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LARGE TIME BEHAVIOR IN INCOMPRESSIBLE NAVIER-STOKES EQUATIONS *

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Abstract. We give a development up to the second order for strong solutions u of incompressible Navier-Stokes equations in $\mathbb{R}^n$, $n \geq 2$. By combining estimates obtained from the integral equation with a scaling technique we prove that, for initial data satisfying some integrability conditions (and small enough, if $n \geq 3$), u behaves like the solution of the heat equation taking the same initial data as u plus a corrector term that we compute explicitely.

Key words. Incompressible Navier-Stokes equations, strong solutions, large time behaviour, asymptotic development, heat equation.

AMS subject classifications. 76D05, 35B40, 35K05.

0. Introduction and main results.

This paper is devoted to the study of the large time behavior of the solutions of the incompressible Navier-Stokes equations in the whole space $\mathbb{R}^n$, $n \geq 2$:

$$(NS) \qquad \begin{cases} u_t - \Delta u + u^i \partial_i u + \nabla p = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^n \\ u(x) \to 0 & |x| \to \infty \\ div\ u = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^n \\ u(x,0) = u_0,\ div\ u_0 = 0 & \text{in } \mathbb{R}^n \end{cases}$$

where $u = (u^1, ..., u^n)$ stands for the velocity of the fluid and p for its pressure.

Let us first recall some facts on solutions of (NS). It is known that for any initial data $u_0 \in (L^2(\mathbb{R}^n))^n$ with $div\ u_0 = 0$ global weak solutions of (NS) exist. This was first proved by Leray ([18], [19]) for $n \leq 3$ and then by Hopf [13] for all n by means of a Galerkin method. By a weak solution of (NS) we mean a function u such that:

$$u \in C_{weak}([0,\infty); (L^2(\mathbb{R}^n))^n) \quad \text{with} \quad div\ u\ = 0$$

$$\partial_i u \in L^2(0,\infty; (L^2(\mathbb{R}^n))^n) \quad i = 1, ..., n$$

$$< u(0), \varphi(0) >= -\int_0^\infty < u, \varphi_t > dt + \int_0^\infty < \nabla u, \nabla \varphi > dt + \int_0^\infty < u \cdot \nabla u, \varphi > dt$$

for every $\varphi \in (C_c^\infty([0,\infty) \times \mathbb{R}^n))^n$ with $div\ \varphi = 0$, where $< >$ denotes the scalar product in $(L^2(\mathbb{R}^n))^n$. From now on, we shall drop the superscrip n and denote by X both the spaces X and X^n.

Weak solutions are known to be unique and smooth (hence, strong) when $n = 2$. For higher dimensions, uniqueness and smoothness remain open problems.

Besides the Leray-Hopf construction there are several methods to prove the existence of weak solutions (see [3], [17], [26]) in $\mathbb{R}^n$. They all construct strong solutions

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u_k of some approximating problems which converge weakly in $L^2_{\rm loc}(0,\infty;H^1(\mathbb{R}^n))$ and strongly in $L^2_{\rm loc}(\mathbb{R}^+\times\mathbb{R}^n)$ to some weak solution u of (NS). Those u_k fulfill the energy inequality, that is,

$$\|u_k(t)\|^2_{L^2(\mathbb{R}^n)} + 2\int_s^t \|\nabla u_k(\sigma)\|^2_{L^2(\mathbb{R}^n)} d\sigma \le \|u_k(s)\|^2_{L^2(\mathbb{R}^n)}$$

for $0 \le s \le t$. In case $n \le 4$ the energy inequality also holds for the limit u for every $t > s$, a. e. $s > 0$ and $s = 0$ (see [14], [26]). Following Leray's terminology, those weak solutions verifying the energy inequality are called turbulent.

For initial data $u_0 \in H^{\frac12}(\mathbb{R}^n)$ (see [16]) or $u_0 \in L^p(\mathbb{R}^n)$, $p > n$ (see [10], [11], [2], [8]) or $u_0 \in L^n(\mathbb{R}^n)$ (see [14]) strong local solutions are known to exist. They turn out to be global if the norm of the initial data is small. We call them strong since they belong to classes where regularity and uniqueness hold. Besides, they satisfy the equation in the classical sense and both the energy inequality and the associated integral equation are satisfied.

There are some uniqueness criteria allowing to relate strong and weak solutions, provided they both verify the energy inequality. For instance, if a weak solution u is known to fulfill the energy inequality and we have a strong solution $w \in C([0,T];(L^n(\mathbb{R}^n))^n)$ (see [25]) or $w \in L^r(0,T;L^q(\mathbb{R}^n))$ (see [23]) for some adequate q, r then w agrees with the weak solution u on $[0,T]$.

We must distinguish the cases $n > 2$ and $n = 2$. For $n = 2$ we shall study the asymptotic behavior of weak solutions of (NS) without smallness assumptions on the data. When $n > 2$ we are concerned with the study of the asymptotic behavior of global strong solutions of (NS) like those constructed in [14] with data in $L^n(\mathbb{R}^n)$ of small norm. If u_0 belongs also to some $L^p(\mathbb{R}^n)$ with $1 < p \le n$, Kato obtains decay rates for the $L^q(\mathbb{R}^n)$ norms, $q \ge p$, similar to those which hold for the heat equation (see also [2]), provided that $\frac{n}{2}(\frac1p - \frac1q) < 1$, which always holds when $n = 2$. We shall remove this restriction and extend the decay estimates to reach the case $p = 1$.

Let us consider first the case $n = 2$. In this case, weak solutions of (NS) turn out to be also strong. For data in $L^p \cap L^2(\mathbb{R}^n)$, $1 \le p \le 2$ an argument involving the use of Fourier transforms allows to prove (see [26], [17]) that the weak solutions of (NS) behave in L^2 like the solutions of the heat equation with the same initial data when $t \to \infty$. We extend this result to L^q with $q \ne 2$:

Theorem 1. *Let u be a weak solution of the two-dimensional (NS) with initial data $u_0 \in L^p \cap L^2(\mathbb{R}^2)$, $1 \le p \le 2$ such that div $u_0 = 0$. Then, for any $q \ge p$,*

i) If $1 < p < 2$

$$\|G(t) * u_0 - u(t)\|_q \le Ct^{-\frac2p+\frac1q+\frac12} \quad t > 0$$

ii) If $p = 1$

$$\|G(t) * u_0 - u(t)\|_q \le Ct^{\frac{-3}{2}+\frac1q} log\, t \quad t > 0$$

iii) If $p = 2$

$$t^{\frac12-\frac1q}\|u(t)\|_q \to 0 \quad as\ t \to \infty$$

where we denote by $G(t)$ the heat kernel.

In this theorem and in the sequel, C denotes a positive constant independient of time.

In cases i) and ii) $\|G(t)*u_0\|_q$ decays at a slower rate than the powers appearing in the right hand side when $t\to\infty$. Therefore, we may say that $G(t)*u_0$ is the first term in the asymptotic development of u when $t\to\infty$.

Both i) and ii) follow easily from the integral equation satisfied by u thanks to the decay estimates on the L^q norms of u obtained by Kato for data of small L^2 norm. Since the L^2 norm of u is known to tend to zero as $t\to\infty$ we can spare this smallness hypothesis.

Kato's estimates together with the covergence to zero of the L^2 norm yield iii), which also holds for $\|G(t)*u_0\|_q$. In fact, at least when $q=2$, this decay estimate turns out to be optimal for both the heat and Navier-Stokes equations in the sense that no uniform decay rate can be found for the L^2 norm of solutions with initial data $u_0\in L^2(\mathbb{R}^2)$ (see [21]). However, we ignore whether it is possible to find functions $g(t)$ and $\delta(t)$ with $\delta(t)\to\infty$ as $t\to\infty$ such that

$$\delta(t)\, t^{\frac{1}{2}-\frac{1}{q}}\|u(t)-g(t)\|_q\to 0 \quad as\ t\to\infty$$

under the only assumption $u_0\in L^2(\mathbb{R}^2)$ (see remarks in Section 1.2). The first term in this case (other than 0) is unknown.

In some cases we can make the above result more precise:

Theorem 2. *Let u be a solution of the two dimensional (NS) with initial data $u_0\in L^p\cap L^2(\mathbb{R}^2)$ $\frac{6}{5}<p<2$ and v a solution of*

$$(\mathcal{L}_2)\quad \begin{cases} v_t-\Delta v=-h^i\partial_i h-\partial_j\nabla E_2 * h^i\partial_i h^j & in\ \mathbb{R}^+\times\mathbb{R}^2\\ div\ v=0 & in\ \mathbb{R}^+\times\mathbb{R}^2\\ v(x,0)=u_0,\ div\ u_0=0 & in\ \mathbb{R}^2\end{cases}$$

where $h(t)=G(t)*u_0$ is the solution of the heat equation with data u_0 and E_2 stands for the fundamental solution of $-\Delta$ in $\mathbb{R}^2$. Then, for any $q\ge p$ we have

$$\|(u-v)(t)\|_q\le Ct^{-\frac{3}{p}+\frac{1}{q}+1}$$

We see that, when $\frac{6}{5}<p<2$, the function v, that can be written as:

$$v(t)=G(t)*u_0-\int_0^t G(t-s)*(h^i\partial_i h+\partial_j\nabla E_2 * h^i\partial_i h^j)(s)ds$$

that is, $G(t)*u_0$ plus a corrector term $I(t)$, approaches u better that $G(t)*u_0$. The restriction on p is needed to make some integrals finite when estimating the difference by using the integral equations.

For initial data $u_0\in L^p\cap L^2(\mathbb{R}^2)$ with $1<p\le\frac{6}{5}$, Theorem 2 implies

$$\|(u-v)(t)\|_q\le Ct^{-\frac{3}{r}+\frac{1}{q}+1}$$

when $q\ge r$, for any $\frac{6}{5}<r<2$. This decay is faster that the decay $t^{-\frac{2}{p}+\frac{1}{q}+\frac{1}{2}}$ observed for $\|G(t)*u_0-u(t)\|_q$. Therefore, the second term in the development, in norm L^q, $q\ge\frac{6}{5}$, is again $I(t)$.

When $p=1$, Theorem 1 yields for $\|G(t)*u_0-u(t)\|_q$, $q\ge 1$, the decay rate $Ct^{\frac{-3}{2}+\frac{1}{q}}log\ t$. By Theorem 2 we get the slower decay rate $Ct^{-\frac{3}{r}+\frac{1}{q}+1}$ with $r>\frac{6}{5}$ for $\|v(t)-u(t)\|_q$ $q\ge r$. However, in this case $p=1$ and provided some integrability hypotheses on the data are added, we can use a scaling technique to find the second term in the development of u. We have:

THEOREM 3. *Set $q \geq 1$. Let u be a solution of the two dimensional (NS) with initial data $u_0 \in (L^2 \cap L^1)(\mathbb{R}^2, 1+|x|)$ such that $div\ u_0 = 0$ and $u_0 \in L^{2r}(\mathbb{R}^2)$ for some r satisfying $q \geq r > \frac{2q}{q+2}$. We set $M = \int_{\mathbb{R}^2} u_0(x)dx = (M^1, M^2)$ and $E_2 = \frac{1}{2\pi} log\ |x|$. Then,*

$$\frac{t^{\frac{3}{2}-\frac{1}{q}}}{log\ t} \|u(t) - MG(t) + R(t)\|_q \to 0 \quad as\ t \to \infty$$

where

$$R(t) = log\ t \left(\frac{M^i M}{2} \partial_i G(t) \ + \ \frac{M^i M^j}{2} \partial_i G(t) * \nabla \partial_j E_2 \right)$$

This result remains true when replacing u with the solution v of $(\mathcal{L}_2)$, so that $R(t)$ is also the second term in the development of v. The term $MG(t)$ comes from $G(t) * u_0$ while $R(t)$ is the contribution due to the integral term $I(t)$. Note that, formally, we have set $h(t) = MG(t)$ in the expresion of $I(t)$. It is clear then that, for u_0 as in Theorem 3,

$$\frac{t^{\frac{3}{2}-\frac{1}{q}}}{log\ t} \|u(t) - v(t)\|_q \to 0 \quad as\ t \to \infty$$

so that, again, v approaches u better than $G(t) * u_0$.

