Models with time-dependent parameters using transform methods: Application to Heston’s model

Alberto Elices

Model Validation Group
Risk Division
Grupo Santander

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Outline of the presentation

- Introduction.
- Characteristic functions of models with time-dependent parameters.
- Application to Heston’s model.
- Case study: Calibration to Eurostoxx 50.
- Application to Forward start options.
- Forward skew of Heston’s model.
- Conclusions.
Introduction

- Exotic valuation: usually carried out with Monte Carlo.
- Calibration: fast analytic models are needed for valuation of vanilla products.
- Analytic models depend on just a few parameters which cannot fit the whole set of market parameters.
- More degrees of freedom are needed in order to calibrate the market across all maturities.
- The most natural way of introducing more parameters is to let them depend on time.
Characteristic functions of models with time-dependent parameters

- **Characteristic function methods:**
  - Useful when the characteristic function is analytic.
  - The Inversion of the characteristic function is carried out through the inverse Fourier transform.

- **Characteristic function:**

\[
\phi_{uv}(X/x_u) = E(e^{iX \cdot x_v}) = \int_{\mathbb{R}^N} e^{iX \cdot x_v} f_{uv}(X/x_u) dx_v
\]
Family of characteristic functions for which the methodology can be applied:

\[ \varphi_{uv}(X/x_u) = \exp\left( C_{uv}(X) + D_{uv}(X) \cdot x_u \right) \]

\[ x(t) = (x_1(t), \ldots, x_N(t)) \quad X = (X_1, \ldots, X_N) \]

\[ D_{uv}(X) = (D_{uv,1}(X), \ldots, D_{uv,N}(X)) \]

The method proposed introduces time-dependent parameters for a wide variety of models which admit analytic characteristic function:

- Merton jump model.

\[ \varphi_{uv}(G/g_u) = \exp\left( C_{uv}(G) + iGg_u \right) \]

\[ g_u : sum of all Poisson distributed jumps up to time t_u. \]
Introduction

- Cox Ingersoll Ross model.

\[ \varphi_{uv}(R/r_u) = \exp(C_{uv}(R) + iD_{uv}(R)r_u) \]

\( r_u \): short rate interest rate at time \( t_u \).

- Heston stochastic volatility model.

\[ \varphi_{uv}(X,V/x_u,v_u) = \exp(C_{uv}(X,V) + D_{uv}(X,V)v_u + iXx_u) \]

\( x_u \): logarithm of underlying. \( v_u \): variance process.

- Hybrids with jumps, stochastic interest rates and volatility.

\[ \varphi_{uv}(X,V,R,G/x_u,v_u,r_u,g_u) = e^{C_{uv} + D_{uv,2}r_u + D_{uv,1}v_u + iXx_u + iGg_u} \]
Characteristic functions of models with time-dependent parameters

- All relevant information of a Markov process with independent increments at an instant \( t_v \) is given by the joint probability distribution: \( \varphi_{0v}(X/x_0) \)

- Objective: Find \( \varphi_{0v}(X/x_0) \) in terms of \( \varphi_{0u}(X/x_0) \) and \( \varphi_{uv}(X/x_u) \)
Characteristic functions of models with time-dependent parameters

- **Characteristic function under search:**
  \[
  \varphi_{0v}(X/X_0) = \int_{\mathbb{R}^N} d\mathbf{x}_v e^{i\mathbf{X} \cdot \mathbf{x}_v} f_{0v}(\mathbf{x}_v/X_0)
  \]

- **Joint density** \( t_0 \to t_v \) **in terms of densities** \( t_0 \to t_u \) **and** \( t_u \to t_v \) **(independent increments):**
  
  \[
  f_{0v}(\mathbf{x}_v/X_0) = \int_{\mathbb{R}^N} d\mathbf{x}_u f_{0u}(\mathbf{x}_u/X_0) f_{uv}(\mathbf{x}_v/X_u)
  \]

