Perturbed Gaussian copula: introducing the skew effect in co-dependence

Gaussian copula models are often used in the industry when single-asset information is quoted but little is known about their joint relation. These models may arise from correlated stochastic Brownian processes with deterministic volatility and correlation. If stochastic volatility is introduced, skewness and fat tails can be included in the co-dependence but analytic tractability is lost. Alberto Elices and Jean-Pierre Fouque show how this analytic tractability is preserved through another copula derived from an asymptotic expansion of the correlated processes with stochastic volatility around the Gaussian copula case.

Copula models arise in the market when quoted information about the behaviour of single assets is available but very little is known about their joint relations. Information about the joint distribution of assets is captured in the prices of basket products written on them. However, as a lot of information is embedded in a single price, some assumptions have to be made about the form of the distribution in order to extract it.

On the other hand, when products that depend on several assets are not available or are not liquid enough, it is necessary to make some assumptions about the joint relations through some parameters whose values might be given as input. The sensitivity to these unobserved parameters allows for estimating model risk and taking conservative positions for trading. Most of the assumptions of co-dependence among assets can be simplified through a copula model. One of the most popular contexts in which copula models have been used is credit (for example, collateralised debt obligations). Another popular application appears in hybrid models that combine two different asset classes for which co-dependence information is not available. The particular application addressed in this article is to quanto options.

The very popular Gaussian copula can be generated by the dynamics of log prices given by correlated Brownian motions. It depends on one parameter: the correlation coefficient between them. In this simple model, the skewness of the marginal distributions is not taken into account. If stochastic volatility is added to the dynamics, the skew of the marginal distributions will affect the co-dependence in a complicated manner. Using an asymptotic expansion for a regime of fast mean-reverting stochastic volatility, it is possible to approximate this interplay between the skew of marginal distributions and co-dependence in a tractable way. We call this the perturbed Gaussian copula. It incorporates two parameters for each underlying: one controls the implied volatility level and the other the slope of the implied volatility with respect to strike.

This article is organised as follows. First, we give a brief description of the perturbed Gaussian copula. Then we present a calibration procedure to match real market skews based on a regular Newton-Raphson method for which approximate initial parameters are estimated through analytical formulas based on asymptotic expansions. The interpretation of the action of the perturbed Gaussian copula compared with the Gaussian copula is then presented. We follow that with a real case study and then conclude.

Perturbed Gaussian copula from fast mean-reverting stochastic volatility

We now briefly explain how the perturbed Gaussian copula is obtained from fast mean-reverting stochastic volatility models by an asymptotic expansion (see Fouque & Zhou, 2008, and Fouque et al, 2011, for more details). Consider the processes \(X_i^{(1)}\), \(X_i^{(2)}\) and \(Y_i\), which follow the dynamics of equation (1), where \(W_i^{(1)}\), \(W_i^{(2)}\) and \(W_i^{(y)}\) are correlated standard Brownian motions with correlations given by equation (2), \(a_i^{(y)}(i = 1, 2)\) are the drifts of \(X_i^{(1)}\), \(m_i\) is the long-term value of \(Y_i\), \(\nu\) is a parameter that controls the volatility of the process \(Y_i\) and \(\epsilon\) is a small constant \((\epsilon << 1)\) that is the inverse of the mean reversion speed (the smaller \(\epsilon\), the faster mean reversion):

\[
\begin{align*}
\frac{dX_i^{(1)}}{\alpha_1 - \frac{1}{2} \sigma_i^2 (Y_i)} dt + f_1 (Y_i) dW_i^{(1)} \\
\frac{dX_i^{(2)}}{\alpha_2 - \frac{1}{2} \sigma_i^2 (Y_i)} dt + f_2 (Y_i) dW_i^{(2)} \\
\frac{dY_i}{\frac{1}{\nu} (m - Y_i)} dt + \frac{2 \epsilon}{\nu} dW_i^{(y)}
\end{align*}
\]

(1)

\[
\begin{align*}
\frac{dW_i^{(1)}}{dt} &= \rho_{1y} dt \\
\frac{dW_i^{(1)}}{dt} &= \rho_{2y} dt \\
\frac{dW_i^{(2)}}{dt} &= \rho_{2y} dt
\end{align*}
\]

(2)