When the mass of the initial data is zero, Theorem 3 reduces to

$$\frac{t^{\frac{3}{2}-\frac{1}{q}}}{log\ t} \|u(t)\|_q \to 0 \quad as\ t \to \infty$$

It was proved in [22] that, if $u_0 \in L^1(\mathbb{R}^2, 1 + |x|^2) \cap L^2(\mathbb{R}^2, |x|^{\frac{1}{2}}) \cap H(\mathbb{R}^2)$, where $H(\mathbb{R}^2)$ denotes the closure in L^2 of $C_0^\infty(\mathbb{R}^2) \cap \{u\ s.t.\ div\ u\ = 0\}$ and the mass of u_0 is zero, that is $\mathcal{F}u_0 = 0$, then $\|u(t)\|_2 \leq C(1 + t)^{-1}$. Solutions may decay faster, even exponentially, depending on the order of the zero of $\mathcal{F}u$ at time zero.

We shall also extend the above theorems to higher dimensions:

THEOREM 4. *For $n \geq 3$, let u be a global strong solution of (NS) with data $u_0 \in L^p \cap L^n(\mathbb{R}^n)$, $1 \leq p < n$, of L^n norm small enough and such that $div\ u_0 = 0$. Then, for $q \geq p$ we have*

$$\|G(t) * u_0 - u(t)\|_q \leq C t^{(-\frac{2}{p}+\frac{1}{q})\frac{n}{2}+\frac{1}{2}}$$

For weak solutions satisfying the energy inequality (or that can be approached by solutions of approximating problems verifying it) the above decay estimate on $\|G(t) * u_0 - u(t)\|_q$ was known to hold for $q = 2$ and $1 \leq p \leq 2$ (see [17], [26]). For divergence free data u_0 belonging to $L^2 \cap L^n(\mathbb{R}^n)$ with small $L^n(\mathbb{R}^n)$ norm there exists a unique strong global solution and at least one global weak solution. They both agree when the weak solution satisfies the energy inequality. This kind of weak solutions are known to exist when $n \leq 4$ (see for instance [26]). Therefore, Theorem 4 extends the results known for weak solutions.

Denoting by $(\mathcal{L}_n)$ the n dimensional analogous of the problem $(\mathcal{L}_2)$ introduced before we get that in some cases the solution v of $(\mathcal{L}_n)$ approaches u better than $G(t) * u_0$, furnishing the second term in the development of u:

THEOREM 5. *For $n \geq 3$, let u be a global strong solution of (NS) with data $u_0 \in L^p \cap L^n(\mathbb{R}^n)$, $\frac{3n}{n+3} < p < n$, whose L^n norm is small enough and such that $div\ u_0 = 0$. Then, for any $q \geq p$ we have*

$$\|(u-v)(t)\|_q \leq Ct^{(-\frac{3}{p}+\frac{1}{q})\frac{n}{2}+1}$$

where v is the solution of $(\mathcal{L}_n)$ with data u_0.

Taking $u_0 \in L^p \cap L^n(\mathbb{R}^n)$, $\frac{2n}{n+2} < p \leq \frac{3n}{n+3}$, of small L^n norm such that $div\ u_0 = 0$, we conclude that

$$\|(u-v)(t)\|_q \leq Ct^{(-\frac{3}{r}+\frac{1}{q})\frac{n}{2}+1}$$

for $q \geq r$ and $\frac{3n}{n+3} < r < n$. This decay rate is faster than the observed for $\|G(t) * u_0 - u(t)\|_q$. However, we ignore what happens when $p \leq q \leq \frac{3n}{n+3}$ or $1 < p \leq \frac{2n}{n+2}$.

Theorems 4 and 5 are obtained estimating the norms by using the integral equations and the known decay estimates. We can handle the case $p = 1$ by using a scaling technique, provided some integrability hypotheses are added:

THEOREM 6. *For $n \geq 3$, let u be a strong solution of (NS) with data $u_0 \in L^1(\mathbb{R}^n, 1+|x|) \cap L^n(\mathbb{R}^n)$, $1 \leq p \leq n$, of L^n norm small enough and such that $div\ u_0 = 0$ and $q \geq 1$. If $u_0 \in L^{2r}(\mathbb{R}^n)$ for some $q \geq r > \frac{nq}{q+n}$ then*

$$t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\|u(t) - MG(t) + m^i\partial_i G(t) + R(t)\|_q \to 0$$

as $t \to \infty$, where

$$M = \int_{\mathbb{R}^n} u_0(x)dx; \quad m_i = \int_{\mathbb{R}^n} x^i u_0(x)dx,\ i = 1, ...n$$

$$R(t) = \Big(\int_0^\infty \int_{\mathbb{R}^n} u^i u(\sigma, y)dyd\sigma\Big)\ \partial_i G(t) + \Big(\int_0^\infty \int_{\mathbb{R}^n} u^i u^j(\sigma, y)dyd\sigma\Big)\partial_i G(t) * \nabla\partial_j E_n$$

and E_n stands for the fundamental solution of $-\Delta$ in $\mathbb{R}^n$.

The same result holds if we replace u by v so that, in particular, we see that v approaches u better than $G(t) * u_0$. Both Theorem 3 and Theorem 6 extend to (NS) the results proved in [27] for the following scalar convection-diffusion equations:

$$u_t - \Delta u + a|u| \cdot \nabla u = 0 \quad in\ \mathbb{R}^+ \times \mathbb{R}^n, \quad a \in \mathbb{R}^n,$$

where the same difference between the case $n = 2$ and $n \geq 3$ appears. Remark that the solutions we are dealing with satisfy the decay estimate $\|u(t)\|_2 \leq C(1+t)^{\frac{-n}{4}}$. When $n \geq 3$ this decay ensures the existence of $\int_0^\infty \int_{\mathbb{R}^n} u^i u^j(\sigma, y)dyd\sigma$ for $i, j = 1, ..., n$. For $n = 2$ we obtain and upper bound for $\int_0^t \int_{\mathbb{R}^n} u^i u^j(\sigma, y)dyd\sigma$ which grows as *log t*. The results in [27] were proved changing to selfsimilar variables and then making eigenvalue expansions in some weighted Sobolev spaces. Our technique can be adapted to yield another proof of them.

Theorem 6 is related to a result obtained by Schonbek in [22]. When $n = 3$ she proved that $k_1(1+t)^{\frac{-5}{4}} \leq \|u(t)\|_2 \leq k_2(1+t)^{\frac{-5}{4}}$ for a special class of data. Theorem 6 implies that $u(t,x) = MG(t,x) + m^i\partial_i G(t,x) + R(t,x) + r(t,x)$ where $\|r(t,x)\|_2 = o(t^{\frac{-5}{4}})$ as $t \to \infty$ and $\|MG(t,x) + m^i\partial_i G(t,x) + R(t,x)\|_2 = Ct^{\frac{-5}{4}}$.

The paper is organized as follows. In the first section we briefly recall some known results which will be of use to us in the sequel. The next two sections are devoted to the proof of Theorems 1 and 2 respectively. In section 4 we study some related linear problems and prove Theorem 3. The last section deals with the asymptotic behavior in higher dimensions.

1. KNOWN RESULTS.

1.1. Strong solutions.

The following results are taken from [14]. They are established by using an iterative scheme (that goes back to Leray). The idea is to convert (NS) to an integral equation

$$u(t) = G(t) * u_0 + \int_0^t \partial_i G(t-s) * P(u_i u(s)) ds = Su(t)$$

where $G(t)$ denotes the heat kernel and P the projection on the space of divergence free vectors, which is a bounded operator from L^p to L^p for $1 < p < \infty$. Taking $u_0 \in L^n(\mathbb{R}^n)$, the sequence $u^{m+1} = Su^m$, $m \geq 1$ with $u^1 = G(t) * u_0$ converges strongly to a solution u of the integral equation on some time interval $[0, T]$. If $\|u_0\|_n$ is small enough we can take $T = \infty$. The construction also yields the estimates below.

THEOREM 1.1.1. ([14]) *Let $u_0 \in L^n(\mathbb{R}^n)$ be such that div $u_0 = 0$ and $\|u_0\|_n$ is small enough. Then, there exists a unique solution u of (NS) such that:*

i)

$$t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{q})} u \in BC([0,\infty); L^q(\mathbb{R}^n)) \quad \forall n \leq q \leq \infty$$

ii)

$$t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{q})+\frac{1}{2}} \nabla u \in BC([0,\infty); L^q(\mathbb{R}^n)) \quad \forall n \leq q < \infty$$

Moreover,

iii)

$$u \in L^r((0,\infty); L^q(\mathbb{R}^n)), \quad \frac{1}{r} = \frac{n}{2}(\frac{1}{n} - \frac{1}{q}), \quad n < q < \frac{n^2}{n-2}$$

iv)

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \|u(s)\|_n ds = 0$$

Remark 1.1.1.- When $n = 2$, $\|u(t)\|_2$ is monotonically nonincreasing (the energy inequality holds) and iv) implies that $\|u(t)\|_2 \to 0$ as $t \to \infty$ if $\|u_0\|_2$ is small.

THEOREM 1.1.2. ([14]) *Let $u_0 \in L^n(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ such that div $u_0 = 0$ and $\|u_0\|_n$ is small enough, with $1 < p < n$. Then,*

i)

$$t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} u \in BC([1,\infty); L^q(\mathbb{R}^n)) \quad \forall p \leq q \leq \infty$$

ii)

$$t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})+\frac{1}{2}} \nabla u \in BC([1,\infty); L^q(\mathbb{R}^n)) \quad \forall p \leq q < \infty$$

if $\frac{n}{2}(\frac{1}{p} - \frac{1}{q}) < 1$ in i) or $\frac{n}{2}(\frac{1}{p} - \frac{1}{q}) + \frac{1}{2} < 1$ in ii),otherwise, we replace them by any positive number less than 1.

iii)

$$\|u(t) - G(t) * u_0\|_p \leq C_\delta t^{\frac{-\delta}{2}} \quad 0 < \delta < Min(1, n(1 - \frac{1}{p}), \frac{n}{p} - 1)$$

Remarks 1.1.2

- In view of 1.1.1.i) and the fact that $u \in BC([0,\infty); L^p(\mathbb{R}^n))$ (see [14]) when $u_0 \in L^p, 1 < p < n$, we can replace in 1.1.2.i) $BC([1,\infty); L^q(\mathbb{R}^n))$ by $BC([0,\infty); L^q(\mathbb{R}^n))$ for any $q \geq p$.

- If $\|u_0\|_n$ is small then $\|u(t)\|_n \to 0$ as $t \to \infty$ (see Masuda's remark in [14]).
- Strong solutions have been also obtained for data $u_0 \in L^r(\mathbb{R}^n)$, $r > n$ (see [10],[8])
- In [2] strong solutions with data $u_0 \in L^2 \cap L^n(\mathbb{R}^n)$ are constructed in a different way. A decay estimate $\|u(t)\|_n \leq C t^{\frac{n}{2}(\frac{1}{2}-\frac{1}{n})}$ without restrictions on the size of $\frac{n}{2}(\frac{1}{2}-\frac{1}{n})$ is also obtained.
- When $n = 2$, 1.1.2.i) holds for any $q \geq p$. If $n \geq 3$ it holds for any $q \geq p$ when $p \geq \frac{n}{2}$ and for $p \leq q \leq \frac{pn}{n-2p}$ when $p < \frac{n}{2}$.

1.2. Weak solutions.

The use of Fourier transform allows to obtain decay estimates for weak solutions with initial data $u_0 \in L^2(\mathbb{R}^n)$ without smallness hypotheses, provided they satisfy the energy inequality or can be approached by solutions of related problems verifying it. The first results in that direction are due to Schonbek [20]. The idea consists in taking the Fourier transform of the equation and splitting the frequency domain in order to get differential inequalities yielding some decay. One does this first for some approached solutions and in the limit we get a decay estimate for u. This result has been succesively improved and extended in [21],[22], [17] and [26]. The following theorem is taken from [26].

