{d}s\,\mathrm{d}y$$

$$=\int_0^{t\delta}\int_{|y|\le\delta\sqrt{t}}\left(G_x\left(1-\frac{s}{t},x-\frac{y}{\sqrt{t}}\right)-G_x(1,x)\right)\frac{u_x(s,y)b(y)}{\log t}\,\mathrm{d}s\,\mathrm{d}y$$

$$-\int_{t\delta}^t\int_{\mathbb{R}}G_x(1,x)\frac{u_x(s,y)b(y)}{\log t}\,\mathrm{d}s\,\mathrm{d}y$$

$$-\int_0^{t\delta}\int_{|y|>\delta\sqrt{t}}G_x(1,x)\frac{u_x(s,y)b(y)}{\log t}\,\mathrm{d}s\,\mathrm{d}y$$

$$+\int_{t\delta}^{t}\int_{\mathbb{R}} G_x\left(1-\frac{s}{t},x-\frac{y}{\sqrt{t}}\right)\frac{u_x(s,y)b(y)}{\log t}\,\mathrm{d}s\,\mathrm{d}y$$

$$+\int_{0}^{t\delta}\int_{|y|>\delta\sqrt{t}} G_x\left(1-\frac{s}{t},x-\frac{y}{\sqrt{t}}\right)\frac{u_x(s,y)b(y)}{\log t}\,\mathrm{d}s\,\mathrm{d}y$$

$$=J_\delta^1+J_\delta^2+J_\delta^3+J_\delta^4+J_\delta^5$$

with $0<\delta<1$. Now, we estimate $J_\delta^i,\ i=1,\dots,5$.
Estimate of $J_{\delta,\lambda}^1$:

$$\|J_\delta^1\|_p\le C\int_0^{t\delta}\int_{|y|\le\delta\sqrt{t}}\left\|G_x\left(1-\frac{s}{t},x-\frac{y}{\sqrt{t}}\right)\right.$$

$$\left.-G_x(1,x)\right\|_{L_x^p}\frac{|u_x|(s,y)|b|(y)}{\log(t)}\,\mathrm{d}s\,\mathrm{d}y.$$

Given $\varepsilon>0$, we can choose $0<\delta<1$ such that

$$\sup_{s\le t\delta,|y|\le\delta\sqrt{t}}\left\|G_x\left(1-\frac{s}{t},x-\frac{y}{\sqrt{t}}\right)-G_x(1,x)\right\|_{L_x^p}\le\varepsilon.$$

Therefore,

$$\|J_\delta^1\|_{L_x^p}\le C\varepsilon\int_0^{t\delta}\int_{|y|\le\delta\sqrt{t}}\frac{|u_x|(s,y)|b|(y)}{\log(t)}\,\mathrm{d}s\,\mathrm{d}y.$$

Taking into account (42) it follows that

$$\int_0^{t\delta}\int_{\mathbb{R}}|u_x|(s,x)|b|(y)\,\mathrm{d}s\,\mathrm{d}y\le C_1+C_2\log(t\delta).$$

Therefore, $\|J_\delta^1\|_{L_x^p}\le C\,\varepsilon$ where C is a constant independent of $t\ge1$ if δ is chosen small enough in the function of ε.
Estimate of J_δ^2:

$$\|J_\delta^2\|_{L_x^p}\le C\|G_x(1)\|_{L_x^p}\int_{t\delta}^{t}\int_{\mathbb{R}}\frac{|u_x|(s,y)|b|(y)}{\log(t)}\,\mathrm{d}s\,\mathrm{d}y$$

$$\le C\int_{t\delta}^{t}\frac{\|u_x(s)\|_\infty\|b\|_1}{\log(t)}\,\mathrm{d}s\le C\int_{t\delta}^{t}\frac{s^{-1}}{\log(t)}\,\mathrm{d}s\le C\frac{\log(t/t\delta)}{\log(t)}$$

that tends to zero when $t\to\infty$, for δ fixed.
Estimate of J_δ^3:

$$\|J_\delta^3\|_{L_x^p}\le\|G_x(1)\|_{L_x^p}\int_0^{t\delta}\int_{|y|\ge\delta\sqrt{t}}\frac{|u_x|(s,y)|b|(y)}{\log(t)}\,\mathrm{d}s\,\mathrm{d}y.$$