The processes \(X_i^{(1)}\) and \(X_i^{(2)}\) represent two correlated lognormal underlyings. The functions \(f(Y)\) are the volatilities of the underlyings. They depend on a common Ornstein-Uhlenbeck stochastic volatility factor \(Y_i\). The correlations \(\rho_{1y}\) control the slope of the skew of each underlying and the parameter \(\nu\) (vol-vol) controls the convexity of the smile. Additional model features are given by equations (3) and (4):

\[
\begin{align*}
\frac{f_1 (y)}{\sigma_1 g (y)} &= \alpha_1 g (y) \\
\frac{f_2 (y)}{\sigma_2 g (y)} &= \alpha_2 g (y)
\end{align*}
\]

(3)

\[
\left\{ g^2 \right\} = \int_{-\infty}^{\infty} g (y)^2 \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{(y - m)^2}{2\nu^2} \right) dy = 1
\]

(4)

The volatility functions \(f(y)\) have common dynamics (the function \(g(y)\)) but different volatility levels \(\alpha_i\). In addition, the condition \(\left\{ g \right\} = 1\) is a normalisation of the function \(g(y)\) with respect to the stationary normal probability distribution \(N(m, \nu^2)\) of the
Ornstein-Uhlenbeck process (see section 3.3.3 of Fouque et al., 2011, for more details).

Our objective is to derive the joint and marginal transition probability distributions $u^i$, $u^j$, and $u^k$ from an initial point $(x_1, x_2, y)$ to the end fixed point $(\xi_1, \xi_2, v)$ for the underlyings and the variance. These transition probability distributions defined by the dynamics of equation (1) are given by equation (5), where $X_i = (X_i^1, X_i^2)$:

$$u^i = P(Y_0 \in d\xi_1, X_2^0 \in d\xi_2, Y_1 = y)$$
$$v_1^j = P(Y_0 \in d\xi_1, X_2^1 = x_1, Y_1 = y)$$
$$v_2^j = P(Y_0 \in d\xi_1, X_2^2 = x_1, Y_1 = y)$$

The solution of (6) is expanded in powers of $v_\xi$ as indicated by equation (11):

$$u^i = u_0 + \sqrt{e} u_1 + e^{3/2} u_2 + \cdots$$

The approximation of the solution will be given by the first two terms: $u^i \sim u_0 + \sqrt{e} u_1$. The perturbation method consists of substituting (11) in equation (6), grouping the terms of the same order (terms multiplying 1/$e$, 1/$e^2$, 1, $e$ and so on), and setting each term to zero to obtain equations for $u_0$, $u_1$, and so on (a detailed presentation of this calculation can be found in Fouque et al., 2011):

$$u^i(t, x_1, x_2; T, \xi_1, \xi_2) \sim \frac{1}{W} u_0 \left(1 + \frac{\tanh\sqrt{e} u_1}{u_0}\right)$$

The end result is that our approximation of $u^i$ to the order $v_\xi$ is independent of $y$ and given by equation (12), where the constant $W$ normalises the density to integrate to one in the whole domain and the function $\tanh$ has been introduced to ensure positivity (see Fouque & Zhou, 2008, for details). The components $u_0$ and $\sqrt{e} u_1$ are given by equations (13) and (14), where $A = 2\pi\sigma\sigma'(T - t)\sqrt{1 - \rho^2}$:

$$u_0 = \frac{1}{A} \exp \left[ -\frac{1}{2} \left( \frac{\xi_1^2}{\sigma_1^2} - 2 \frac{\xi_1 \xi_2}{\sigma_1 \sigma_2} \frac{\xi_2}{\sigma_2} + \frac{\xi_2^2}{\sigma_2^2} \right) - 2 \frac{\xi_1 \xi_2}{\sigma_1 \sigma_2} \frac{\xi_2}{\sigma_2} + \frac{\xi_2^2}{\sigma_2^2} \right]$$

$$v_\xi = - (T - t) \left[ R_1 \left( \frac{\xi_1^2}{\sigma_1^2} - \frac{\xi_2^2}{\sigma_2^2} \right) + R_2 \left( \frac{\xi_1}{\sigma_1} \frac{\xi_2}{\sigma_2} \right) \right]$$

The zero-order component $u_0(t, x_1, x_2, T, \xi_1, \xi_2)$ in (13) is a bi-normal distribution in terms of $\xi$, the correlation $\rho$ and the average means $\bar{x}$, given by equation (15):

$$\bar{x}_i = x_i + \int_t^T \alpha_i(t) ds - \frac{1}{2} \sigma_i^2 (T - t)$$

which shows that the expectation of the end point is the initial point plus the integral of the drift, $\alpha_i^0$, minus the convexity correction of a regular lognormal process with constant volatility $\sigma_i^0$.