Theorem 1.2.1. ([26])

Let u be a weak solution of the incompressible (NS) equations which satisfies the energy inequality (or can be approached by solutions of approximated problems satisfying it) for any $n \geq 2$. Then, for every $u_0 \in L^2(\mathbb{R}^n)$ with div $u_0 = 0$:

i) $\|u(t)\|_2 \to 0$ *as* $t \to \infty$

If further $\|G(t) * u_0\|_2^2 \leq C(1+t)^{-\alpha_0}$ *forall* $t \geq 0$ *then :*

ii) $\|u(t)\|_2^2 \leq C(1+t)^{-\alpha}$ *with* $\alpha = Min(\frac{n}{2}+1, \alpha_0)$, $t \geq 0$

iii) $\|u(t) - G(t) * u_0\|_2^2 \leq h(t)(1+t)^{-d}$ $\forall t \geq 0$ *where:*

$$d = \frac{n}{2} + 1 - 2Max(1-\alpha_0, 0) \quad (d > \alpha = \alpha_0 \; if \; \frac{n}{2} + 1 > \alpha_0)$$

$$h(t) = \begin{cases} \varepsilon(t) \to 0 \quad t \to \infty & if \quad \alpha = 0 \\ Cln^2(t+c) & if \quad \alpha = 1 \\ C & if \quad \alpha \neq 0, 1 \end{cases}$$

Remark 1.2.1.- Therefore, if $u_0 \in L^p \cap L^2$, $1 \leq p \leq 2$ we have $\alpha_0 = \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$ and

$$\|u(t) - G(t) * u_0\|_{L^2(\mathbb{R}^2)} \leq g(t)(1+t)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{2})} \quad \forall t \geq 0$$

with

$$g(t) = \begin{cases} \varepsilon(t) \to 0 \qquad t \to \infty & \text{if} \quad p = 2 \\ Cln(t+c)(1+t)^{\frac{-1}{2}} & \text{if} \quad p = 1 \\ (1+t)^{(\frac{1}{2}-\frac{n}{2p})} & \text{if} \quad 1 < p < 2 \end{cases}$$

Remark 1.2.2.- Some results on the behavior of weak solutions in exterior domains are also known. See for instance [1] and the references therein.

Remark 1.2.3.- As we said in the introduction, lower bounds for the decay of the L^2 norm and faster decay rates when $n = 2$ and the mass of the data is zero have been obtained in [22].

Remark 1.2.4.- Concerning the case $p = 2$ in Remark 1.2.1, its is known that both $\|G(t) * u_0\|_2$ and $\|u(t)\|_2$ tend to zero as t goes to infinity. Moreover, in both cases no uniform decay rate can be found. This is well known for the heat equation, where $\|G(t) * u_0\|_2$ can decay at an arbitrarily slow algebraic rate or even exponentially, by choosing an adequate $u_0 \in L^2(\mathbb{R}^n)$ (see [21], for instance). For Navier-Stokes in dimensions 2 and 3, the proof of the lack of uniformity is due to Schonbek [21]. When $n \geq 3$ a fonction $\delta(t) \to \infty \quad t \to \infty$ can be found in such a way that

$$\delta(t)\|G(t) * u_0 - u(t)\|_2 \to 0 \quad as\ t \to \infty$$

By Theorem 2 (iii) of [17] we may choose $\delta(t) = t^{\frac{n}{4}-\frac{1}{2}}$. However, we ignore if such a $\delta(t)$ exists when $n = 2$.

1.3. Solutions with singular data.

When $n = 2$ solutions of this kind have been constructed in [11] and its asymptotic behavior is studied in [12], [6], where the following result is proved:

THEOREM 1.3.1. *Let u be a solution of (NS) with initial data $u_0 \in L^{2,\infty}(\mathbb{R}^2)$ such that div $u_0 = 0$ and $v_0 =$ curl $u_0 \in M(\mathbb{R}^2)$. If the total variation of v_0 is small (see [12]) or, more generally, the mass of v_0, $|\int_{\mathbb{R}^2} v_0|$ is small (see [6]), then*

$$t^{\frac{1}{2}-\frac{1}{q}}\ \|u(t) - G(t) * u_0\|_q \to 0 \quad t \to \infty \quad \forall q > 2$$

$$t^{1-\frac{1}{q}}\ \|\nabla u(t) - \nabla G(t) * u_0\|_q \to 0 \quad t \to \infty \quad \forall 1 < q < \infty$$

We denote by $M(\mathbb{R}^2)$ the space of finite measures and by $L^{2,\infty}(\mathbb{R}^2)$ the usual Lorenz space.

The energy of these solutions is not necessarily finite. If we take $u_0 \in L^2(\mathbb{R}^2) \subset L^{2,\infty}(\mathbb{R}^2)$ with *div* $u_0 = 0$ and without hypothesis on *curl* u_0 then Theorem 1.2.1 asserts that

$$\|u(t) - G(t) * u_0\|_{L^2(\mathbb{R}^2)} \to 0 \quad as\ t \to \infty$$

but it gives no information nor on L^q norms with $q \neq 2$ neither on the behavior of $\nabla u(t)$.

In higher dimensions solutions with data in Morrey spaces have been constructed in [24] and [15]. However, few things are known on the asymptotic behavior.

2. n = 2 : FIRST TERM.

Let us take $u_0 \in L^2(\mathbb{R}^2)$ such that *div* $u_0 = 0$ and let u be the corresponding weak solution of (NS), which is known to be unique and smooth. Since $\|u(t)\|_2 \to 0$ as $t \to \infty$ (Theorem 1.2.1) we can choose t_0 such that $\|u(t_0)\|_2$ is small enough to apply the estimates of Theorem 1.1.1

If we also assume $u_0 \in L^p(\mathbb{R}^2)$ for some $1 \leq p < 2$, then (see [17]) we know that $u(t) \in L^p$ for $t \geq 0$. Therefore, the decay estimates furnished by Theorem 1.1.2 also apply.

First, we are going to extend Theorem 1.1.2 to the case $p = 1$. In order to do that we need a previous result:

LEMMA 2.1. *Let $G(t)$ be the n-dimensional heat kernel. Then, for every $i = 1, ..., n$, and every $t > 0$, $\partial_i G(t)$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ and*

$$\|\partial_i G(t)\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq Ct^{\frac{-1}{2}}$$

Proof. There are several equivalent definitions of $\mathcal{H}^1$ (see [9]):

$$\mathcal{H}^1(\mathbb{R}^n) = \{u \in L^1(\mathbb{R}^n)\ s.t.\ R_i * u \in L^1(\mathbb{R}^n),\ i = 1, ..., n\}$$

$$= \{u \in L^1(\mathbb{R}^n)\ s.t.\ \operatorname*{Sup}_{s>0} |h_s * u| \in L^1(\mathbb{R}^n)\}$$

where $R_i(x) = \frac{x_i}{|x|^n}$ and $h_s(x) = s^{-n}h(\frac{x}{s})$ with $h \in \mathcal{S}(\mathbb{R}^n)$ such that $0 \le h \le 1$ and $\int h = 1$. We may endow $\mathcal{H}^1(\mathbb{R}^n)$ with either of the equivalent norms

$$\|u\|_{L^1(\mathbb{R}^n)} + \sum_{i=1}^n \|R_i * u\|_{L^1(\mathbb{R}^n)}$$

or

$$\|u\|_{L^1(\mathbb{R}^n)} + \|\mathrm{Sup}_{s>0}|h_s * u|\|_{L^1(\mathbb{R}^n)}$$

Let us fix any $i \in 1, ..., n$. In order to prove that $\partial_i G(t) \in \mathcal{H}^1$ it suffices to verify that $\operatorname*{Sup}_{s>0} |h_s * \partial_i G(t)| \in L^1(\mathbb{R}^n)$ where $h_s(x) = s^{-n}h(\frac{x}{s})$ and $h \in \mathcal{S}(\mathbb{R}^n)$ is such that $0 \le h \le 1$ and $\int_{\mathbb{R}^n} h(x)dx = 1$. We take $h = G(1)$ and then

$$|h_s * \partial_i G(t)(x)| = |\partial_i (G(t+s, x))| = \frac{|x_i|}{(4\pi(t+s))^{\frac{n}{2}}} e^{\frac{-|x|^2}{4(t+s)}}$$

so that

$$\operatorname*{Sup}_{s>0} |h_t * \partial_i G(t)| = |\partial_i G(t)| \in L^1(\mathbb{R}^n)$$

and

$$\|\partial_i G(t)\|_{\mathcal{H}^1} \le Ct^{\frac{-1}{2}} \qquad \square$$

Proposition 2.1. *Let u be a solution of (NS) in dimension two, with initial data $u_0 \in L^1 \cap L^2(\mathbb{R}^2)$ such that div $u_0 = 0$. Then:*

i) $u(t) \in L^q \quad t > 0, \quad 1 \le q \le 2$

ii) $\|u(t)\|_q \le Ct^{-1+\frac{1}{q}} \quad t > 0, \quad q > 1$

iii) $\|u(t)\|_1 \le C \quad t \ge 0.$

Remark 2.1 These estimates extend the estimates known for $q = 2$ (see [26], [17]).

Proof. i) It is well known that $u \in L^\infty([0,\infty); L^2(\mathbb{R}^2))$. By taking the divergence of the equation we get the following equation for the pressure :

$$-\Delta\, p = \partial_j(u^i \partial_i u^j)$$

so that, up to a function of time, the pressure is given by $p = E_2 * \partial_j(u^i\partial_i u^j)$ where E_2 denotes the fundamental solution of $-\Delta$ in $\mathbb{R}^2$. Let us write the associated integral equation:

$$\begin{aligned} u(t) &= G(t) * u_0 + \int_0^t \partial_i G(t-s) * u^i u(s) ds \\ &+ \int_0^t \partial_i G(t-s) * \partial_j \nabla E_2 * u^i u^j(s) ds \end{aligned}$$

Since $u_0 \in L^1$, $G(t) * u_0 \in L^q$ for all $q \ge 1$ and $t > 0$. On the other hand, $u(s) \in L^2$ implies $u^i u(s) \in L^1$ and

$$\| \int_0^t \partial_i G(t-s) * u^i u(s) ds\|_q \ \le\ C \int_0^t (t-s)^{-1+\frac{1}{q}-\frac{1}{2}} \|u(s)\|_2^2 ds \le Ct^{\frac{1}{q}-\frac{1}{2}}$$

provided that $1 \leq q < 2$. Therefore, the first integral belongs to L^q for all $1 \leq q < 2$.

As far as the second one is concerned, since $\partial_i G(t)$ (Lemma 2.1) belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ and $\partial_j \nabla E_2$ is a Calderon-Zygmund kernel we conclude that (see [5]) $\partial_i G(t-s) * \partial_j \nabla E_2 \in L^1$ and

$$\|\partial_i G(t-s) * \partial_j \nabla E_2\|_1 \leq C \|\partial_i G(t-s)\|_{\mathcal{H}^1} \leq C(t-s)^{\frac{-1}{2}}$$

Then

$$\| \int_0^t \partial_i G(t-s) * \partial_j \nabla E_2 * u^i u^j(s) ds\|_1 \leq \int_0^t (t-s)^{\frac{-1}{2}} \|u(s)\|_2^2 ds \leq C t^{\frac{1}{2}}$$

In the same way, since $\partial_j \nabla E_2$ is a Calderon-Zygmund kernel, we have (see [5])

$$\|\partial_i G(t-s) * \partial_j \nabla E_2\|_q \leq C_q \|\partial_i G(t-s)\|_q \quad 1 < q < \infty$$

so that we get :

$$\| \int_0^t \partial_j \nabla E_2 * \partial_i G(t-s) * u^i u^j(s) ds\|_q \leq \int_0^t (t-s)^{-1+\frac{1}{q}-\frac{1}{2}} \|u(s)\|_2^2 ds \leq C t^{\frac{1}{q}-\frac{1}{2}}$$

for $1 < q \leq 2$. Thus, the second integral belongs also to L^q for all $1 \leq q < 2$.

Remark 2.2 It is known (see [4]) that if $A \in (L^p)^n$, $B \in (L^{p'})^n$ are such that $div\ A = 0$, $curl\ B = 0$. Then, $A \cdot B$ belongs to the Hardy space $\mathcal{H}^1$ and

$$\|A \cdot B\|_{\mathcal{H}^1} \leq C \|A\|_p \|B\|_{p'}$$

In our case, for almost every $s \geq 0$ and every $j = 1, ..., n$ we have $u(s), \nabla u^j(s) \in L^2$ with $div\ u = 0$ and $curl(\nabla\ u^j) = 0$. Therefore, $u^i \partial u^j(s) \in \mathcal{H}^1$. Taking into account that $\partial_j \nabla E_2$ is a Calderon-Zygmund kernel we conclude that $\partial_j \nabla E_2 * u^i \partial_i u^j(s) \in L^1$ for almost every $s > 0$.

ii) Taking norms in the integral equation :

$$\begin{aligned} u(t) \quad &= G(t) * u_0 \ + \ \textstyle\int_0^t \partial_i G(t-s) * u^i u(s) ds \\ &+ \ \textstyle\int_0^t \partial_i G(t-s) * \partial_j \nabla E_2 * u^i u^j(s) ds \end{aligned}$$

we get

$$\|u(t)\|_q \ \leq \ \|G(t) * u_0\|_q \ + \ \int_0^t \|\partial_i G(t-s) * u^i u(s)\|_q ds$$

Taking into account some classical estimates on the heat kernel when $n = 2$:

$$\|\partial^\alpha G(t) * a\|_q \ \leq \ C t^{-\frac{|\alpha|}{2}-\frac{1}{r}+\frac{1}{q}} \|a\|_r \quad q \geq r$$

together with the fact that

$$\|\partial_j \nabla E_2 * u^i u^j(s)\|_r \ \leq \ C \|u^i u^j(s)\|_r \quad r > 1$$

it follows that

$$\|u(t)\|_q \ \leq \ C t^{-1+\frac{1}{q}} \ + \ C \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{r}+\frac{1}{q}} s^{\frac{-2}{p}+\frac{1}{r}} ds$$

thanks to the estimates:

$$\|u(t)\|_{2r}^2 \;\leq\; Ct^{2(\frac{-1}{p}+\frac{1}{2r})}$$

valid for $2 > p > 1$ if $2r \geq p$. To prove these estimates it suffices to observe that (Theorem 1.2.1) the L^2 norm of u tends to zero as t goes to infinity and then apply Theorem 1.1.1 and Remark 1.1.2.

