Variational methods for image reconstruction and parameter identification

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General framework

Medium $\mathcal{R}$ with defects $\Omega$: How many? How big? Where?
Material parameters for $\Omega$?

Applications

- Medicine (tumors, fracture, clots)
- Geophysics (gas, oil reservoirs)
- Structure analysis (cracks, damage)
Outline

- The inverse scattering problem.
- Constrained optimization reformulation.
- Descent method:
  - Parameters
  - Objects: Initialization.
  - Objects: Correction.
  - Combined.

- Alternatives.

- Conclusion
Scattering problem

An emitted incident wave $u_{inc}$ interacts with a medium $\mathcal{R}$ containing objects $\Omega$. 

Forward problem

- The shape, size, location and material parameters of the objects are known.
- What is the wave field at the receptors "×"
- Well posed problem: unique solution that depends continuously on the initial data.
Scattering problem

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Forward problem

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- **What is the wave field at the receptors "×"**
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Scattering problem

An emitted incident wave $u_{inc}$ interacts with a medium $\mathcal{R}$ containing objects $\Omega$.

Inverse problem

- The wave field is measured at receptors $u_{meas}$.
- Find objects $\Omega$ and material parameters such that
  \[ u = u_{meas} \text{ on } \Gamma_{meas}, \quad u = \text{sol. forward problem} \]
- Ill posed problem: it may not have a solution, it may not be unique and it may not depend continuously on the data.
How do we distinguish the objects from the background?

They differ in the constitutive parameters:

- elastic constants in problems of acoustic scattering,
- electric conductivity and permittivity in electromagnetic scattering,
- thermic diffusivity and conductivity in thermal scattering.

Incident wave?

Acoustic, electromagnetic, thermal waves.
Simplifying assumptions:

- We work with acoustic waves.
- The incident waves are time-harmonic
  \[ U_{inc}(x, t) = \text{Re}[e^{-i\omega t} u_{inc}(x)]. \]
- \( u_{inc} \) is a plane wave propagating in the direction \( d \),
  \[ u_{inc}(x) = e^{ikx \cdot d}. \]
- The solution of the forward problem is time-harmonic
  \[ U(x, t) = \text{Re}[e^{-i\omega t} u(x)]. \]
Forward Helmholtz problem

The incident wave interacts with the objects (penetrable objects) $\Omega$. It generates a scattered wave $u_{sc}$ outside $\mathbb{R}^n \setminus \Omega$ and a transmitted wave $u_{tr}$ in $\Omega$. The total wave

$$u = u_{inc} + u_{sc} \quad \text{in} \quad \mathbb{R}^n \setminus \Omega \quad \text{and} \quad u = u_{tr} \quad \text{in} \quad \Omega$$

is a solution of:

$$\begin{cases}
\Delta u + k_e^2 u = 0 & \text{in} \quad \mathbb{R}^n \setminus \Omega \\
\Delta u + k_i^2 u = 0 & \text{in} \quad \Omega \\
u^- = u^+, \quad \partial_n u^- = \partial_n u^+ & \text{on} \quad \partial \Omega \\
\lim_{r \to \infty} r^{(n-1)/2} (\partial_r (u - u_{inc}) - i k_e (u - u_{inc})) = 0
\end{cases}$$

where $k_e, k_i > 0$ are known.
Constrained optimization reformulation

Inverse problem

Find $\Omega, k_i$ such the solution $u$ of the forward problem satisfies

$$u = u_{meas} \quad \text{in } \Gamma_{meas}.$$ 

Variational reformulation

Find $\Omega, k_i$ minimizing

$$J(\Omega, k_i) = \frac{1}{2} \int_{\Gamma_{meas}} |u - u_{meas}|^2 d\ell$$

where $u$ is the solution of the forward problem for objects $\Omega$ and parameters $k_i$.

- $\Omega, k_i$ are the design variable.
- The forward problem is the constraint.
### Constrained optimization reformulation

#### Inverse problem

Find $\Omega, k_i$ such the solution $u$ of the forward problem satisfies

$$u = u_{\text{meas}} \quad \text{in } \Gamma_{\text{meas}}.$$  

#### Variational reformulation

Find $\Omega, k_i$ minimizing

$$J(\Omega, k_i) = \frac{1}{2} \int_{\Gamma_{\text{meas}}} |u - u_{\text{meas}}|^2 d\ell$$

where $u$ is the solution of the forward problem for objects $\Omega$ and parameters $k_i$.