The first-order component $v_\xi u_1$ in (14) is expressed in terms of partial derivatives of the zero-order component, where the constants $R_1$, $R_2$, are calibrated to market and $R_{12}$, $R_{21}$, $Q_{12}$, and $Q_{21}$ are

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given by equations (16):

\[
\begin{align*}
R_{12} & = \left( \frac{\partial^2}{\partial x_1 \partial x_2} \right) \mathbf{R} \mathbf{p} \\
R_{21} & = \left( \frac{\partial^2}{\partial x_2 \partial x_1} \right) \mathbf{R} \mathbf{p} \\
Q_{12} & = \left( \frac{\partial^2}{\partial x_1 \partial x_2} \right) \mathbf{Q} \\
Q_{21} & = \left( \frac{\partial^2}{\partial x_2 \partial x_1} \right) \mathbf{Q}
\end{align*}
\]

where equations (3) and (4) have been used. The same techniques applied to the marginal transition density functions \( v_i \) yield equations (17) and (18):

\[
v_i^t(t, x_i, T, \xi_i) = \frac{1}{W_i} \left[ 1 + \tanh \left( -\frac{R}{p_i} \frac{\partial^3}{\partial x_i} \left( \frac{\partial^2}{\partial x_i} - \frac{\partial^2}{\partial x_j} \right) \right) \right] \tag{17}
\]

\[
p_i(t, x_i, T, \xi_i) = \frac{1}{\sigma_i \sqrt{2\pi(T-t)}} \exp \left( -\frac{(\xi_i - \bar{\xi}_i)^2}{2\sigma_i^2(T-t)} \right) \tag{18}
\]

where the zero-order approximation is the regular normal density function given by (18), \( \bar{\xi}_i \) is again given by equation (15), and \( W_i \) are the normalising constants so that the marginal densities integrate to one in the whole domain.

The perturbed copula function is given by the ratio of the joint density and the product of the two marginal densities as expressed by equation (19), where \( z_i \) are related to \( \xi_i \) through the quantile function given by equation (20):

\[
f_{\text{cop}}(z_1, z_2) = \frac{u^t(t, x_1, x_2, T, \xi_1, \xi_2)}{v_i(t, x_1, T, \xi_1) v_i(t, x_2, T, \xi_2)} \tag{19}
\]

\[
z_i = P(X_i(t) = x_i | X_s(t) = x) \tag{20}
\]

The variables \( z_i \) in the interval \([0, 1]\) are the marginal cumulative probabilities of \( \xi_i \). It is proved in Fouque & Zhou (2008) that \( f_{\text{cop}} \) is indeed a copula function. At first sight, it may seem that the perturbed copula function may provide more flexibility than the Gaussian copula, for it depends on the parameters \( R_i \) and \( \sigma_i \) in addition to the correlation \( \rho \). Indeed, this is not the case, because the parameters \( R_i \) and \( \sigma_i \) get calibrated to market to fit the skew effect in the dependence (see below). Once they are calibrated to market, the only remaining parameter left is the correlation \( \rho \).

**Calibration**

The calibration procedure involves finding the parameters \( \sigma_i \) (implied volatility level) and \( R_i \) (implied volatility slope) for both underlyings (two degrees of freedom for each underlying). These values along with the correlation \( \rho \) (which is not calibrated but input by the trader) will allow calculating the joint and marginal densities and therefore the copula function. The method proposed is an exact calibration to two vanilla options for each underlying using a simple Newton-Raphson algorithm. Vanilla options are valued using numerical integration of the vanilla payout with respect to the perturbed marginal density function given by equation (17). For the algorithm to converge, initial values close enough to the solution are estimated through asymptotic expansions for vanilla option prices similar to those described above for the perturbed copula. The derivation of the expression of these initial values will be omitted here but it can be found in Elices & Fouque (2010). Following Fouque et al (2011), the implied volatilities are found to be, to the first order of approximation, an affine function of the log-moneyness-to-maturity ratio as follows:

\[
\sigma_i^{\text{imp}}(K) = \frac{\ln(K)}{T-t} + b \tag{21}
\]