We split the integral appearing in the inequality as follows :

a)

$$\int_{\frac{t}{2}}^{t} (t-s)^{-\frac{1}{2}-\frac{1}{r}+\frac{1}{q}} s^{\frac{-2}{p}+\frac{1}{r}} \;\leq\; C\, t^{\frac{1}{2}+\frac{1}{q}-\frac{2}{p}}$$

choosing r such that $\frac{1}{2}-\frac{1}{r}+\frac{1}{q} > 0$, that is, $q \geq r > \frac{2q}{q+2}$.

b)

$$\int_{0}^{\frac{t}{2}} (t-s)^{-\frac{1}{2}-\frac{1}{r}+\frac{1}{q}} s^{\frac{-2}{p}+\frac{1}{r}} \;\leq\; C\, t^{\frac{1}{2}+\frac{1}{q}-\frac{2}{p}}$$

choosing r such that $\frac{-2}{p}+\frac{1}{r}+1 > 0$, that is, $1 < r < \frac{p}{2-p}$, which is possible if $p > 1$.

Therefore,

$$\|u(t)\|_q \;\leq\; Ct^{-1+\frac{1}{q}} + Ct^{\frac{1}{2}+\frac{1}{q}-\frac{2}{p}} \;\leq\; Ct^{-1+\frac{1}{q}}$$

if $1 < p \leq \frac{4}{3}$, where C is a constant depending on q and on the data.

iii) Taking norms in

$$\begin{aligned} u(t) \;&=\; G(t) * u_0 \;+\; \textstyle\int_0^t \partial_i G(t-s) * u^i u(s) ds \\ &+\; \textstyle\int_0^t \partial_i G(t-s) * \partial_j \nabla E_2 * u^i u^j(s) ds \end{aligned}$$

we get :

$$\|G(t) * u_0\|_1 \leq C$$

$$\| \int_0^t \partial_i G(t-s) * u^i u(s) ds\|_1 \leq C \int_0^t (t-s)^{\frac{-1}{2}} s^{\frac{-2}{p}+1} \leq Ct^{\frac{3}{2}-\frac{2}{p}} \leq C$$

if $p = \frac{4}{3}$ and also

$$\| \int_0^t \partial_i G(t-s) * \partial_j \nabla E_2 * u^i u^j(s) ds\|_1 \leq \int_0^t \|\partial_i G(t-s) * \partial_j \nabla E_2\|_1 \|u^i u^j(s) ds\|_1 \leq C$$

when $p = \frac{4}{3}$ since $\partial_i G$ belongs to the Hardy space $\mathcal{H}^1$. □

We prove now that, in a first approximation and for some classes of initial data, the solutions of the incompressible Navier-Stokes equations behave like the solutions of the heat equation with the same initial data.

Theorem 2.1. *Let u be a weak solution of (NS) in dimension $n = 2$, with initial data $u_0 \in L^p \cap L^2(\mathbb{R}^2)$, $1 \leq p \leq 2$ such that div $u_0 = 0$. Then, for any $q \geq p$*

i) If $1 < p < 2$

$$\|G(t) * u_0 - u(t)\|_q \leq Ct^{-\frac{1}{p}+\frac{1}{q}} t^{-\frac{1}{p}+\frac{1}{2}} \quad t > 0$$

ii) If $p = 1$

$$\|G(t) * u_0 - u(t)\|_q \leq Ct^{-1+\frac{1}{q}} t^{-\frac{1}{2}} log\, t \quad t > 0$$

iii) If $p = 2$

$$t^{\frac{1}{2}-\frac{1}{q}}\|u(t)\|_q \to 0 \quad as\ t \to \infty$$

Remark 2.4 All the estimates were known when $q = 2$ for any $1 \le p \le 2$ ([26],[17]). We can replace the powers of t by powers of $t+1$ (and also $log\ t$ by $log\ (t+1)$) when $p \le q \le 2$. In case $p = 2$, iii) is known to hold for the solutions of the heat equation, hence, it holds for the difference $G(t) * u_0 - u(t)$ but this gives no extra information.

Proof. i) From the integral equation we get :

$$\|G(t) * u_0 - u(t)\|_q \le C \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{r}+\frac{1}{q}} s^{\frac{-2}{p}+\frac{1}{r}}$$

for $q \ge r > 1$ and $2r \ge p$. As in the proof of ii) in Theorem 2.1 we split the integral in two intervals $[0, \frac{t}{2}]$ and $[\frac{t}{2}, t]$. By choosing an adequate r we conclude that:

$$\|G(t) * u_0 - u(t)\|_q \le Ct^{-\frac{1}{p}+\frac{1}{q}} t^{-\frac{1}{p}+\frac{1}{2}}$$

ii) Taking norms in the integral equation we get

$$\begin{aligned} \|G(t) * u_0 - u(t)\|_q \quad &\le \int_0^t \|\partial_i G(t-s) * u^i u(s)\|_q ds \\ &+ \int_0^t \|\partial_i G(t-s) * \partial_j \nabla E_2 * u^i u^j(s)\|_q ds \end{aligned}$$

Since

$$\|\partial_i G(t-s) * \partial_j \nabla E_2\|_{r'} \le \begin{cases} C\|\partial_i G(t-s)\|_{r'} & 1 < r' < \infty \\ C\|\partial_i G(t-s)\|_{\mathcal{H}^1} & r' = 1 \end{cases}$$

both integrals are bounded by

$$C \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{r}+\frac{1}{q}} s^{-2+\frac{1}{r}}$$

for $q \ge r,\ 2r \ge 1,\ r \ge 1$. We split this integral as follows :

a)

$$\int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}-\frac{1}{r}+\frac{1}{q}} s^{-2+\frac{1}{r}} \ \le\ C\ t^{\frac{1}{2}+\frac{1}{q}-2}$$

choosing r such that $\frac{1}{2} - \frac{1}{r} + \frac{1}{q} > 0$, that is, $q \ge r > \frac{2q}{q+2}$.

b)

$$\int_0^{\frac{t}{2}} (t-s)^{-\frac{1}{2}-\frac{1}{r}+\frac{1}{q}} s^{-2+\frac{1}{r}} \ \le\ C\ t^{\frac{1}{2}+\frac{1}{q}-2} log\ t$$

choosing $r = 1$.

Therefore, ii) holds.

iii) It is known that $\|u(t)\|_2 \to 0$ as $t \to \infty$. Interpolating and taking into account that $t^{\frac{1}{2}-\frac{1}{r}}\|u(t)\|_r \le C$ (see Theorem 1.1.1) we get:

$$t^{\frac{1}{2}-\frac{1}{q}}\|u(t)\|_q \le C(\|u(t)\|_2)^{1-\alpha}(t^{\frac{1}{2}-\frac{1}{r}}\|u(t)\|_r)^{\alpha} \le C(\|u(t)\|_2)^{1-\alpha}$$

for any $2 < q < r$, which yields the result. ☐

3. n = 2 : SECOND TERM.

Let u be again a weak solution of (NS) and v a solution of:

$$(\mathcal{L}_2)\quad \begin{cases} v_t - \Delta v = -h^i\partial_i h - \partial_j \nabla E_2 * h^i \partial_i h^j & \text{in } \mathbb{R}^+ \times \mathbb{R}^2 \\ div\ v = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^2 \\ v(x,0) = u_0,\ div\ u_0 = 0 & \text{in } \mathbb{R}^2 \end{cases}$$

where $h(t) = G(t) * u_0$ so that v can be written as h plus a corrector term. Let us see whether the difference $u(t) - v(t)$ tends to zero faster than $u(t) - h(t)$. We assume again that $u_0 \in L^2(\mathbb{R}^2) \cap L^p(\mathbb{R}^2),\ 1 \le p < 2$.

Taking norms in the integral equation satisfied by the difference $u(t) - v(t)$ we get:

$$\begin{aligned} \|u(t) - v(t)\|_q \ &\le \ \|\textstyle\int_0^t \partial_i G(t-s) * (u^i(u-h) + h(u^i - h^i))(s) ds\|_q \\ &+ \|\textstyle\int_0^t \partial_i G(t-s) * \partial_j \nabla E_2 * (u^i(u-h) + h(u^i-h^i))(s)ds\|_q \\ &\le \ \textstyle\int_0^t \|\partial_i G(t-s)\|_{r'}(\|u(s)\|_{2r} + \|h(s)\|_{2r})\|(u-h)(s)\|_{2r} ds \\ &+ \ \textstyle\int_0^t \|\partial_i G(t-s) * \partial_j \nabla E_2\|_{r'}(\|u(s)\|_{2r} + \|h(s)\|_{2r})\|(u-h)(s)\|_{2r} ds \end{aligned}$$

for $q \ge r, r' \ge 1$ such that $\frac{1}{r} + \frac{1}{r'} = \frac{1}{q}$. In view of the estimates:

$$\begin{aligned} \|u(t)\|_{2r} \ &\le \ Ct^{-\frac{1}{p}+\frac{1}{2r}} \\ \|h(t)\|_{2r} \ &\le \ Ct^{-\frac{1}{p}+\frac{1}{2r}} \\ \|(u-h)(t)\|_{2r} \ &\le \ Ct^{-\frac{1}{p}+\frac{1}{2r}}t^{-\frac{1}{p}+\frac{1}{2}} \end{aligned}$$

valid when $q \ge p,\ q \ge r \ge 1,\ 2r \ge p$ and the fact that

$$\|\partial_i G(t-s) * \partial_j \nabla E_2\|_{r'} \le \begin{cases} C\|\partial_i G(t-s)\|_{r'} & 1 < r' < \infty \\ C\|\partial_i G(t-s)\|_{\mathcal{H}^1} & r' = 1 \end{cases}$$

we get:

$$\|u(t) - v(t)\|_q \ \le C \int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{r}+\frac{1}{q}} s^{-\frac{2}{p}+\frac{1}{r}} s^{-\frac{1}{p}+\frac{1}{2}} ds$$

We split the integral as follows:

a)

$$\int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}-\frac{1}{r}+\frac{1}{q}} s^{-\frac{3}{p}+\frac{1}{r}+\frac{1}{2}} \ \le \ Ct^{-\frac{3}{p}+\frac{1}{q}+1}$$

if $r > \frac{2q}{q+2}$

b)

$$\int_0^{\frac{t}{2}} (t-s)^{-\frac{1}{2}-\frac{1}{r}+\frac{1}{q}} s^{-\frac{3}{p}+\frac{1}{r}+\frac{1}{2}} \ \le \ Ct^{-\frac{3}{p}+\frac{1}{q}+\frac{1}{2}}$$

if $-\frac{3}{p} + \frac{1}{r} + \frac{3}{2} > 0$, that is, $r \le q$ and $1 \le r < \frac{2p}{3(2-p)}$, provided $p > \frac{6}{5}$

Therefore,

$$\|(u-v)(t)\|_q \le Ct^{-\frac{1}{p}+\frac{1}{q}}t^{-\frac{1}{p}+\frac{1}{2}}t^{-\frac{1}{p}+\frac{1}{2}}$$

if $2 > p > \frac{6}{5}$. This decay is faster that the corresponding to $u(t) - h(t)$.

Thus, we have proved the following:

THEOREM 3.1. *Let u be a solution of the two dimensional (NS) with initial data $u_0 \in L^p \cap L^2(\mathbb{R}^2)$ $\frac{6}{5} < p < 2$ and v a solution of $(\mathcal{L}_2)$. Then, for any $q \geq p$ we have*

$$\|(u-v)(t)\|_q \leq Ct^{-\frac{3}{p}+\frac{1}{q}+1}$$

4. n = 2: EXPLICIT SECOND TERM.