- $\Omega, k_i$ are the design variable.
- The forward problem is the constraint.
Optimization Strategy

Descent method

- Find initial guesses for the parameter $k_{i,0}$ and the objects $\Omega_0$.
- Generate sequences of approximations $k_{i,n}$, $\Omega_n$ along which the functional $J(\Omega_n, k_{i,n})$ decreases.
- Stop when variations are negligible.
For functions $J(x)$ of a vector variable $x$:

- Choose an initial guess $x_0$.

- Generate a sequence $x_n$ along which $J(x_n)$ decreases. We move in the direction of steepest descent $-\nabla J(x_{n-1})$:

$$x_n = x_{n-1} - \alpha \nabla J(x_{n-1}).$$

To find it, we choose a direction $v$ for a negative derivative:

$$\frac{dJ(x_{n-1} + tv)}{dt} = v \nabla J(x_{n-1}) < 0 \Rightarrow v = -\nabla J(x_{n-1}).$$
How to find descent directions for material parameters

- \( J(k_i), k_i \) material parameter.

- Choose a perturbation function \( v \) and study \( J(t) = J(k_i + tv) \).

- Fixed \( k_i \) and \( v \), this is a function of a real parameter \( t \):

\[
\frac{dJ(t)}{dt} = \frac{dJ(k_i + tv)}{dt} = Re \left[ \int_{\Gamma_{meas}} (u(t) - u_{meas}) \frac{du(t)}{dt} dl \right],
\]

\( u(t) \) being the solution of the forward problem with coefficient \( k_i + tv \) inside the objects \( \Omega \).

- Computing \( \frac{du(t)}{dt} \) is avoided introducing an adjoint field \( p \):

\[
\frac{dJ(t)}{dt} \bigg|_{t=0} = Re \left[ \int_{\Omega} v u(0) p dl \right].
\]

- Choosing \( v = -Re[u(0) p] \), \( \frac{dJ(t)}{dt} \bigg|_{t=0} < 0 \) and \( J(k_i + tv) < J(k_i) \) for \( t \) small.
Given a guess \( k_i \), the corrected guess is \( k'_i = k_i - t \text{Re}[u \phi] \), with

### Adjoint problem

\[
\begin{align*}
\Delta p + k^2_e p &= (u_{meas} - u)\delta_{\Gamma_{meas}} \quad \text{in } \mathbb{R}^n \setminus \Omega \\
\Delta p + k^2_i p &= 0 \quad \text{in } \Omega \\
p^- &= p^+ , \quad \partial_n p^- = \partial_n p^+ \quad \text{on } \partial \Omega \\
\lim_{r \to \infty} r^{(n-1)/2}(\partial_r p - ik_e p) &= 0
\end{align*}
\]

### Forward problem

\[
\begin{align*}
\Delta u + k^2_e u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega \\
\Delta u + k^2_i u &= 0 \quad \text{in } \Omega \\
u^- &= u^+ , \quad \partial_n u^- = \partial_n u^+ \quad \text{on } \partial \Omega \\
\lim_{r \to \infty} r^{(n-1)/2}(\partial_r (u - u_{inc}) - ik_e (u - u_{inc})) &= 0
\end{align*}
\]

Carpio-Rapun 2008
How to find descent directions for the domains?

Topological derivative (Sokolowski-Zokowski 1999)

The topological derivative of a functional $J(\mathcal{R})$ at the point $x \in \mathcal{R}$ is

$$D_T(x, \mathcal{R}) = \lim_{\varepsilon \to 0} \frac{J(\mathcal{R} \setminus B_\varepsilon(x)) - J(\mathcal{R})}{\text{Vol}(B_\varepsilon(x))}$$

- It is a scalar function defined for $x \in \mathcal{R}$.
- It measures the sensitivity of the functional to placing a small object at $x$.
- $D_T(x, \mathcal{R}) << 0 \implies$ large probability to find an object

If $x \in \mathcal{R}$ and $h(\varepsilon) = \text{Vol}(B_\varepsilon(x))$

$$J(\mathcal{R} \setminus B_\varepsilon(x)) = J(\mathcal{R}) + h(\varepsilon)D_T(x, \mathcal{R}) + o(h(\varepsilon)) \quad \text{cualdo} \quad \varepsilon \to 0$$
Correction
Given a guess $\Omega_{ap}$, the corrected guess is

$$\Omega_{ap} + \{\text{points with large negative TD}\}.$$ 

Topological derivative
For $x \in \mathbb{R}^n \setminus \Omega_{ap}$, the topological derivative of

$$J(\mathbb{R}^n \setminus \Omega_{ap}) = \frac{1}{2} \int_{\Gamma_{meas}} |u - u_{meas}|^2$$

is

$$D_T(x, \mathbb{R}^n \setminus \Omega_{ap}) = \text{Re} \left[ (k_i^2 - k_e^2) u(x)p(x) \right]$$

where $u$ and $p$ solve forward and adjoint problems with $\Omega = \Omega_{ap}$

It changes when $\Omega_{ap}$ changes.