Here, \( F_{it} = S_{it} \exp(\int t^t dx^0) \) is the forward value of the underlying \( i \) at time \( T \). For pricing purposes, the risk-neutral measure will be used so that \( \alpha^{\text{uni}}(t, q^0) \) where \( r_i \) is the domestic risk-free rate of the currency used as numeraire and \( q^0 \) is the dividend yield (or the foreign risk-free rate if a foreign exchange pair is considered) for underlying \( i \). Consequently, up to first order, the implied volatility behaves as a straight line where \( a \) is the slope and \( b \) is the intercept, and the independent variable is the log-moneyness-to-maturity ratio, \( \ln(K/F_{it})/(T-t) \). The parameters \( a \) and \( b \) are estimated through a linear regression of the volatility with respect to the log-moneyness-to-maturity ratio:

\[
R_i = ab^3 \quad \sigma_i = b - ab^2 = \frac{1}{2} \tag{22}
\]

It is shown in chapter five of Fouque et al (2011) that the parameters of the model \( R_i \) and \( \sigma_i \) are given in terms of the calibrated slope \( a \) and intercept \( b \) by equation (22).

**Interpretation of perturbed copula**

This section interprets the calibration and the effect of the perturbed Gaussian copula compared with the Gaussian copula. In terms of pricing, the interpretation is carried out applying the perturbed Gaussian copula approach to the valuation of foreign exchange quanto options with an underlying of the dollar price of a commodity, it is modelled as a foreign exchange underlying (i.e. the dollar price of gold or of the dollar price of gold against the euro). Equation (23) shows the payout function of this option, where \( S_s \) is the price of gold in dollars, \( X_T \) is the price of dollars per euro, \( D_{\text{USD}} \) is the discount factor of the dollar curve and \( K \) is the strike price. If the dollar money-market account is chosen as numeraire, both \( S_s \) and \( X_T \) are denominated in the numeraire currency (the dollar) and their drifts are simply calculated as the difference between their domestic (dollar) and the foreign interest rates at maturity (although \( S_s \) is a commodity, it is modelled as a foreign exchange underlying). As the option is quanto, \( K^*(S_s - K) \) will be paid in euros but should be discounted with the numeraire in dollars. Therefore, it must be converted to dollars first, multiplying by the euro/dollar exchange rate \( X_s \). The spot price of gold in dollars is \( S_s = 981.3 \) and the spot price of the euro/dollar exchange rate is \( X_s = 1.422 \) (market data from March 17, 2009):

\[
p = \mathbb{E}\left[ \frac{S_T - K}{K} X_T D_{\text{USD}} \right] \tag{23}
\]

The expectation given by equation (23) is calculated through the double integral given by equation (24), where \( \xi_s = \ln(S_s), \xi_T = \ln(X_T) \) and \( f_{\text{cop}}^{\text{imp}} \) is the joint probability density function of both underlyings given by equation (25) (see chapter two of Meucci, 2005). The function \( f_{\text{cop}} \) is the copula function defined in equation (19), \( z_s \) are given by equation (20) and \( f_{\text{cop}}^{\text{imp}} \) are the empirical marginal distributions. The latter are obtained through equation (26) (see Gatheral, 2006, to find out where this equation comes from), where \( P(K, T, \sigma^{\text{imp}}) \) is the Black-Scholes price of a put option of underlying \( i \) with strike \( K \), maturity \( T \) and \( \sigma^{\text{imp}}(K, T) \) is the interpolated implied volatility for
strike level $K$ of underlying $i$ at maturity $T$:

$$p = \frac{DF^\text{USD}_T}{K^i} \int \frac{e^{\frac{s_i - K}{R}}}{\kappa^i} f^{\text{Joint}}(\xi_1, \xi_2) d\xi_1 d\xi_2$$

$$f^{\text{Joint}}(\xi_1, \xi_2) = f^{\text{cop}}(\xi_1 - \xi_2) f_1^{\text{Marg}}(\xi_1) f_2^{\text{Marg}}(\xi_2)$$

$$f_1^{\text{Marg}}(\xi_1) = \frac{1}{DF^\text{USD}_T} \frac{d^2 P}{dK^i} \left[ \frac{e^{\xi_1 - T} \alpha^{\text{logN}}(\xi_1 - T)}{\kappa^i} \right]$$

The interpretation of the perturbed copula will be carried out for a scenario where both underlyings (gold price/dollar and euro/dollar) have left skew (higher implied volatility at lower strikes as shown in left plot of figure 1). The volatility surfaces of the underlyings are generated with a Heston model and calibrated to the 25% delta call and put. Other combinations of skew are symmetric with respect to the one considered. Figure 1 presents the left skew scenario generated for the dollar price of gold. The calibrated parameters are $\kappa_1 = 0.0892, R_1 = 1.31 \times 10^{-4}$. The left plot of figure 1 shows the implied volatility of the original Heston-generated skew (labelled ‘Original’) and the perturbed-copula calibrated skew (labelled ‘PerCop’) versus the log-moneyness-to-maturity ratio. The plots join points of the same height. Lines close to each other indicate steeper evolution of the function. When the function is in the interval [0, 0.5], the contour line levels are spaced in logarithmic scale (linear in the exponents).