In this section we shall obtain an explicit approximation up to the second term of both u and v. Let us consider the following three problems :

$$(\mathcal{P}_1) \begin{cases} w_t - \Delta w = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^2 \\ div\ w = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^2 \\ w(x,0) = u_0,\ div\ u_0 = 0 & \text{in } \mathbb{R}^2 \end{cases}$$

$$(\mathcal{P}_2) \begin{cases} w_t - \Delta w = -f^i \partial_i f & \text{in } \mathbb{R}^+ \times \mathbb{R}^2 \\ div\ w = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^2 \\ w(x,0) = 0 & \text{in } \mathbb{R}^2 \end{cases}$$

$$(\mathcal{P}_3) \begin{cases} w_t - \Delta w = -\partial_j \nabla E_2 * f^i \partial_i f^j & \text{in } \mathbb{R}^+ \times \mathbb{R}^2 \\ div\ w = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^2 \\ w(x,0) = 0 & \text{in } \mathbb{R}^2 \end{cases}$$

where $w = (w^1, w^2)$ and $f = (f^1, f^2)$. We assume that $u_0 \in L^1(\mathbb{R}^2)$, $\partial_j \nabla E_2 * f^i \partial_i f^j$, $f^i \partial_i f \in L^1(0,\infty; L^1(\mathbb{R}^2))$ and $div\ f = 0$. Let us denote by w_i the solution of each $(\mathcal{P}_i)$. Then, $w = w_1 + w_2 + w_3$ is a solution of

$$(\mathcal{P}) \begin{cases} w_t - \Delta w = -f^i \partial_i f - \partial_j \nabla E_2 * f^i \partial_i f^j & \text{in } \mathbb{R}^+ \times \mathbb{R}^2 \\ div\ w = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^2 \\ w(x,0) = u_0,\ div\ u_0 = 0 & \text{in } \mathbb{R}^2 \end{cases}$$

and satisfies the integral equation:

$$w(t) = G(t) * u_0 - \int_0^t G(t-s) * f^i \partial_i f(s) ds - \int_0^t G(t-s) * \nabla \partial_j E_2 * f^i \partial_i f^j(s) ds$$

Therefore, we may think of the solutions u of (NS) or v of $(\mathcal{L}_2)$ as being the sum of three terms :

- a term w_1 which solves $(\mathcal{P}_1)$
- a term w_2 which solves $(\mathcal{P}_2)$ with $f = u$ (resp. $f = h$)
- a term w_3 which solves $(\mathcal{P}_3)$ with $f = u$ (resp. $f = h$)

We are going to study the asymptotic behavior of the solutions of these problems in order to get information on u and v when the initial data $u_0 \in L^1 \cap L^2(\mathbb{R}^2)$.

4.1. Problem $(\mathcal{P}_1)$**.**

The solution of this heat equation is $w_1 = G(t) * u_0$, whose asymptotic behavior is well known. In case $u_0 \in L^1(\mathbb{R}^2)$

$$t^{1-\frac{1}{q}} \|G(t) * u_0 - MG(t)\|_q \to 0 \quad as\ t \to \infty$$

for any $1 \le q \le \infty$, where $M = \int_{\mathbb{R}^2} u_0$. Thus, the first term in the development of w_1 when $t \to \infty$ is $MG(t)$. If $u_0 \in L^1(1 + |x|; \mathbb{R}^2)$, we know further that:

$$t^{1-\frac{1}{q}+\frac{1}{2}} \|G(t) * u_0 - MG(t)\|_q \le C\|u_0\|_{L^1(|x|,\, \mathbb{R}^2)}$$

In fact, setting $M^i = \int_{\mathbb{R}^2} x^i u_0$ we have

$$t^{\frac{3}{2}-\frac{1}{q}} \|G(t) * u_0 - MG(t) + M^i \partial_i G(t)\|_q \to 0 \quad as\ t \to \infty$$

We can obtain more terms when more moments of the initial data are finite (see [7]).

In case $u_0 \in L^p(\mathbb{R}^2),\ 1 < p < \infty$ we have

$$t^{\frac{1}{p}-\frac{1}{q}} \|G(t) * u_0\|_q \to 0 \quad as\ t \to \infty$$

for any $p \le q \le \infty$, so that the first term in the development is zero.

This convergence is clear when $u_0 \in \mathcal{D}(\mathbb{R}^n)$, since $u_0 \in L^1(\mathbb{R}^n)$ and then $\|G(t) * u_0\|_p \le C(1+t)^{-1+\frac{1}{p}}$. Thanks to the fact that $\|G(t) * u_0\|_p$ decreases as time grows, we can extend the result to any $u_0 \in L^p(\mathbb{R}^n)$ by density. Given $u_0 \in L^p(\mathbb{R}^n)$, we take a sequence $u_{0,k} \subset \mathcal{D}(\mathbb{R}^n)$ such that $u_{0,k} \to u$ in $L^p(\mathbb{R}^n)$. Let u_k be the solution of the heat equation with data $u_{0,k}$. Then,

$$\|u(t)\|_p \le \|u_k(t)\|_p + \|u(t) - u_k(t)\|_p \le \|u_k(t)\|_p + \|u_0 - u_{0,k}\|_p$$

Given $\varepsilon > 0$, we can choose k large enough to have

$$\|u_0 - u_{0,k}\|_p \le \frac{\varepsilon}{2}$$

and fixing that k we get

$$\|u_k(t)\|_p \le \frac{\varepsilon}{2}$$

for $t \ge t_\varepsilon$. Therefore, $\|u(t)\|_p \to 0$ as $t \to \infty$. Once this is proved, it is clear that

$$t^{\frac{1}{p}-\frac{1}{q}} \|G(t) * u_0\|_q \to 0 \quad as\ t \to \infty$$

for any $p \le q \le \infty$.

4.2. Problem $(\mathcal{P}_2)$**.**

In order to describe the asymptotic behavior of w_2 we are going to use a scaling technique. In the following we shall drop the subscript 2 and write only w. Since we want to take $f = u$, u being a weak solution of (NS) with data $u_0 \in L^1 \cap L^2(\mathbb{R}^2)$ we shall assume that

$$f(t) \in BC(0, \infty; L^2(\mathbb{R}^2)); \quad \|f(t)\|_2 \le C(1+t)^{\frac{-1}{2}} \quad t \ge 0$$

so that we can rewrite the integral expression for w

$$w(t) \; = \; -\int_0^t \partial_i G(t-s) * f^i f(s) ds$$

By rescaling, we see that the functions $w_\lambda(t,x) = \lambda^2 w(\lambda^2 t, \lambda x)$ satisfy

$$w_\lambda(t) \; = \; -\lambda^{-1} \int_0^t \partial_i G(t-s) * f^i_\lambda f_\lambda(s) ds$$

We remark that :

$$\|w_\lambda(t)\|_q = \lambda^{2(1-\frac{1}{q})} \|w(\lambda^2 t)\|_q$$

Thus, if we want to make precise the asymptotic behavior of w when $t \to \infty$ it suffices to find a function g such that:

$$\|w_\lambda(t) - g_\lambda(t)\|_q \delta(\lambda) \to 0 \quad as\ \lambda \to \infty$$

for some $\delta(\lambda)$ tending to ∞ as $\lambda \to \infty$, where $g_\lambda(t,x) = \lambda^2 g(\lambda^2 t, \lambda x)$. That implies:

$$\|w(t) - g(t)\|_q t^{1-\frac{1}{q}} \delta(t^{\frac{1}{2}}) \to 0 \quad as\ t \to \infty$$

It is easy to prove that

$$\|w_\lambda\|_q \; = \; \lambda^{-1} \| \int_0^t \partial_i G(t-s) * f^i_\lambda f_\lambda(s) ds\|_q$$

is bounded by $C\ log \lambda^2\ \lambda^{-1}$. Therefore, the same kind of bound should hold for g_λ but the difference $\|g_\lambda(t) - w_\lambda(t)\|_q$ should go to zero faster. Under certain conditions it is possible to take $g(t) = (\frac{M^i M}{2}) log\ t\ \partial_i G(t)$ with $M = (M^1, M^2)$ and $\delta(t) = \frac{t^{\frac{1}{2}}}{log\ t}$. More precisely, we prove the following:

THEOREM 4.1. *Let w be a solution of $(\mathcal{P}_2)$ and $q \geq 1$. Let us assume that :*

$$i) \quad (1+t)^{\frac{1}{2}+\varepsilon} \|f(t) - G * u_0(t)\|_2 \leq C \quad t \geq 0$$

for some $\varepsilon > 0$

$$ii) \quad \|f(t)\|_{2r} \; \leq \; C(1+t)^{-1+\frac{1}{2r}} \qquad t \geq 0$$

where $q \geq r > \frac{2q}{q+2}$.

Then

$$\frac{t^{\frac{3}{2}-\frac{1}{q}}}{log(t)} \|w(t) - log\ t\ \partial_i G(t)(\frac{M^i M}{2})\|_q \to 0 \quad as\ t \to \infty$$

where $M = (M^1, M^2) = \int_{\mathbb{R}^2} u_0(x) dx$ provided that $|x|^{\frac{\alpha}{2}} u_0 \in L^1$ and $|x|^{\frac{\alpha}{2}} u_0 \in L^2$ for some $\alpha \geq 2$.

Remark 4.1 It follows from i) that ii) holds replacing 2r by 2 and that

$$t^{\frac{1}{2}} \|f(t) - MG(t)\|_2 \to 0 \quad as\ t \to \infty$$

Proof. Setting:

$$g_\lambda(1) = \lambda^{-1}\ log\ \lambda^2\ \partial_i G(1)(\frac{M^i M}{2}) = (log(\cdot)\ \partial_i G(\cdot)(\frac{M^i M}{2}))_\lambda(1)$$

we see that:

$$\frac{t^{1+\frac{1}{2}-\frac{1}{q}}}{log(t)}\| \int_0^t \partial_i G(t-s) * f^i f(s)ds - log(t)\ \partial_i G(t)(\frac{M^iM}{2}))\|_q \to 0 \quad as\ t\to\infty$$

is equivalent to

$$\|\lambda^{-1}\int_0^1 \partial_i G(1-s) * f^i_\lambda f_\lambda(s)ds - g_\lambda(1)\|_q \ = \ o(1)log(\lambda^2)\lambda^{-1} \quad as\ \lambda\to\infty$$

Making the change of variables $z(s,y) = e^s f(e^s-1, e^{\frac{s}{2}}y)$ it follows from i) that

$$\lim_{\sigma\to\infty}\{\frac{1}{\sigma}\int_0^\sigma\int_{\mathbb{R}^2} z^i z\} = M^iM\int G(1)^2 = \frac{M^iM}{2}$$

Thus, we must prove:

$$\|\int_0^1 \partial_i G(1-s) * \frac{f^i_\lambda f_\lambda(s)}{log\ \lambda^2}ds - \partial_i G(1)\lim_{\sigma\to\infty}\{\frac{1}{\sigma}\int_0^\sigma\int_{\mathbb{R}^2} z^i z\}\|_q \to 0 \quad as\ \lambda\to\infty$$

We remark first that :

$$\int_0^1 (\partial_i G(1-s) * \frac{f^i_\lambda f_\lambda(s)}{log\ \lambda^2})(x)ds \ = \ \int_0^1\int_{\mathbb{R}^2} \partial_i G(1-s, x-y)\frac{f^i f(\lambda^2 s, \lambda x)}{log\ \lambda^2}\lambda^4 dsdy \ =$$

$$= \int_0^{\lambda^2}\int_{\mathbb{R}^2} \partial_i G(1-\frac{s}{\lambda^2}, x-\frac{y}{\lambda})\frac{f^i f(s,y)}{log\ \lambda^2}dsdy \ =$$

$$= \int_0^{log(\lambda^2+1)}\int_{\mathbb{R}^2} \partial_i G(1-\frac{e^s-1}{\lambda^2}, x-\frac{ye^{\frac{s}{2}}}{\lambda})\frac{z^i z(s,y)}{log\ \lambda^2}dsdy$$