- **Substituting** \( f_{0v}(\mathbf{x}_v/X_0) \) **in** \( \varphi_{0v}(X/X_0) \):
  
  \[
  \varphi_{0v}(X/X_0) = \int_{\mathbb{R}^N} d\mathbf{x}_u f_{0u}(\mathbf{x}_u/X_0) \int_{\mathbb{R}^N} d\mathbf{x}_v e^{i\mathbf{X} \cdot \mathbf{x}_v} f_{uv}(\mathbf{x}_v/X_u) \varphi_{uv}(X/X_u) = \exp(C_{uv}(X) + D_{uv}(X) \cdot \mathbf{x}_u)
  \]

\[
\]
Characteristic functions of models with time-dependent parameters

- After substituting $\varphi_{uv}(X/x_u)$:

$$\varphi_{0v}(X/x_0) = \int_{\mathbb{R}^N} dx_u f_{0u}(x_u/x_0) \exp(C_{uv}(X) + D_{uv}(X) \cdot x_u)$$

$$= \exp(C_{uv}(X)) \int_{\mathbb{R}^N} dx_u f_{0u}(x_u/x_0) \exp(i(i^{-1}D_{uv}(X)) \cdot x_u)$$

$$= \exp\left(C_{uv}(X) + C_{0u}(i^{-1}D_{uv}(X)) + D_{0u}(i^{-1}D_{uv}(X)) \cdot x_0\right)$$
Characteristic functions of models with time-dependent parameters

- **Identifying terms:**

\[
\varphi_{0v}(X/x_0) = \exp(C_{0v}(X) + D_{0v}(X) \cdot x_0)
\]

\[
\begin{align*}
C_{0v}(X) &= C_{uv}(X) + C_{0u}\left(i^{-1}D_{uv}(X)\right) \\
D_{0v}(X) &= D_{0u}\left(i^{-1}D_{uv}(X)\right)
\end{align*}
\]
Application to Heston’s model

- **Heston process:**

\[
\begin{aligned}
    dS_t &= \mu S_t dt + \sqrt{\nu_t} S_t dW_t \\
    d\nu_t &= \kappa(\theta - \nu_t) dt + \sigma \sqrt{\nu_t} dY_t \\
    d\langle W_t, Y_t \rangle &= \rho dt
\end{aligned}
\]

- The two state variables for Heston’s process are the logarithm of the stock price \( x_t = \log(S_t) \) and the variance process \( \nu_t \):

\[
X_u = (x(t_u), \nu(t_u))
\]

- These two state variables translate into \( X \) and \( V \) for the characteristic function:

\[
X = (X, V)
\]
Application to Heston’s model

- Joint characteristic function for Heston process:

\[
\varphi_{uv}(X,V/x(t_u),\nu(t_u)) = e^{C_{uv}(X,V)+D_{uv,2}(X,V)\nu(t_u)+D_{uv,1}(X,V)x(t_u)}
\]

\[
D_{uv,2}(X,V) = \frac{\kappa - \rho \sigma X_i + d}{\sigma^2} \left( \frac{g - \tilde{g}e^{-d\tau}}{1 - \tilde{g}e^{-d\tau}} \right)
\]

\[
D_{uv,1}(X,V) = iX
\]

\[
C_{uv}(X,V) = i\mu X\tau + \frac{\kappa \theta}{\sigma^2} \left( -2 \ln \left( \frac{1 - \tilde{g}e^{-d\tau}}{1 - \tilde{g}} \right) + (\kappa - \rho \sigma X_i - d)\tau \right)
\]

\[
\tilde{g} = \frac{\kappa - \rho \sigma X_i - d - iV\sigma^2}{\kappa - \rho \sigma X_i + d - iV\sigma^2}
\]

\[
g = \frac{\kappa - \rho \sigma X_i - d}{\kappa - \rho \sigma X_i + d}
\]

\[
d = \sqrt{\left(\kappa - \rho \sigma X_i\right)^2 + \sigma^2 X(i + X)}
\]
Application to Heston’s model

- Characteristic function with time-dependent parameters at maturity $t_v$:

$$
\phi_{0v}(X,V/x(t_0),\nu(t_0)) = e^{C_{0v}(X,V)+D_{0v,2}(X,V)\nu(t_0)+D_{uv,1}(X,V)x(t_0)}
$$

$$
\begin{align*}
C_{0v}(X,V) &= C_{uv}(X,V) + C_{0u}(X,i^{-1}D_{uv,2}(X,V)) \\
D_{0v,2}(X,V) &= D_{0u,2}(X,i^{-1}D_{uv,2}(X,V)) \\
D_{0v,1}(X,V) &= iX
\end{align*}
$$

| 0 | $\tau_{0u} = t_u$ | $t_u$ | $\tau_{uv} = t_v - t_u$ | $t_v$ |
Application to Heston’s model