Forward problem $\Omega = \Omega_{ap}$:

\[
\begin{align*}
\Delta u + k_e^2 u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega_{ap} \\
\Delta u + k_i^2 u &= 0 \quad \text{in } \Omega_{ap} \\
 u^- &= u^+, \quad \partial_n u^- = \partial_n u^+ \quad \text{on } \partial \Omega_{ap} \\
\lim_{r \to \infty} r^{(n-1)/2}(\partial_r(u - u_{inc}) - ik_e(u - u_{inc})) &= 0
\end{align*}
\]

Adjoint problem with $\Omega = \Omega_{ap}$:

\[
\begin{align*}
\Delta p + k_e^2 p &= (u_{meas} - u)\delta_{\Gamma_{meas}} \quad \text{in } \mathbb{R}^n \setminus \Omega_{ap} \\
\Delta p + k_i^2 p &= 0 \quad \text{in } \Omega_{ap} \\
p^- &= p^+, \quad \partial_n p^- = \partial_n p^+ \quad \text{on } \partial \Omega_{ap} \\
\lim_{r \to \infty} r^{(n-1)/2}(\partial_r p - ik_e p) &= 0
\end{align*}
\]

The boundary conditions affect $u$ and $p$. The forward field sees $u_{inc}$. The adjoint field sees $u_{meas}$.
First guess for the objects

- Assume $k_i$ known. Set $\Omega_{ap} = \emptyset$.
- Compute
  \[
  D_T(x, \mathbb{R}^n) = \text{Re } [(k_i^2 - k_e^2) u(x)p(x)]
  \]
  where $u$ and $p$ are solutions of forward and adjoint problems with $\Omega_{ap} = \emptyset$.
- Set $\Omega_{ap} = \{ x \in \mathbb{R}^n | D_T(x, \mathbb{R}^n) < -C_1 \}$.

\[
\begin{cases}
  \Delta u + k_e^2 u = 0 & \text{in } \mathbb{R}^n \\
  \lim_{r \to \infty} r^{(n-1)/2} (\partial_r (u - u_{inc}) - ik_e (u - u_{inc})) = 0
\end{cases}
\]
If $k_e$ is constant, $u = u_{inc}(x) = e^{ik_e x \cdot d}$

\[
\begin{cases}
  \Delta p + k_e^2 p = (u_{meas} - u) \partial_{\Gamma_{meas}} & \text{in } \mathbb{R}^n \\
  \lim_{r \to \infty} r^{(n-1)/2} (\partial_r p - ik_e p) = 0
\end{cases}
\]
If $k_e$ is constant, $p = \int_{\Gamma_{meas}} G_k e(x - y) (u_{meas} - u)(y) \, dy$
Examples
"×" = receptors, 24 directions in $[0, 2\pi)$, 
$k_e = 2$ and $k_i = 1/2$. Noise in the data = 0.1%

- Similar results if + receptors, + far, + incident waves, + noise
The results depend on the wave length.
Top row $k_e = 2$ and $k_i = 1/2$. Low row $k_e = 4$ and $k_i = 1$.

$$k = \omega \sqrt{\frac{\rho}{\mu}}.$$
Heterogeneous materials
Correction of the initial objects

Previous examples with $\Omega = \emptyset$.

Initial approximation $\Omega_1$. We have superimposed the topological derivative when $\Omega = \Omega_1$. 
Monotone strategy

Idea

We add to the current approximation $\Omega_{ap}$ the points at which the TD takes the largest negative values. Both J and TD decrease.

Algorithm

1. Compute the TD when $\Omega = \emptyset$
2. Set $\Omega_1 = \{x, D_T(x, \mathbb{R}^n) < -C_1\}$, $C_1 > 0$
3. For $j=1:j_{\text{max}}$
   - Compute the TD in $\mathbb{R}^n \setminus \Omega_j$
   - Set $\Omega_{j+1} \supset \Omega_j$
   $$\Omega_{j+1} = \Omega_j \cup \{x, D_T(x, \mathbb{R}^n \setminus \Omega_j) < -C_{j+1}\}$$
Piecewise constant parameters
Piecewise constant parameters
Piecewise constant parameters
Heterogenous material and limited data
Non monotone strategy

The previous scheme generates an increasing sequence of domains. If at some point an spurious region is included it cannot be removed.