Therefore, it is enough to prove that:

$$\int_0^{\lambda^2}\int_{\mathbb{R}^2} \partial_i G(1-\frac{s}{\lambda^2}, x-\frac{y}{\lambda})\frac{f^i f(s,y)}{log\ \lambda^2}dsdy \ - \ \partial_i G(1,x)\int_0^{\lambda^2}\int_{\mathbb{R}^2}\frac{f^i f(s,y)}{log\ \lambda^2}dsdy$$

and

$$\partial_i G(1,x)\int_0^{\lambda^2}\int_{\mathbb{R}^2}\frac{f^i f(s,y)}{log\ \lambda^2}dsdy \ - \ \partial_i G(1,x)\lim_{\sigma\to 0}\{\frac{1}{\sigma}\int_0^\sigma\int_{\mathbb{R}^2} z^i z(s,y)\}$$

tend to zero in L^q_x when $\lambda\to\infty$. Since

$$\int_0^{\lambda^2}\int_{\mathbb{R}^2}\frac{f^i f(s,y)}{log\ \lambda^2}dsdy = \frac{1}{log\ \lambda^2}\int_0^{log(1+\lambda^2)}\int_{\mathbb{R}^2} z^i z(s,y)dsdy$$

the last convergence is clear. We must only prove the first one. In order to do that we split the integral as follows:

$$\int_0^{\lambda^2}\int_{\mathbb{R}^2}(\partial_i G(1-\frac{s}{\lambda^2}, x-\frac{y}{\lambda}) - \partial_i G(1,x))\frac{f^i f(s,y)}{log\ \lambda^2}dsdy \ =$$

$$= \int_0^{\lambda^2\delta} \int_{|y|\leq\lambda\delta} (\partial_i G(1-\frac{s}{\lambda^2}, x-\frac{y}{\lambda}) - \partial_i G(1,x)) \frac{f^i f(s,y)}{log\ \lambda^2} dsdy +$$

$$+ \int_{\lambda^2\delta}^{\lambda^2} \int_{\mathbb{R}^2} \partial_i G(1-\frac{s}{\lambda^2}, x-\frac{y}{\lambda}) \frac{f^i f(s,y)}{log\ \lambda^2} dsdy \ + \ \int_{\lambda^2\delta}^{\lambda^2} \int_{\mathbb{R}^2} \partial_i G(1,x) \frac{f^i f(s,y)}{log\ \lambda^2} dsdy$$

$$+ \int_0^{\lambda^2\delta} \int_{|y|>\lambda\delta} \partial_i G(1-\frac{s}{\lambda^2}, x-\frac{y}{\lambda}) \frac{f^i f(s,y)}{log\ \lambda^2} dsdy + \int_0^{\lambda^2\delta} \int_{|y|>\lambda\delta} \partial_i G(1,x) \frac{f^i f(s,y)}{log\ \lambda^2} dsdy$$

$$= I^1_{\delta,\lambda} \ + \ I^2_{\delta,\lambda} \ + \ J^2_{\delta,\lambda} \ + \ I^3_{\delta,\lambda} \ + \ J^3_{\delta,\lambda}$$

Estimate $I^1_{\delta,\lambda}$

$$\|I^1_{\delta,\lambda}\|_{L^q_x} \leq C \int_0^{\lambda^2\delta} \int_{|y|<\lambda\delta} \|\partial_i G(1-\frac{s}{\lambda^2}, x-\frac{y}{\lambda}) - \partial_i G(1,x)\|_{L^q_x} \frac{|f^i||f|(s,y)}{log\ \lambda^2} dsdy$$

Thanks to the continuity of traslations in L^q_x and the continuity with respect to t, given $\varepsilon > 0$ we can choose $\delta > 0$ such that

$$\operatorname*{Sup}_{s\leq\lambda^2\delta,\ |y|\leq\lambda\delta} \|\partial_i G(1-\frac{s}{\lambda^2}, x-\frac{y}{\lambda}) - \partial_i G(1,x)\|_{L^q_x} \leq \varepsilon$$

Therefore,

$$\|I^1_{\delta,\lambda}\|_{L^q_x} \leq \ \varepsilon \int_0^{\lambda^2\delta} \int_{|y|\leq\lambda\delta} \frac{|f|^2(s,y)}{log\ \lambda^2} dsdy \ \leq \ \varepsilon \int_0^{log(1+\lambda^2\delta)} \int_{\mathbb{R}^2} \frac{|z|^2(s,y)}{log\ \lambda^2} dsdy \ \leq \varepsilon C$$

uniformly with respect to λ, since

$$\|z(s)\|_2 = e^{\frac{s}{2}} \ \|f(e^s-1)\|_2 \ \leq C$$

Estimate $J^2_{\delta,\lambda}$

$$\|J^2_{\delta,\lambda}\|_{L^q_x} \leq C\|\nabla G(1)\|_{L^q_x} \int_{\lambda^2\delta}^{\lambda^2} \int_{\mathbb{R}^2} \frac{|f|^2(s,y)}{log\ \lambda^2} dsdy \ \leq$$

$$\leq \ \|\nabla G(1)\|_{L^q_x} \int_{log(1+\lambda^2\delta)}^{log(1+\lambda^2)} \int_{\mathbb{R}^2} \frac{|z|^2(s,y)}{log\ \lambda^2} dsdy \ \leq C \frac{log(\frac{1+\lambda^2}{1+\lambda^2\delta})}{log\ \lambda^2}$$

Therefore, it tends to zero when $\lambda \to \infty$ and δ is fixed.

Estimate $I^2_{\delta,\lambda}$

$$\|I^2_{\delta,\lambda}\|_{L^q_x} \ \leq \ \frac{1}{log(\lambda^2)} \int_{log(1+\lambda^2\delta)}^{log(1+\lambda^2)} \| \int_{\mathbb{R}^2} \partial_i G(1-\frac{e^s-1}{\lambda^2}, x-\frac{e^{\frac{s}{2}}y}{\lambda}) z^i z(y,s) dy\|_{L^q_x} ds$$

We first observe that :

$$\int_{\mathbb{R}^2} \partial_i G(1-\frac{e^s-1}{\lambda^2}, x-\frac{e^{\frac{s}{2}}y}{\lambda}) z^i z(y,s) dy \ =$$

$$= (\partial_i G(1 - \frac{e^s - 1}{\lambda^2}, y) *_y (\lambda e^{\frac{-s}{2}})^2 \; z^i z(s, \lambda e^{\frac{-s}{2}} y))(x)$$

Thus,

$$\| \int_{\mathbb{R}^2} \partial_i G(1 - \frac{e^s - 1}{\lambda^2}, x - \frac{e^{\frac{s}{2}} y}{\lambda}) z^i z(y, s) dy \|_{L^q_x} \; \leq$$

$$\leq \; C(1 - \frac{e^s - 1}{\lambda^2})^{-(\frac{1}{r} - \frac{1}{q}) - \frac{1}{2}} \| (\lambda e^{\frac{-s}{2}})^2 z^i z(s, \lambda e^{\frac{-s}{2}} y) \|_{L^r_x}$$

for $r \leq q$. Since

$$\|z(s)\|_{2r} = e^{\frac{(s}{2)(2-\frac{1}{r})}} \; \|f(e^s - 1)\|_{2r} \; \leq C$$

it follows that

$$\|(\lambda e^{\frac{-s}{2}})^2 z^i z(\lambda e^{\frac{-s}{2}} y, s)\|_{L^r_x} \; \leq \; \|z(s)\|^2_{L^{2r}_x} (\lambda e^{\frac{-s}{2}})^{2(1-\frac{1}{r})} \; \leq \; C(\lambda e^{\frac{-s}{2}})^{2(1-\frac{1}{r})}.$$

Therefore,

$$\|I^2_{\delta,\lambda}\|_{L^q_x} \; \leq \; \frac{C\lambda^{2(1-\frac{1}{r})}}{log(\lambda^2)} \int_{log(1+\lambda^2\delta)}^{log(1+\lambda^2)} (1 - \frac{e^s - 1}{\lambda^2})^{-(\frac{1}{r} - \frac{1}{q}) - \frac{1}{2}} (e^{\frac{-s}{2}})^{2(1-\frac{1}{r})} ds \; \leq$$

$$\leq \; \frac{C\lambda^{2(1-\frac{1}{r})} (\lambda^2\delta + 1)^{(-1+\frac{1}{r})}}{log(\lambda^2)} \int_{log(1+\lambda^2\delta)}^{log(1+\lambda^2)} (1 - \frac{e^s - 1}{\lambda^2})^{-(\frac{1}{r} - \frac{1}{q}) - \frac{1}{2}} ds \; \leq$$

$$\leq \; \frac{C}{log \; \lambda^2} (\frac{\lambda^2}{1 + \delta\lambda^2})^{2-\frac{1}{r}} \int_{\delta}^{1} (1-t)^{-(\frac{1}{r} - \frac{1}{q}) - \frac{1}{2}} dt \; \leq \; \frac{C}{log \; \lambda^2}$$

where C is a constant depending on δ but not on λ, taking $r > \frac{2q}{q+2}$ in order to have $-(\frac{1}{r} - \frac{1}{q}) + \frac{1}{2} > 0$.

We conclude that

$$\|I^2_{\delta,\lambda}\|_{L^q_x} \; \leq \; \frac{C}{log\lambda^2} \to 0 \quad as \; \lambda \to \infty$$

for a fixed δ.

<u>*Estimate* $J^3_{\delta,\lambda}$</u>

$$\|J^3_{\delta,\lambda}\|_{L^q_x} \; \leq \; \|\nabla G(1)\|_{L^q_x} \int_0^{\lambda^2\delta} \int_{|y| \geq \delta\lambda} \frac{|f|^2(s,y)}{log \; \lambda^2}$$

We must prove that:

$$\int_0^{\lambda^2\delta} \int_{|y| \geq \delta\lambda} \frac{|f|^2(s,y)}{log \; \lambda^2} \to 0 \quad as \; \lambda \to \infty$$

for δ fixed. For any $\alpha \geq 2$ we have

$$\| |y|^{\frac{\alpha}{2}} G * u_0(t,y) \|^2_2 \; \leq \; C(t^{\frac{\alpha}{2}-1} + (t+1)^{-1}) \; \leq \; C(t^{\frac{\alpha}{2}-1})$$

provided that $u_0,\ |x|^{\frac{\alpha}{2}} u_0 \in L^1$ and $|x|^{\frac{\alpha}{2}} u_0 \in L^2$. It follows that if $f = G * u_0$

$$\int_0^{\lambda^2\delta} \int_{|y|\geq\delta\lambda} \frac{|y|^\alpha |f|^2(y,s)}{\lambda^\alpha\ log\ \lambda^2} \ \leq\ C\frac{(\lambda^2\delta)^{\frac{\alpha}{2}}}{\lambda^\alpha\ log\ \lambda^2} \ \leq\ \frac{C}{log\ \lambda^2}$$

Since $\|f(t) - G * u_0(t)\|_2 \leq C(1+t)^{\frac{-1}{2}-\varepsilon}$

$$\int_0^{\lambda^2\delta} \int_{|y|\geq\delta\lambda} \frac{|f(s,y) - G * u_0(s,y)|^2}{log\ \lambda^2} \ \leq\ \frac{C}{log\ \lambda^2}$$

Therefore, for a fixed δ

$$\|J^3_{\delta,\lambda}\|_{L^q_x} \to 0 \quad as\ \lambda \to \infty$$

<u>*Estimate* $I^3_{\lambda,\delta}$</u>

$$\|I^3_{\lambda,\delta}\|_{L^q_x} \ \leq\ \int_0^{\lambda^2\delta} \| \int_{|y|\geq\delta\lambda} \partial_i G(1 - \frac{s}{\lambda^2}, x - \frac{y}{\lambda}) \frac{f^i f(s,y)}{log\ \lambda^2} \|_{L^q_x} \ \leq$$

As in estimate $I^2_{\lambda,\delta}$ but taking $r = 1$ we get

$$\|I^3_{\lambda,\delta}\|_{L^q_x} \leq\ C \int_0^{\lambda^2\delta} (1 - \frac{s}{\lambda^2})^{-1+\frac{1}{q}-\frac{1}{2}} \| \frac{f^i f(s,y)}{log\ \lambda^2} \|_{L^1(|y|\geq\lambda\delta)}$$

$$\leq\ C \int_0^{\lambda^2\delta} \int_{|y|\geq\delta\lambda} \frac{|f|^2(s,y)}{log\ \lambda^2} dyds$$

and we finish in the same way as in estimate $J^3_{\lambda,\delta}$. ∎

4.3. Problem $(\mathcal{P}_3)$**.**

In the following we shall drop the subscript 3 and write only w. As before we shall assume that

$$f(t) \in BC(0,\infty; L^2(\mathbb{R}^2)); \quad \|f(t)\|_2 \leq C(1+t)^{\frac{-1}{2}} \quad t \geq 0$$

so that we can rewrite the integral expression for w

$$w(t)\ =\ -\int_0^t \partial_i G(t-s) * \nabla\partial_j E_2 * f^i f^j(s) ds$$

By rescaling, we see that the functions $w_\lambda(t,x) = \lambda^2 w(\lambda^2 t, \lambda x)$ satisfy

$$w_\lambda(t)\ =\ -\lambda^{-1} \int_0^t \partial_i G(t-s) * \nabla\partial_j E_2 * f^i_\lambda f^j_\lambda(s) ds$$

Since $\|\partial_i G(t-s) * \nabla\partial_j E_2\|_1 \leq C\|\partial_i G(t-s)\|_{\mathcal{H}^1}$ it is easy to prove that

$$\| \int_0^t \partial_i G(t-s) * \nabla\partial_j E_2 * f^i_\lambda f^j_\lambda(s) ds\|_q$$

is bounded by $C\ log\lambda^2\ \lambda^{-1}$.