Taking $t_0 \in (0, t\delta)$ we have

$$\int_0^{t\delta} \int_{|y| \geq \delta\sqrt{t}} \frac{|u_x|(s,y)|b|(y)}{\log(t)} \, \mathrm{d}y$$

$$\leq \int_0^{t_0} \frac{\|u_x\|_1}{\log(t)} \|b(y)\|_\infty \, \mathrm{d}s + \int_{t_0}^{t\delta} \frac{\|u_x\|_\infty}{\log(t)} \int_{|y| \geq \delta\sqrt{t}} |b|(y) \, \mathrm{d}y.$$

By (3)

$$\|J_\delta^3\|_{L_x^p} \leq \frac{C}{\log t} + C \frac{\log(t\delta/t_0)}{\log(t)} \|b(y)\|_{L^1(|y| > \delta\sqrt{t})}.$$

Therefore, as $b \in L^1(\mathbb{R})$, we have $\|J_\delta^3\|_{L_x^p} \to 0$ as $t \to \infty$.
Estimate of J_δ^4:

$$\|J_\delta^4\|_p \leq \int_{t\delta}^t \int_{\mathbb{R}} \left\| G_x \left(1 - \frac{s}{t}, x - \frac{y}{\sqrt{t}}\right) \right\|_{L_x^p} \frac{|u_x|(s,y)|b|(y)}{\log(t)} \, \mathrm{d}s \, \mathrm{d}y$$

$$\leq \frac{C}{\log(t)} \int_{t\delta}^t \left(1 - \frac{s}{t}\right)^{-(1/2)(1-1/p)-1/2} s^{-1} \, \mathrm{d}s \leq \frac{C}{\log(t)} \to 0.$$

Estimate of J_δ^5:

$$\|J_\delta^5\|_{L_x^p} \leq \int_0^{t\delta} \left\| \int_{|y| \geq \delta\sqrt{t}} G_x \left(1 - \frac{s}{t}, x - \frac{y}{\sqrt{t}}\right) \frac{u_x(s,y)b(y)}{\log(t)} \, \mathrm{d}y \right\|_{L_x^p} \mathrm{d}s$$

$$\leq \int_0^{t\delta} \left(1 - \frac{s}{t}\right)^{-(1/2)(1-1/p)-1/2} \left\| \frac{u_x(s,y)b(y)}{\log(t)} \right\|_{L^1(|y| \geq \delta\sqrt{t})}$$

$$\leq C_\delta \int_0^{t\delta} \int_{|y| \geq \delta\sqrt{t}} \frac{|u_x|(s,y)|b|(y)}{\log(t)} \, \mathrm{d}y \, \mathrm{d}s \to 0,$$

as $t \to \infty$, repeating the same argument as in the estimate J_δ^3. Thus (43) holds.

(b) By Lemma 4

$$\frac{t^{(1/2)(1-1/p)+1/2}}{\log(t)} \|w_2(t)\|_p \leq C$$

for every $p \in [1, \infty]$. Therefore, $t^{(1/2)(q-1/p)-1/2} \|w_2(t)\|_p \leq C t^{(1/2)(q-3)} \log(t) \to 0$ as $t \to \infty$ if $2 < q < 3$. □

3.2. *Case* $q > 3$

In this section we prove Theorem 4 for $N = 1$ and $q > 3$.

Proposition 4. *If* $u_0 \in L^1(\mathbb{R}) \cap L^q(\mathbb{R})$ *and* $u_0 = (v_0)_x$ *for some* $v_0 \in L^1(\mathbb{R})$, *then*

$$t^{(1/2)(1-1/p)+1/2} \left\| w(t) - \left(\int_0^\infty \int_{\mathbb{R}} u_x(\sigma, y) b(y)\, \mathrm{d}y\, \mathrm{d}\sigma \right) G_x(t) \right\|_p \to 0 \tag{44}$$

as $t \to \infty$, *for every* $p \in [1, \infty]$.

Remark 10. Since $\int_0^\infty \int_{\mathbb{R}} |b|(y)|u_x|(s,y)\, \mathrm{d}s\, \mathrm{d}y \leq \infty$, now

$$\mathscr{K} = \lim_{\sigma \to \infty} \left\{ \frac{1}{\log \sigma} \int_0^\sigma \int_{\mathbb{R}} b(y) u_x(s,y)\, \mathrm{d}s\, \mathrm{d}y \right\} = 0.$$

Proof of Proposition 4. If $bu_x \in L^1(\mathbb{R}^N \times (0,\infty))$ we repeat the same argument as was used in the proof of section (b) of Proposition 1.