- Valuation of vanilla options:

\[ C = DF_T E\left((S_T - K)^+\right) = DF_T \left( E\left(e^{\xi T} 1_{\{\xi_T > \ln K\}}\right) - K E\left(1_{\{\xi_T > \ln K\}}\right) \right) \]

- Characteristic function for cash or nothing option:

\[ \varphi_{0T}^{CN} (X/X_0) = \varphi_{0T} (X,0/X_0) = \int_{\mathbb{R}} e^{iX \cdot \xi_T} f_{0T}(x_T, \nu_T / x_0) dx_T d\nu_T \]

- Inversion formula: cumulative density in terms of characteristic function.

\[ P(x > a) = \frac{1}{2} + \frac{1}{2\pi} \int_0^{\infty} \frac{\varphi(X)}{e^{iX a} - e^{-iX a}} dX \]
Application to Heston’s model

- **Characteristic function for asset or nothing option:**

\[
\phi_{0T}^{AN}(X/x_0) = \frac{\phi_{0T}(X-i,0/x_0)}{\phi_{0T}(-i,0/x_0)} = \frac{E(e^{i(X-i)x_T})}{E(e^{x_T})} = \frac{E(e^{iXx_T}e^{x_T})}{E(S_T)}
\]

\[
\phi_{0T}^{AN}(X/x_0,v_0) = \int_{R} e^{iXx_T} dx_T \int_{R} f_{0T}(x_T,v_T/x_0,v_0) \frac{e^{x_T}}{E(S_T)} dv_T
\]

- **Final expression of vanillas:**

\[
C = DF_T\left(E(S_T)P^{AN}(x_T > \ln K) - KP^{CN}(x_T > \ln K)\right)
\]
Application to Heston’s model

- Valuation of FX quanto options \( (S_T \text{ in USD per EUR}) \):

\[
C = D F_T^S E \left( (S_T - K)^+ S_T \right) = D F_T^S \left\{ E \left( e^{2x_T 1_{\{x_T > \ln K\}}} \right) - K E \left( e^{x_T 1_{\{x_T > \ln K\}}} \right) \right\}
\]

- Characteristic function for asset^2 or nothing option:

\[
\varphi_{0T}^{A^2N} \left( \frac{X}{x_0} \right) = \frac{\varphi_{0T} (X - 2i/x_0)}{\varphi_{0T} (-2i/x_0)} = \frac{E \left( e^{i(X-2i)x_T} \right)}{E \left( e^{2x_T} \right)} = \frac{E \left( e^{iX_T} e^{2x_T} \right)}{E \left( S_T^2 \right)}
\]

- Final expression for FX quanto vanillas:

\[
C = D F_T^S \left( E \left( S_T^2 \right) P_{A^2N}^{A^2N} \left( x_T > \ln K \right) - KE \left( S_T \right) P_{AN}^{AN} \left( x_T > \ln K \right) \right)
\]
A bootstrapping algorithm is proposed:

- Periods in between vanilla maturities are chosen to let parameters change.
- 1. \( n = 1 \)
- 2. Search model parameters \((\theta, \kappa, \sigma, \rho)\) from \( T_{i-1} \) to \( T_i \) to fit vanillas at \( T_i \) minimizing the following objective function:

\[
FO = \sum_{i=1}^{M} \sum \frac{w_i}{w_j} \left( price_{model}^i - price_{market}^i \right)^2
\]

N.B. \( w_i \) chosen to give more weight to options closer to ATM.
- 3. The parameters up to \( T_i \) are fixed
- 4. \( n = n + 1 \)
- 5. Return to step 2
Case study: Calibration to Eurostoxx 50.