It suffices to rewrite step by step the proof of theorem 4.1 replacing the L^q norms of $\partial_i G(t-s)$ by the L^q norms of $\partial_i G(t-s) * \nabla\partial_j E_2$ to get:

THEOREM 4.2. *Let w be a solution of $(\mathcal{P}_3)$ and $q \geq 1$. Let us assume that :*

$$i) \quad (1+t)^{\frac{1}{2}+\varepsilon}\|f(t) - G * u_0(t)\|_2 \leq C \quad t \geq 0$$

for some $\varepsilon > 0$

$$ii) \quad \|f(t)\|_{2r} \ \leq \ C(1+t)^{-1+\frac{1}{2r}} \quad t \geq 0$$

where $q \geq r > \frac{2q}{q+2}$. Then

$$\frac{t^{\frac{3}{2}-\frac{1}{q}}}{log\ t}\|w(t) - (\frac{M^i M^j}{2})log\ t\ \partial_i G(t) * \nabla\partial_j E_2\|_q \to 0 \quad as\ t \to \infty$$

provided that $u_0 \in L^1 \cap L^2(\mathbb{R}^2, |x|)$

Remark 4.2 If we take u to be a solution of (NS) and v a solution of $(\mathcal{L}_2)$ with initial data $u_0 \in (L^1 \cap L^2)(\mathbb{R}^2, 1+|x|) \cap L^{2r}(\mathbb{R}^2)$ then we can apply the theorem with $f = u$ or $f = G(t) * u_0$ and $M = \int_{\mathbb{R}^2} u_0(x)dx$ to obtain the behavior of the third term, which is the same for both of them.

4.4. Conclusion.

Putting together the previous results we obtain :

THEOREM 4.3. *Let u be a solution of te two dimensional (NS) with initial data $u_0 \in (L^1 \cap L^2)(\mathbb{R}^2, 1+|x|)$ such that div $u_0 = 0$ and set $M = \int_{\mathbb{R}^2} u_0(x)dx$. Then, for a given $q \geq 1$*

$$\frac{t^{\frac{3}{2}-\frac{1}{q}}}{log(t)}\|u(t) - MG(t) + log(t)\ \big(\partial_i G(t)(\frac{M^i M}{2})\ +\ \partial_i G(t) * \nabla\partial_j E_2\ (\frac{M^i M^j}{2})\big)\|_q$$

tends to zero as t goes to infinity provided that $u_0 \in L^{2r}(\mathbb{R}^2)$ for some $q \geq r > \frac{2q}{q+2}$.

Remark 4.3 When $u_0 \in (L^1 \cap L^2)(\mathbb{R}^2, 1+|x|)$ we can take $r = 1$ and then the result holds for $1 \leq q < 2$.

5. HIGHER DIMENSIONS : n > 2.

In the sequel, we shall be concerned with solutions u of (NS) taking data $u_0 \in L^p \cap L^n$ such that *div* $u_0 = 0$, $\|u_0\|_n$ is small and $1 \leq p \leq n$. For that kind of data unique global strong solutions are known to exist.

5.1. Decay estimates.

We shall improve here Theorem 1.1.2 i) by proving the following result :

THEOREM 5.1. *Let u be a strong solution of (NS) with data $u_0 \in L^p \cap L^n(\mathbb{R}^n)$, $1 \leq p \leq n$, of small L^n norm such that div $u_0 = 0$. Then,*

$$\|u(t)\|_q \leq Ct^{(-\frac{1}{p}+\frac{1}{q})\frac{n}{2}}$$

if $q \geq p$.

Proof. Taking norms in the integral equation associated to (NS) we get

$$\|u(t)\|_q \;\leq\; Ct^{(-\frac{1}{p}+\frac{1}{q})\frac{n}{2}} \;+\; C\int_0^t (t-s)^{-\frac{1}{2}+(-\frac{1}{r}+\frac{1}{q})\frac{n}{2}} s^{(\frac{-2}{k}+\frac{1}{r})\frac{n}{2}}$$

for $q \geq p,\ q \geq r,\ 2r \geq k,\ r \geq 1$. Here we have used the fact that

$$\|\partial_i G * \partial_j \nabla E_2\|_r \;\leq\; \begin{cases} C\|\partial_i G\|_r & r > 1 \\ C\|\partial_i G\|_{\mathcal{H}^1} & r = 1 \end{cases}$$

and the estimates (see Theorem 1.1.2 and remarks 1.1.2):

$$\|u(t)\|_{2r}^2 \;\leq\; Ct^{2(\frac{-1}{k}+\frac{1}{2r})\frac{n}{2}}$$

known to be valid for $1 \leq k \leq n$, $2r \geq k \geq p$ and, if $k < \frac{n}{2}$, $2r \leq \frac{kn}{n-2k}$. We have also used some classical estimates on the heat kernel :

$$\|\partial^\alpha G(t) * a\|_q \;\leq\; Ct^{-\frac{|\alpha|}{2}+(-\frac{1}{r}+\frac{1}{q})\frac{n}{2}}\|a\|_r$$

for $q \geq r$. Remark that $2r \geq k$ yields a restriction on r if $k > 2$.

We split the integral as follows :

a)

$$\int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}+(-\frac{1}{r}+\frac{1}{q})\frac{n}{2}} s^{(\frac{-2}{k}+\frac{1}{r})\frac{n}{2}} \;\leq\; C\, t^{\frac{1}{2}+(\frac{1}{q}-\frac{2}{k})\frac{n}{2}}$$

choosing r such that $\frac{1}{2}+(-\frac{1}{r}+\frac{1}{q})\frac{n}{2} > 0$, that is, $q \geq r > \frac{nq}{q+n}$.

b)

$$\int_0^{\frac{t}{2}} (t-s)^{-\frac{1}{2}+(-\frac{1}{r}+\frac{1}{q})\frac{n}{2}} s^{(\frac{-2}{k}+\frac{1}{r})\frac{n}{2}} \;\leq\; C\, t^{\frac{1}{2}+(\frac{1}{q}-\frac{2}{k})\frac{n}{2}}$$

choosing r such that $(\frac{-2}{k}+\frac{1}{r})\frac{n}{2}+1 > 0$, that is, $1 \leq r < \frac{nk}{2(n-k)}$.

Those conditions imply $k > \frac{2n}{n+2}$ and $q \geq \frac{k}{2}$.

Supposing the above restrictions to be verified we should get

$$\|u(t)\|_q \;\leq\; Ct^{(-\frac{1}{p}+\frac{1}{q})\frac{n}{2}} + Ct^{\frac{1}{2}+(\frac{1}{q}-\frac{2}{k})\frac{n}{2}} \;\leq\; Ct^{(-\frac{1}{p}+\frac{1}{q})\frac{n}{2}}$$

if $k = \frac{2pn}{n+p}$, where C is a constant depending on q and on the data. This is valid for any $q \geq \frac{k}{2} = \frac{pn}{n+p}$. Since $\frac{np}{n+p} \leq p$ it is valid for any $q \geq q$. On the other hand $k = \frac{2pn}{n+p} > \frac{2n}{n+2}$. It remains to check that, when $k < \frac{n}{2}$, the conditions $q \geq r > \frac{nq}{q+n}$ and $1 \leq r < \frac{nk}{2(n-k)}$ are compatible with the restriction $2r \leq \frac{kn}{n-2k}$. Several posibilities arise :

- $p \geq \frac{n}{3}$. Since $k \geq p$ the restriction is innecessary so that we can find an adequate r in both cases.

- $p < \frac{n}{3}$. We have $\frac{nk}{2(n-k)} < \frac{kn}{2(n-2k)}$, so that we can find an adequate r for the case b). It remains the condition $\frac{nq}{q+n} \leq \frac{kn}{2(n-2k)}$, that is, $q(2n-5k) \leq kn$. When $p \geq \frac{n}{4}$ we get $2n-5k \leq 0$ and the inequality holds for any $q \geq p$. When $p < \frac{n}{4}$ we need to take $q \leq \frac{pn}{n-4p}$ in order to find some r for a).

Therefore, the decay estimate

$$(*) \quad \|u(t)\|_q \leq Ct^{(-\frac{1}{p}+\frac{1}{q})\frac{n}{2}}$$

holds now for $q \geq p$ if $p \geq \frac{n}{4}$ and for $\frac{pn}{n-4p} \geq q \geq p$ when $1 \leq p < \frac{n}{4}$. If $n \geq 4$ we have concluded since $p < \frac{n}{4}$ is excluded.

If we iterate this process using these new decay rates when estimating the integrals appearing in the integral equation we obtain that (*) holds for $q \geq p$ if $p \geq \frac{n}{8}$ and for $\frac{pn}{n-8p} \geq q \geq p$ when $1 \leq p < \frac{n}{8}$. The last possibility is again excluded when $n \geq 8$.

In general, assuming (*) to hold for $q \geq p$ if $p \geq \frac{n}{z}$ and for $\frac{pn}{n-zp} \geq q \geq p$ when $1 \leq p < \frac{n}{z}$ we get from the integral equation that (*) also holds for $q \geq p$ if $p \geq \frac{n}{2z}$ and for $\frac{pn}{n-2zp} \geq q \geq p$ when $1 \leq p < \frac{n}{2z}$ so that when $n \leq 2z$ we have concluded. Thus, we can get the right decay estimate for any $q \geq p$ by repeating this procedure. The number of iterations we need depends on the dimension. □

5.2. First term.

The integral equation yields:

$$\|G(t) * u_0 - u(t)\|_q \leq C \int_0^t (t-s)^{-\frac{1}{2}+(-\frac{1}{r}+\frac{1}{q})\frac{n}{2}} s^{(\frac{-2}{p}+\frac{1}{r})\frac{n}{2}}$$

for $q \geq p$, $q \geq r \geq 1$ and $2r \geq p$, that is, $q \geq p$. As we did in the above proof we split the integral in two intervals $[0, \frac{t}{2}]$ and $[\frac{t}{2}, t]$. By choosing an adequate r we conclude that:

THEOREM 5.2. *Let u be a strong solution of (NS) with data $u_0 \in L^p \cap L^n(\mathbb{R}^n)$, $\frac{nq}{n+q} < p \leq n$, of small L^n norm such that div $u_0 = 0$. Then,*

$$\|G(t) * u_0 - u(t)\|_q \leq C t^{(-\frac{1}{p}+\frac{1}{q})\frac{n}{2}} t^{-\frac{n}{2p}+\frac{1}{2}}$$

if $q \geq p$.