Therefore, we only have to prove that $bu_x \in L^1(\mathbb{R}^N \times (0,\infty))$.

We estimate separately $\int_0^a \int_{\mathbb{R}} |b(x)u_x(t)|\, \mathrm{d}x\, \mathrm{d}s$ and $\int_a^\infty \int_{\mathbb{R}} |b(x)u_x(t)|\, \mathrm{d}x\, \mathrm{d}s$. For the first integral, we have

$$\int_0^a \int_{\mathbb{R}} |b(x)u_x(t)|\, \mathrm{d}x\, \mathrm{d}s \leq \int_0^a \|b(x)\|_\infty \|u_x(t)\|_1\, \mathrm{d}s \leq C \int_0^a t^{-1/2} \leq C.$$

Let us show that

$$\int_0^\infty \int_{\mathbb{R}} |b(x)u_x(t)|\, \mathrm{d}x\, \mathrm{d}s \leq C.$$

To do this, we use the fact that u satisfies the integral equation

$$u(t) = \hat{v}(t) + d \int_0^t T(t-\tau)[\partial_x(F(u(\tau,x)))]\, \mathrm{d}\tau, \tag{45}$$

where $F(u) = |u|^{q-1}u$, with $q > 3$, $\hat{v}(t) = T(t)u_0$ and $T(\cdot)$ is the semigroup generated by the linear equation

$$\hat{v}_t - (a(x)\hat{v}_x)_x = 0,$$

$$\hat{v}(0,x) = u_0(x). \tag{46}$$

Differentiating the integral equation we get

$$u_x(t) = \hat{v}_x(t) + d \int_0^t \partial_x[T(t-\tau)\partial_x(F(u(\tau,x)))]\, \mathrm{d}\tau.$$

First, we see that

$$\left\| \int_0^t \partial_x[T(t-\tau)\partial_x(F(u(\tau,x)))]\, \mathrm{d}\tau \right\|_{L_x^p} \leq C t^{-1/2(1-1/p)-1}. \tag{47}$$

We split the integral as follows:

$$J = \int_0^t \partial_x[T(t-\tau)\partial_x(F(u(\tau,x)))]\,d\tau = J_1 + J_2$$

$$= \int_0^{t/2} \partial_x[T(t-\tau)\partial_x(F(u(\tau,x)))]\,d\tau + \int_{t/2}^t \partial_x[T(t-\tau)\partial_x(F(u(\tau,x)))]\,d\tau.$$

Owing to estimates (2) and (3), it follows that

$$\|J_2\|_p \le C\int_{t/2}^t (t-\tau)^{-1/2}\|u(\tau)\|_\infty^{q-1}\|u_x(\tau)\|_p\,d\tau$$

$$\le C\int_{t/2}^t (t-\tau)^{-1/2}\tau^{-(q-1)/2}\tau^{-(1/2)(1-1/p)-1/2}\,d\tau$$

$$\le Ct^{-((q-1)/2)-(1/2(1-1/p))-1/2}t^{1/2} = Ct^{-(1/2(1-1/p))-1/2}t^{-(q-2)/2}.$$

As a consequence, we deduce that $\|J_2\|_p \le Ct^{-(1/2)(1-1/p)-1/2}t^{-(q-2)/2}$ with $q>3$. In the interval $[0,t/2]$ we have that

$$J_1 = \int_0^{t/2} \partial_x[T(t-\tau)\partial_x(F(u(\tau,x)))]\,d\tau = -\int_0^{t/2}\int_{\mathbb{R}} S_{\xi,x}(t-\tau,x,\xi)F(u(\tau,\xi))\,d\xi\,d\tau,$$

where S is the fundamental solution associated with (46).

We need the following lemma.

Lemma 5. *Let S be the fundamental solution for Eq.* (46). *Then*

$$\|S_{\xi,x}(t-\tau,x,\xi)\|_p \le C(t-\tau)^{-(1/2)(1-1/p)-1}$$

for all $p\in[1,\infty]$.

Proof of Lemma 5. Given $(\tau,\xi)\in(0,\infty)\times\mathbb{R}^N$, let $S=S(t,\tau,x,\xi)$ be the solution of

$$u_t - (a(x)u_x)_x = 0 \quad \text{in } (0,\infty)\times\mathbb{R},$$
$$u(\tau) = \delta_0(x-\xi),$$

where $\langle\delta_0(x-\xi),\psi(x)\rangle = \psi(\xi)$ for every $\psi\in BC(\mathbb{R})$.