The left plots of figure 2 show the perturbed joint probability density function (numerator of the perturbed copula function) of both underlyings given by equation (12) (the axes are the log-moneyness of each underlying). The middle plots of figure 2 present the copula function given by equation (19) (the axes show the normalised cumulative density function in the interval [0, 1] of each underlying). Finally, the right plots of figure 2 display the ratio between the perturbed and Gaussian copula functions. These plots might be replicated using the calibrated parameters given in figure 1 and calculating $R_2$ and $Q_2$ according to equation (16). The upper plots of figure 2 correspond to a positive 0.6 correlation scenario and the lower plots to a negative –0.6 correlation scenario.

The upper left plot of figure 2 shows an oval form positively sloped that corresponds to the density of a bi-normal distribution with positive correlation. However, the maximum density (red colour) is not reached at the centre of the oval form (as would be the case for a bi-normal distribution) but it is displaced along the diagonal direction towards the upper right corner (higher values of both underlyings). This happens because both underlyings are left skewed and the point of maximum density is pushed towards higher values of the underlyings. On the other hand, left tails of both underlyings get fatter (more density is moved to lower values of the underlyings). For the negative correlation scenario (bottom left plot of figure 2), the oval form is negatively sloped (along the anti-diagonal direction). However, the left skew effect of both underlyings continues to displace the maximum density towards higher values of the underlyings (along the diagonal direction towards the upper right corner) and fattening tails for lower values of the underlyings.

The middle plots of figure 2 show the copula function for...
both correlation scenarios. When the copula function is equal to one, the joint probability density function is the product of the two marginals indicating independence or no co-dependence. When the copula function is greater than one, the density is increased, indicating that there is more co-dependence (the opposite happens with a copula function less than one). See that in the middle plots of figure 2 there are several contour lines very close to level one. These solid cyan (dark blue) lines parallel to the diagonal direction (upper middle plot) and anti-diagonal direction (lower middle plot) indicate where the copula function is equal to one.

The middle upper plot of figure 2 shows how the copula function increases the probability density along the main diagonal (red and yellow colours) because the correlation is positive (copula values less than one are in the anti-diagonal corners of the plot). In the middle upper plot it can be seen that there is slightly more density for low underlying values (lower left corner) than the opposite corner for the effect of the left skew. The middle lower plot of figure 2 shows that the copula is greater than one in the anti-diagonal direction because the underlyings are negatively correlated. In this plot it can be seen more clearly how the left skew effect pushes the copula function towards lower values of the underlyings (see the cyan colour at the lower left corner of the lower-mid plot where the copula gets back to the level equal to one).

The right plots of figure 2 show the ratio between the perturbed and Gaussian copulas to allow for comparison. Again, several contour lines around level one have been plotted to create a solid line to help identify this level. The upper right plot of figure 2 (positive correlation) shows that the effect of the perturbed copula is to increase the density of the left tails (lower levels) of both underlyings. The left tail of the horizontal variable is increased the most (see the red colour in the upper left side of the plot) for higher values of the vertical variable (where there is less co-dependence). As the vertical variable gets lower, the ratio decreases but it is greater than one. A similar effect happens for the left tail of the vertical variable (lower side of the plot): the ratio is the biggest (red colour) for high values of the horizontal variable where the co-dependence is the smallest. See that the biggest ratio (red colour) is located on the extremes of the anti-diagonal direction (where the co-dependence is the smallest).

The lower right plot of figure 2 corresponds to the negative correlation scenario. Again, the ratio of the left tails of both underlyings is increased. See that the lower left corner of the plot has the red colour associated with the biggest ratio, precisely where the co-dependence is the smallest. For higher values of the co-dependence (the anti-diagonal), the densities are displaced towards the left tails of both underlyings (towards the lower left corner of the plot).