- Time dependent Heston model is calibrated to the following Eurostoxx 50 volatility surface:

<table>
<thead>
<tr>
<th>K \ Mat</th>
<th>1m</th>
<th>3m</th>
<th>6m</th>
<th>9m</th>
<th>1y</th>
<th>2y</th>
<th>3y</th>
<th>4y</th>
<th>5y</th>
<th>10y</th>
</tr>
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<tbody>
<tr>
<td>0.85</td>
<td>23.0</td>
<td>18.7</td>
<td>18.5</td>
<td>18.6</td>
<td>19.1</td>
<td>19.7</td>
<td>20.6</td>
<td>21.5</td>
<td>22.2</td>
<td>25.8</td>
</tr>
<tr>
<td>0.90</td>
<td>18.9</td>
<td>16.7</td>
<td>17.0</td>
<td>17.2</td>
<td>17.8</td>
<td>18.8</td>
<td>19.8</td>
<td>20.8</td>
<td>21.5</td>
<td>25.3</td>
</tr>
<tr>
<td>0.95</td>
<td>15.2</td>
<td>14.7</td>
<td>15.5</td>
<td>16.0</td>
<td>16.6</td>
<td>17.8</td>
<td>19.0</td>
<td>20.0</td>
<td>20.8</td>
<td>24.7</td>
</tr>
<tr>
<td>1.00</td>
<td>12.2</td>
<td>13.2</td>
<td>14.1</td>
<td>14.8</td>
<td>15.5</td>
<td>16.9</td>
<td>18.2</td>
<td>19.3</td>
<td>20.2</td>
<td>24.2</td>
</tr>
<tr>
<td>1.05</td>
<td>11.6</td>
<td>12.3</td>
<td>13.1</td>
<td>13.9</td>
<td>14.4</td>
<td>16.1</td>
<td>17.5</td>
<td>18.7</td>
<td>19.5</td>
<td>23.7</td>
</tr>
<tr>
<td>1.10</td>
<td>13.3</td>
<td>12.3</td>
<td>12.6</td>
<td>13.2</td>
<td>13.7</td>
<td>15.4</td>
<td>16.9</td>
<td>18.1</td>
<td>19.0</td>
<td>23.2</td>
</tr>
<tr>
<td>1.15</td>
<td>15.6</td>
<td>12.9</td>
<td>12.4</td>
<td>12.7</td>
<td>13.2</td>
<td>14.8</td>
<td>16.3</td>
<td>17.5</td>
<td>18.5</td>
<td>22.7</td>
</tr>
</tbody>
</table>

- To avoid problems with discrete dividend payments, what is calibrated is the forward delivered at the last maturity rather than the underlying itself.

- Two calibrations are carried out:
  - Left: constrained calibration (esp. with respect to $\sigma$ and $\kappa$).
  - Right: unconstrained calibration

<table>
<thead>
<tr>
<th>$v_0$</th>
<th>$\theta$</th>
<th>$\kappa$</th>
<th>$\sigma$</th>
<th>$\rho$</th>
</tr>
</thead>
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<tr>
<td>max</td>
<td>1</td>
<td>1</td>
<td>20</td>
<td>1</td>
</tr>
<tr>
<td>min</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$v_0$</td>
<td>$\theta$</td>
<td>$\kappa$</td>
<td>$\sigma$</td>
<td>$\rho$</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Case study: Calibration to Eurostoxx 50.

- Maximum error for both calibrations: 8 bp for most OTM options.
- Both calibrations are equivalent from a qualitative point of view:
  - Market is pricing in increasing uncertainty of volatility: $\sigma$ is constant while $\kappa$ reduces (left) vs $\kappa$ is constant while $\sigma$ increases (right).
  - Market is pricing in increasing volatility (from 11% to around 45% at 10y) and increasing skew.

![Graphs showing calibration results](image-url)
Application to Forward start options.