Therefore $z=S_\xi$ satisfies

$$\begin{aligned} &z_t - (a(x)z_x)_x = 0 \quad \text{in } (0,\infty)\times\mathbb{R}\\ &z(\tau) = \delta_0'(x-\xi). \end{aligned} \tag{48}$$

Making $v_x = S_\xi$, v verifies

$$v_t - a(x)v_{xx} = 0 \quad \text{in } (0,\infty)\times\mathbb{R}$$
$$v(\tau) = \delta_0(x-\xi). \tag{49}$$

On the other hand, applying Theorem 4.7 of [11] to (49) it follows that

$$\|v_x(t)\|_q = \|z(t)\|_q \le C(t-\tau)^{-(1/2)(1-1/q)-1/2}. \tag{50}$$

If we consider $(t+\tau)/2$ we have

$$\left\| z\left(\frac{t+\tau}{2}\right)\right\|_q \le C\left(\frac{t-\tau}{2}\right)^{-(1/2)(1-1/q)-1/2}.$$

On the other hand, z satisfies

$$z_t - (a(x)z_x)_x = 0 \quad \text{in } (0,\infty)\times\mathbb{R},$$
$$z\left(\frac{t+\tau}{2}\right) = f \in L^q(\mathbb{R})$$

for any $q \in [1,\infty]$.

Applying again Theorem 4.7 of [11] to the solution of this problem we obtain

$$\|z_x(t)\|_p \le C\left(\frac{t-\tau}{2}\right)^{-(1/2)(1/q-1/p)-1/2} \|f\|_q$$

and therefore by (50) we have

$$\|z_x(t)\|_p = \|S_{\xi,x}(t-\tau)\|_p \le C(t-\tau)^{-(1/2)(1/q-1/p)-(1/2)}(t-\tau)^{-(1/2)(1-1/q)-1/2}.$$

This concludes the proof of Lemma 5. □

Then, thanks to Lemma 5 we obtain

$$\|J_1\|_p \le C\int_0^{t/2}(t-\tau)^{-(1/2)(1-1/p)-1}\|u(\tau)\|_q^q\,\mathrm{d}\tau$$
$$\le C\left(\frac{t}{2}\right)^{(-1/2)(1-1/p)-1}\int_0^{t/2}(\tau+1)^{-1/2(q-1)}\,\mathrm{d}\tau$$

if $u_0 \in L^q(\mathbb{R})$. Therefore, $\|J_1\|_p \le Ct^{-(1/2)(1-1/p)-1}$ since $q>2$. Thus,

$$\int_a^\infty\int_{\mathbb{R}}|b(x)||J|\,\mathrm{d}x\,\mathrm{d}s \le \int_a^\infty \|b(x)\|_1\|J\|_\infty\,\mathrm{d}x\,\mathrm{d}s \le C\int_a^\infty t^{-3/2} \le C.$$

To conclude the proof of Proposition 4 we prove that

$$\int_a^\infty \int_{\mathbb{R}} |b(x)\hat{v}_x(t)| \, \mathrm{d}x \, \mathrm{d}s \le C.$$

Since $\hat{v}$ is the solution of the linear problem, we have

$$\hat{v}_x(t) = \partial_x[T(t)u_0] = \int_{\mathbb{R}} S_x(t,0,x,\xi)u_0(\xi)\, \mathrm{d}\xi.$$

Now, we suppose that $u_0 = (v_0)_x$ for some $v_0 \in L^1(\mathbb{R})$ so that

$$\int_{\mathbb{R}} S_x(t,0,x,\xi)u_0(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}} S_{\xi,x}(t,0,x,\xi)v_0(\xi)\,\mathrm{d}\xi.$$

Therefore by Lemma 5

$$\|\hat{v}_x(t)\|_p \le Ct^{-1/2(1-1/p)-1}\|v_0\|_1. \tag{51}$$

Taking into account (51) we deduce

$$\int_a^\infty \int_{\mathbb{R}} |b(x)\hat{v}_x(t)|\,\mathrm{d}x\,\mathrm{d}s \le C\int_a^\infty \|b(x)\|_1 \|\hat{v}_x(t)\|_\infty \,\mathrm{d}s$$

$$\le C\int_a^\infty t^{-3/2} \le C. \qquad \square$$

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