With regard to the pricing of the foreign exchange quanto option described at the beginning of this section, the perturbed copula (compared with the Gaussian copula) provides positive premium corrections for almost every scenario as correlation varies from –0.6 to 0.6 for this left skew scenario where both underlyings have left skews. This is not surprising as the joint distribution of figure 2 (left plots) is deformed towards the upper right corner (the direction in which the option payout
increases) and the tails do not pay out as it is a call option. For other skew configurations, the premium corrections given by the perturbed copula might not be as clear.

Case study: foreign exchange quanto options
This section compares the perturbed copula, the Gaussian copula and a Monte Carlo method with local volatility and constant instantaneous correlation for a set of scenarios built out of a real market scenario for the same foreign exchange call option on the dollar price of gold quantoed to euros considered above. The local volatility model replicates the implied volatility surfaces of each underlying (as does the Gaussian copula) and the constant correlation \( \rho \) given by equation (2) and used for the Monte Carlo method is equal to the correlation input for the Gaussian copula. This scenario corresponds to dollar price of gold highly left skewed (a risk reversal of around 10%) and the euro/dollar very mildly right skewed (almost a smile). Five correlations (0.6, 0.3, 0, –0.3, –0.6), two maturities (one and two years) and five moneyness levels (0.70, 0.85, at-the-money, 1.15 and 1.20) with strikes \( K = (656.46, 797.13, 937.79, 1,078.47, 1,125.36) \) are considered. The spot price of gold in dollars and the euro/dollar are \( S_0 = 937.79 \) and \( X_0 = 1.4029 \). Further details about calibration parameters and the resulting distributions of the perturbed copula and the underlyings may be found in Elices & Fouque (2010).

Table A shows the difference in basis points between the perturbed and Gaussian copulas varying by moneyness, maturity and correlation. To properly compare perturbed and Gaussian copulas, the correlations that appear in the columns of table A refer to those input in the Gaussian copula. The correlations used for the perturbed copula are implied so that the quanto forward, \( E(S_T X_T) \), is equal for both the Gaussian and perturbed copulas. The upper group of rows show different moneyness levels for one-year maturity and the lower group of rows for two-year maturity. Differences increase with maturity and they can get beyond 40bp of the notional. As the overall effect of the skew of the dollar price of gold is to move the joint distribution towards higher values of the horizontal axis (in this direction the payout is higher), the corrections given by the perturbed copula are positive for almost every case. The fact that the left tail gets fatter does not have a big impact as the payout for the left tail is zero (a call option is considered).

The same scenarios were priced with a Monte Carlo method with local volatility and constant instantaneous correlation and compared with the Gaussian copula.\(^7\) The constant instantaneous correlation input for the Monte Carlo method is the same as the correlation input for the Gaussian copula. The maximum difference does not go beyond 17bp. This means that the Monte Carlo method with local volatility is rather equivalent to the Gaussian copula.

Conclusion
The perturbed copula approach of Fouque & Zhou (2008) has been successfully applied to valuation of derivatives that depend on two underlyings for which prior information about their co-dependence is unknown. The application of this perturbed copula allows introducing the skew information of the underlyings in the co-dependence through a few intuitive and easy-to-interpret parameters (volatility and skew for each underlying and correlation between both underlyings). These parameters are calibrated using exact fit through a Newton-Raphson search algorithm for which approximate initial values are provided.

The effect of the perturbed copula is interpreted compared with the Gaussian copula in terms of the direction of the skew and the correlation of each underlying. Intuitive criteria are provided to qualitatively predict the effects of the perturbed copula in the pricing of a derivative.

A real market case study is also analysed for foreign exchange quanto options to a third currency. It has been seen that the price impact of considering the skew in the co-dependence is moderate but not negligible, going beyond 40bp of the notional amount in some cases. This raises a moderate concern about the model risk of quanto options in the presence of skew.

The cases analysed have also been compared with a Monte Carlo local volatility model that replicates the implied volatility surfaces of both underlyings and a constant instantaneous correlation equal to the correlation used in the Gaussian copula. The conclusion is that this method is rather equivalent to the Gaussian copula. This means that the regular widely used local volatility model does not incorporate the skew effect in the co-dependence and some potential model risk might be unidentified.

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\[^7\] This is not a completely fair comparison as the terminal correlations used by the copula are not equal to the instantaneous correlations input in the Monte Carlo. However, in this situation the impact is not too big.

<table>
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<tr>
<th>Scenarios</th>
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</tbody>
</table>

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