- **Forward start option:**
  \[ p = DF_{t_v} E\left((e^{x_v} - Ke^{x_u})^+\right) = DF_{t_v} E\left(e^{x_u}\right) E\left(e^{\tilde{x}_v} - K\right)^+ \]

- As the forward start option starts at \( t_u \), only the period from \( t_u \) to \( t_v \) is considered:
  \[ x_v = x_u + \tilde{x}_v \quad \tilde{x}_t = \begin{cases} 0 & t < u \\ x_t - x_u & t \geq u \end{cases} \]

- **Objective:** obtain characteristic function \( \tilde{\varphi}_{0v}(\tilde{X}/0,\nu_0) \) of \( \tilde{x}_t \) at \( t_v \) given \( \nu_0 \), in terms of \( \varphi_{0u}(X,V/x_0,\nu_0)\bigg|_{X=0} \) (the distribution of the variance at \( t_u \)) and \( \varphi_{uv}(X,V/x_0 = 0,\nu_u) \) (the distribution of \( \tilde{x}_t \) given the variance \( \nu_u \) at \( t_u \)).

\[
\begin{array}{c|c|c} 
0 & \varphi_{0u}(X,V/x_0,\nu_0)\bigg|_{X=0} & \varphi_{uv}(X,V/x_0 = 0,\nu_u) \\
\tau_{0u} = t_u & t_u & \tau_{uv} = t_v - t_u \end{array}
\]
Application to Forward start options.

- **Density** \( \tilde{f}(\tilde{x}_t/0, \nu_0) \) of \( \tilde{x}_t \) given \( \nu_0 \):
  \[
  \tilde{f}(\tilde{x}_t/0, \nu_0) = \int_{\mathbb{R}^+} d\nu_u f_{0u}(\nu_u/\mathbf{x}_0) f_{uv}(\tilde{x}_t/0, \nu_u)
  \]

- **Characteristic function** \( \tilde{\varphi}(\tilde{X}/0, \nu_0) \) of \( \tilde{x}_t \) given \( \nu_0 \):
  \[
  \tilde{\varphi}(\tilde{X}/0, \nu_0) = \int_{\mathbb{R}^2} e^{i\tilde{x}_t \cdot \tilde{X}} \tilde{f}(\tilde{x}_t/0, \nu_0) d\tilde{x}_t
  \]

- **Substituting and exchanging the order of integration:**
  \[
  \tilde{\varphi}(\tilde{X}/0, \nu_0) = \int_{\mathbb{R}^+} d\nu_u f_{0u}(\nu_u/\mathbf{x}_0) \int_{\mathbb{R}^2} e^{i\tilde{x}_t \cdot \tilde{X}} f_{uv}(\tilde{x}_t/0, \nu_u) d\tilde{x}_t
  \]
  \[
  \varphi_{uv}(\tilde{X}/0, \nu_u) = e^{C_{uv}(\tilde{X}) + D_{uv, 2}(\tilde{X}) \nu_u}
  \]
Application to Forward start options.

- Characteristic function \( \tilde{\varphi}(\tilde{X}/0, \nu_0) \) of \( \tilde{x}_t \) given \( \nu_0 \):

\[
\tilde{\varphi}(\tilde{X}/0, \nu_0) = e^{C_{uv}(\tilde{X})} \int_{\mathbb{R}^+} d\nu_u f_{0u}(\nu_u / x_0) e^{i(-iD_{uv,2}(\tilde{X}))\nu_u} \varphi_{0u}\left(0,-iD_{uv,2}(\tilde{X},\tilde{V})/x_0\right)
\]

- Characteristic function \( \tilde{\varphi}(\tilde{X}/0, \nu_0) \) of \( \tilde{x}_t \) given \( \nu_0 \):

\[
\tilde{\varphi}(\tilde{X}, \tilde{V}/0, \nu_0) = e^{C_{uv}(\tilde{X}, \tilde{V})} \cdot \left[ \varphi_{0u}\left(0,-iD_{uv,2}(\tilde{X}, \tilde{V})/x_0\right) \right]
\]

where \( \varphi_{0u} = e^{C_{0u}(0,-iD_{uv,2}(\tilde{X}, \tilde{V}))+D_{0u,2}(0,-iD_{uv,2}(\tilde{X}, \tilde{V}))\nu_0} \).
Final expression of the joint characteristic function for a forward start option using Heston’s model:

\[
\tilde{\phi}\left(\tilde{X}, \tilde{V} / 0, \nu_0\right) = e^{C_{uv}(\tilde{X}, \tilde{V}) + C_{0u}(0, -iD_{uv,2}(\tilde{X}, \tilde{V})) + D_{0u,2}(0, -iD_{uv,2}(\tilde{X}, \tilde{V}))}\nu_0
\]