Algorithm

1. Compute the TD when $\Omega = \emptyset$ (as before).
2. Take $\Omega_1 = \{x, D_T(x, \mathbb{R}^n) < -C_1\}, \ C_1 > 0$ (as before).
3. For $j=1:j_{\text{max}}$
   - Compute the TD in $\mathbb{R}^n$ when $\Omega = \Omega_j$,
   - Select $\Omega_{j+1}
     \begin{align*}
     \text{if } x \in \mathbb{R}^n \setminus \Omega_j \text{ and } TD < -C_j & \implies x \in \Omega_{j+1} \\
     \text{if } x \in \Omega_j \text{ and } TD > C'_j & \implies x \notin \Omega_{j+1}
     \end{align*}
Reconstruction of an annular region. Points can be added or removed at each stage. The hole can be seen at the sixth iteration.
Correction of parameters and objects

Initialization

- Initial guess $k_{i,0}$ for the parameters $k_i$: a small perturbation of $k_e$.
- Initial guess $\Omega_0$ for the objects $\Omega$: Set $k_i = k_{i,0}$ and compute the TD with object $\Omega = \emptyset$:
  \[
  D_T(x, \mathbb{R}^2) = \text{Re} \left[ (k_{i,0}^2 - k_e^2) u(x) p(x) \right]
  \]
  where $u$ and $p$ are forward and adjoint fields with object $\Omega = \emptyset$ and parameter $k_{i,0}$.

Iteration

- Update $k_{i,j}(x)$ using the gradient technique:
  \[
  k_{i,j+1}(x) = k_{i,j}(x) - t \text{Re}[u(x) p(x)],
  \]
  where $u$, $p$ solve forward and adjoint problems with object $\Omega_j$ and parameter $k_{i,j}(x)$,
- Update $\Omega_{j+1}$ computing the TD with object $\Omega_j$ and parameter $k_{i,j+1}(x)$.
Piecewise constant parameters
Heretogeneous background

Figure 6: (a) Geometrical configuration and (b) function $k_0$ for the examples with space-dependent parameters.

Figure 7: (a) TD with $\Omega = 0$, $a_i = 0.9$ and $k_i = 1.8$. (b) Initial guess $\Omega = \Omega_i$ and TD when $\Omega_i = \Omega_2$, $a_i = a_i^2$ and $k_i = k_i^2$, $d = 1, 2$. (c) Final reconstruction at the 10th step.

Figure 8: (a) Evolution of the parameters $a_i$. The true values are $a_i^1 = 0.7$ and $a_i^2 = 1.2$. (b) Evolution of the parameters $k_i$. The true values are $k_i^1 = 0.75$ and $k_i^2 = 1.5$. (c) Cost functional at each iteration.
Heterogeneous objects

Figure 9: (a) TD with $\Omega_4 = \emptyset$. (b) Initial guess $\Omega_4 = \Omega_1$ and TD when $\Omega_4 = \Omega_1$. (c) Final reconstruction at the 9th step.

Figure 10: (a) Function $k_3$. (b) Reconstructed function $k_3$ at the 9th step. (c) Error.

The symmetry of the parameters is recovered.
Alternatives

- Parameter identification?
- Initial guess of the objects?
- Correction of the initial guess of the object:
  - Contour deformation following a vector field.
    - Problem: the number of objects must be known from the beginning.
  - Level-set based deformations:
    - It allows for topological changes in the set of objects.
    - Problems: First guess, Speed?
    - Santosa 1996, Dorn 2005
Conclusions

- Iterative method to approximate objects buried in a medium and their parameters in problems of acoustic scattering.
- The first iteration gives reasonable approximations of the number of objects, their location and size. Further iterations, catch smaller objects if missed, and may detect the presence of holes.
- The method can be extended to other time-harmonic scattering problems, to non-time harmonic problems involving thermal waves, and to electric impedance tomography.
- The strategy to approximate the material parameters may be used in combination with other variational methods (level sets) to correct initial guesses of the objects.
Joint work with M.L. Rapun, ETSIA, UPM, Madrid.

- For tomography, Inverse Problems 28, 095010 (2012).