5.3. Second term.

Let us define u, h and v as in Section 3. We get from the integral equation:

$$\begin{aligned} \|u(t) - v(t)\|_q \;&\leq\; \|\textstyle\int_0^t \partial_i G(t-s) * (u^i(u-h) + h(u^i - h^i))(s) ds\|_q \\ &+ \|\textstyle\int_0^t \partial_i G(t-s) * \partial_j \nabla E_2 * (u^i(u-h) + h(u^i - h^i))(s) ds\|_q \\ &\leq\; C \textstyle\int_0^t (t-s)^{-\frac{1}{2}+(-\frac{1}{r}+\frac{1}{q})\frac{n}{2}} s^{(-\frac{1}{p}+\frac{1}{2r})\frac{n}{2}} s^{(-\frac{1}{p}+\frac{1}{2r})\frac{n}{2}} s^{-\frac{n}{2p}+\frac{1}{2}} \end{aligned}$$

for $q \geq r \geq 1,\ 2r \geq p$, where we have used the estimates:

$$\begin{aligned} \|u(t)\|_{2r} \;&\leq\; C t^{(-\frac{1}{p}+\frac{1}{2r})\frac{n}{2}} \\ \|(u-h)(t)\|_{2r} \;&\leq\; C t^{(-\frac{1}{p}+\frac{1}{2r})\frac{n}{2}} t^{-\frac{n}{2p}+\frac{1}{2}} \end{aligned}$$

when $\frac{2n}{n+2} < p \leq n$. We split the integral as follows:

a)

$$\int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}+(-\frac{1}{r}+\frac{1}{q})\frac{n}{2}} s^{(-\frac{3}{p}+\frac{1}{r})\frac{n}{2}+\frac{1}{2}} \;\leq\; C t^{(-\frac{3}{p}+\frac{1}{q})\frac{n}{2}+1}$$

if $r > \frac{nq}{q+n}$

b)

$$\int_0^{\frac{t}{2}} (t-s)^{-\frac{1}{2}+(-\frac{1}{r}+\frac{1}{q})\frac{n}{2}} s^{(-\frac{3}{p}+\frac{1}{r})\frac{n}{2}+\frac{1}{2}} \;\leq\; C t^{(-\frac{3}{p}+\frac{1}{q})\frac{n}{2}+1}$$

if $(-\frac{3}{p}+\frac{1}{q})\frac{n}{2}+\frac{3}{2}>0$, that is, $r \leq q$ and $1 \leq r < \frac{np}{3(n-p)}$, provided $p > \frac{3n}{n+3.}$

Therefore,

THEOREM 5.3. *Let u be a strong solution of (NS) with data $u_0 \in L^p \cap L^n(\mathbb{R}^n)$, $\frac{3n}{n+3} < p < n$ of small L^n norm such that $div\ u_0 = 0$. Then*

$$\|(u-v)(t)\|_q \leq Ct^{(-\frac{1}{p}+\frac{1}{q})\frac{n}{2}}t^{-\frac{n}{2p}+\frac{1}{2}}t^{-\frac{n}{2p}+\frac{1}{2}}$$

if $q \geq p$, where v is the solution of $(\mathcal{L}_n)$ with data u_0.

This decay is faster that the corresponding to $u(t)-h(t)$.

5.4. Explicit second term.

Let u be a strong solution of (NS) with data $u_0 \in L^1 \cap L^n(\mathbb{R}^n)$ of small L^n norm such that $div\ u_0 = 0$. Making the change of variables

$$u_\lambda(x,t) = \lambda^n(\lambda^2 t, \lambda x,), \quad \lambda > 0$$

we obtain for u_λ the integral equation:

$$\begin{aligned} u_\lambda(t) &= G(t) * u_{u_\lambda,0} - \lambda^{1-n}\textstyle\int_0^t \partial_i G(t-s) * u_\lambda^i \partial_i u_\lambda(s) ds \\ &-\lambda^{1-n}\textstyle\int_0^t G(t-s) * \nabla\partial_j E_n * u_\lambda^i \partial u_\lambda^j(s) ds \\ &= w_{1,\lambda} + w_{2,\lambda} + w_{3,\lambda} \end{aligned}$$

where E_n stands for the fundamental solution of $-\Delta$ in $\mathbb{R}^n$. We have denoted by $w_{i,\lambda}$ the rescaled solutions of the n-dimensional analogous of problems $(\mathcal{P}_i)$ with $f = u$.

Problem $\mathcal{P}_1$

The first term w_1 is the solution of the heat equation with data u_0. If, for instance, $u_0 \in L^1(1+|x|;\mathbb{R}^n)$ we know that:

$$t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\|G(t)*u_0 - MG(t) + m^i\partial_i G(t)\|_q \to 0 \quad as\ t \to \infty$$

where $M = \int_{\mathbb{R}^n} u_0$ and $m^i = \int_{\mathbb{R}^n} x^i u_0$ for $i = 1,...,n$. (see [7])

Problem $\mathcal{P}_2$

It is easy to prove that

$$\|w_{2,\lambda}\|_q \;=\; \lambda^{-1}\|\int_0^t \partial_i G(t-s) * \frac{f_\lambda^i f_\lambda(s)}{\lambda^{n-2}} ds\|_q$$

is bounded by $C\lambda^{-1}$. Keeping the notations of section 4.2 we shall see that it is possible to take $g(t) = (\int_0^\infty \int_{\mathbb{R}^n} u^i u(y,\sigma) d\sigma dy)\ \partial_i G(t)$ and $\delta(t) = t^{\frac{1}{2}}$. More precisely,

PROPOSITION 5.1. *Let w_2 be the solution of $(\mathcal{P}_2)$ and $q \geq 1$. Then*

$$t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\|w(t) + (\int_0^\infty \int_{\mathbb{R}^n} u^i u(\sigma,y) dy d\sigma)\ \partial_i G(t)\|_q \to 0 \quad as\ t \to \infty$$

provided that $u_0 \in L^{2r}(\mathbb{R}^n)$ for some $q \geq r > \frac{nq}{q+n}$.

Proof. We must prove that

$$\|\int_0^1 \partial_i G(1-s) * \frac{u_\lambda^i u_\lambda}{\lambda^{n-2}} - (\int_0^\infty \int_{\mathbb{R}^n} u^i u(\sigma,y) d\sigma dt)\partial_i G(1)\|_q \to 0 \quad as\ \lambda \to \infty$$

Since

$$\int_0^1 \partial_i G(1-s) * \frac{u^i_\lambda u_\lambda}{\lambda^{n-2}} \;=\; \int_0^{\lambda^2}\int_{\mathbb{R}^n} \partial_i G(1-\frac{s}{\lambda^2}, x-\frac{y}{\lambda}) u^i u$$

this is equivalent to proving :

$$\| \int_0^{\lambda^2}\int_{\mathbb{R}^n} \partial_i G(1-\frac{s}{\lambda^2}, x-\frac{y}{\lambda}) u^i u - (\int_0^{\infty}\int_{\mathbb{R}^n} u^i u(\sigma,y) d\sigma dt)\partial_i G(1)\|_q \to 0 \quad as\ \lambda\to\infty$$

On the other hand,

$$\big(\int_0^{\infty}\int_{\mathbb{R}^n} u^i u(\sigma,y) d\sigma dt \;-\; \int_0^{\lambda^2}\int_{\mathbb{R}^n} u^i u(\sigma,y) d\sigma dt\big)\partial_i G(1,x) \to 0$$

in L^q_x as $\lambda\to\infty$, so that it suffices to prove that:

$$\| \int_0^{\lambda^2}\int_{\mathbb{R}^n} (\partial_i G(1-\frac{s}{\lambda^2}, x-\frac{y}{\lambda}) - \partial_i G(1,x)) u^i u\|_q \to 0 \quad as\ \lambda\to\infty$$

We split it as in the proof of Theorem 4.1. We have

$$\|I^1_{\delta,\lambda}\|_{L^q_x} \le C\varepsilon \int_0^{\lambda^2\delta}\int_{|y|\le\lambda\delta} |u|^2(s,y) ds dy \;\le\; \varepsilon C$$

if δ is small enough since (see Theorem 5.1 and Remark 1.1.2)

$$\int_0^{\infty}\int_{\mathbb{R}^n} |u|^2(s,y) ds dy \;\le\; C\int_0^{\infty} (1+s)^{\frac{-n}{2}} ds \;\le\; C$$

for $n>2$. Next,

$$\|J^2_{\delta,\lambda}\|_{L^q_x} \le C\|\nabla G(t)\|_{L^q_x} \int_{\lambda^2\delta}^{\lambda^2}\int_{\mathbb{R}^n} |u|^2(s,y) ds dy \quad \to 0$$

as $\lambda\to\infty$ for a fixed δ and

$$\|I^2_{\delta,\lambda}\|_{L^q_x} \le C\int_{\lambda^2\delta}^{\lambda^2} \| \int_{\mathbb{R}^n} \partial_i G(1-\frac{s}{\lambda^2}, x-\frac{y}{\lambda}) u^i u(s,y)\ dy\|_{L^q_x} ds$$

$$\le C\lambda^{n(1-\frac{1}{r})} \int_{\lambda^2\delta}^{\lambda^2} (1-\frac{s}{\lambda^2})^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})-\frac{1}{2}} s^{-\frac{n}{2}(2-\frac{1}{r})} \;\le\; C\lambda^{2-n}$$

if $q\ge r>\frac{nq}{q+n}$ and $u_0\in L^{2r}(\mathbb{R}^n)$ (see Theorem 5.1 and Remark 1.1.2). Since $n>2$, it tends to zero as $\lambda\to\infty$ for a fixed δ.

Concerning $\|J^3_{\delta,\lambda}\|_{L^q_x}$ we have

$$\|J^3_{\delta,\lambda}\|_{L^q_x} \;\le\; C\|\nabla G(t)\|_{L^q_x} \int_0^{\lambda^2\delta}\int_{|y|\ge\delta\lambda} |u|^2(s,y) ds dy$$

Since

$$\int_0^{\infty}\int_{\mathbb{R}^n} |u|^2(s,y) ds dy \;\le\; C$$

it follows that
$$\int_0^\infty \int_{|y|\geq \delta\lambda} |u|^2(s,y)dsdy \quad \to 0 \quad as\ \lambda \to \infty$$

Finally,

$$\|I^3_{\lambda,\delta}\|_{L^q_x} \leq \int_0^{\lambda^2\delta} \| \int_{|y|\geq\delta\lambda} \partial_i G(1-\frac{s}{\lambda^2}, x-\frac{y}{\lambda}) u^i u(y,s)\|_{L^q_x} \leq$$

$$\leq C\int_0^{\lambda^2\delta} (1-\frac{s}{\lambda^2})^{-1+\frac{1}{q}-\frac{1}{2}} \|u^i u(y,s)\|_{L^1(|y|\geq\lambda\delta)} \leq C\int_0^{\lambda^2\delta}\int_{|y|\geq\delta\lambda} |u|^2(y,s)dyds$$

and we finish in the same way as before. □

Problem $\mathcal{P}_3$

In this case, by slightly modifying the proof above we get

PROPOSITION 5.2. *Let w the solution of $(\mathcal{P}_3)$ with $f=u$ and $q\geq 1$. Then*

$$t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\|w(t) + (\int_0^\infty \int_{\mathbb{R}^n} u^i u^j(\sigma,y)dyd\sigma)\ \partial_i G(t) * \nabla\partial_j E_n\|_q \to 0 \quad as\ t\to\infty$$

provided that $u_0 \in L^{2r}(\mathbb{R}^n)$ for some $q \geq r > \frac{nq}{q+n}$.

As a consequence of these results we get :

THEOREM 5.4. *Let u be a strong solution of (NS) with data $u_0 \in L^1(\mathbb{R}^n, 1+|x|) \cap L^n(\mathbb{R}^n)$ of small L^n norm and $q \geq 1$. Then*

$$t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\|u(t) - MG(t) - m^i\partial_i G(t) + R(t)\|_q \to 0$$

as $t\to\infty$, where

$$R(t) = \big(\int_0^\infty \int_{\mathbb{R}^n} u^i u(\sigma,y)dyd\sigma\big)\ \partial_i G(t) + \big(\int_0^\infty\int_{\mathbb{R}^n} u^i u^j(\sigma,y)dyd\sigma\big)\partial_i G(t) * \nabla\partial_j E_n$$

provided that $u_0 \in L^{2r}(\mathbb{R}^n)$ for some $q\geq r > \frac{nq}{q+n}$.

Remark 5.2 If instead of the problems $(\mathcal{P}_i)$ corresponding to solutions of (NS) we consider those corresponding to solutions v of $(\mathcal{L}_n)$ with $f = G(t) * u_0$ replaced by $f=u$, the analogous of Propositions 5.1 and 5.2 also hold. Therefore, if $u_0 \in L^1(1+|x|;\mathbb{R}^n)\cap L^n(\mathbb{R}^n)$, $q\geq 1$ and $u_0 \in L^{2r}(\mathbb{R}^n)$ for some $q \geq r \geq \frac{nq}{q+n}$ the solution v of $(\mathcal{L}_n)$ satisfies

$$t^{\frac{1}{2}+\frac{n}{2}(1-\frac{1}{q})}\|v(t) - MG(t) - m^i\partial_i G(t) + R(t)\|_q \to 0$$

as $t\to\infty$, where

$$R(t) = \big(\int_0^\infty\int_{\mathbb{R}^n} (G(t)*u_0)^i(G(t)*u_0)(\sigma,y)dyd\sigma\big)\ \partial_i G(t)$$

$$+\big(+\int_0^\infty\int_{\mathbb{R}^n}(G(t)*u_0)^i(G(t)*u_0)^j(\sigma,y)dyd\sigma\big)\partial_i G(t)*\nabla\partial_j E_n$$

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