\[
\tilde{\phi}\left(\tilde{X}, \tilde{V} / 0, \nu_0\right) = e^{\tilde{C}(\tilde{X}, \tilde{V}) + \tilde{D}(\tilde{X}, \tilde{V})}\nu_0
\]

\[
\begin{cases}
\tilde{C}(\tilde{X}, \tilde{V}) = C_{uv}(\tilde{X}, \tilde{V}) + C_{0u}(0, -iD_{uv,2}(\tilde{X}, \tilde{V})) \\
\tilde{D}(\tilde{X}, \tilde{V}) = D_{0u,2}(0, -iD_{uv,2}(\tilde{X}, \tilde{V}))
\end{cases}
\]

The marginal distribution of the underlying \(\tilde{X}_t\) is obtained by setting \(\tilde{V}\) equal to 0.
Forward skew of Heston’s model.

Consider the price of the forward start option when the underlying $\tilde{x}_t$ is driven by BS process with constant vol:

\[ p = DF_{t,v} E\left((e^{x_v} - Ke^{x_u})^+\right) = DF_{t,v} E\left(e^{x_u}\right)E\left((e^{\tilde{x}_v} - K)^+\right) \]

\[ \tilde{x}_t = \left(\mu - \frac{1}{2}\sigma_{BS}^2\right)dt + \sigma_{BS} dW_t \]

It is understood by forward skew the implied volatility surface that results when the forward start option price above, equals the price of the same forward start option when $\tilde{x}_t$ is a Heston process.
Forward skew of Heston’s model.

- Lower maturity options are more sensitive to the variance distribution as the forward start term increases.

- Constrained calibration (left) seems a lot more reasonable than unconstrained calibration (right).
Forward skew of Heston’s model.

- Longer maturity options are less sensitive to the variance distribution as the forward start term increases.

- Constrained and unconstrained calibrations seem to agree a lot more for longer maturity options.
Forward skew of Heston’s model.

- Between both calibrations: big difference for short maturity forward start options.
  - Both calibrations fit the marginal distribution of the underlying but,
  - the variance distribution is not specifically calibrated in either case.

- Market volatility surface:
  - Gives info about the marginal distribution of the underlying.
  - No info is given about the distribution of the variance (this info could be given by forward start or cliquet option quotes).
Forward skew of Heston’s model.

- What’s different from both calibrations?
- Consider the instantaneous volatility (obtained applying Itô):
  \[ \tilde{\sigma}_t = \sqrt{v_t} \]

\[
d\tilde{\sigma}_t = \left( \frac{4\kappa\theta - \sigma^2}{8\tilde{\sigma}_t} - \frac{\kappa}{2} \tilde{\sigma}_t \right) dt + \frac{1}{2} \sigma dY_t
\]

- Calibrations with \( \sigma \) greater than one can lead to very negative drift:
  - Constrained: \( \sigma \) cannot be higher than 1.5.
  - Unconstrained: \( \sigma \) far exceeds 1 at higher maturities:
    - Variance is biased towards values near zero at long maturities.
    - Forward implied volatility is lower and forward skew does not make sense.
Forward skew of Heston’s model.

- **Calibration of the uncertainty of volatility (κ and σ):**
  - Left: constraining σ to moderate values and calibrating κ seems to provide a better forward skew.
  - Right: fixing κ and calibrating σ may bias the forward skew towards artificially lower implied volatilities.

![Graph](image1.png)

**Calibration of STOXX50E: ATM + 3 vanillas around. var₀ = 0.0174**

![Graph](image2.png)

**Calibration of STOXX50E: ATM + 3 vanillas around. var₀ = 0.0175**
Conclusions

- A new method to introduce piecewise constant time-dependent parameters using transform methods is presented:
  - The characteristic function of the underlying for a time horizon is calculated in terms of the characteristic functions of the sub-periods where the parameters change.
  - Analytic tractability is preserved for a wide family of models such as hybrids with stochastic vol, interest rates an jumps.
- The method has been applied to Heston’s model.
- Two calibrations were carried out on the Eurostoxx 50.
- The method has also been applied to valuation of forward start options.
- The forward skew of both calibrations is explored.