ON SOME CONJECTURES ABOUT FREE AND NEARLY FREE DIVISORS

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Dedicated to Gert-Martin Greuel on the occasion of his 70th birthday

Abstract. In this paper infinite families of irreducible either free divisors or nearly free divisors in the complex projective plane, which are not rational curves, are given. Moreover, there corresponding local singularites can have an arbitrary number of branches. All these examples contradict to some of the conjectures proposed by A. Dimca and G. Sticlaru in [14]. Our examples say nothing about the most remarkable conjecture by A. Dimca and G. Sticlaru, which predicts that every rational cuspidal plane curve is either free or nearly free.

1. Introduction

The notion of free divisor was introduced by K. Saito [22] in the study of discriminants of versal unfoldings of germs of isolated hypersurface singularites. Since then many interesting and unexpected applications to Singularity Theory and Algebraic Geometry have been appearing. In this paper we are mainly focused on complex projective plane curves and we adapt the corresponding notions and results to this set-up. The results contained in this paper have needed a lot of computations in order to get the correct statements. All of them have been done using the computer algebra system `Singular` [9] through `Sagemath` [25]. We thank `Singular`'s team for such a great mathematical tool and especially to Gert-Martin for his dedication to `Singular` developement.

Let $S := \mathbb{C}[x, y, z]$ be the polynomial ring endowed with the natural graduation $S = \bigoplus_{m=0}^{\infty} S_m$ by homogeneous polynomials. Let $f \in S_d$ be a homogeneous

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polynomial of degree d in the polynomial ring, let $C \subset \mathbb{P}^2$ be defined by $f = 0$. Assume that C is reduced. We denote by J_f the Jacobian ideal of f, which is the homogeneous ideal in S spanned by f_x, f_y, f_z. We denote by $M(f) = S/J_f$ the corresponding graded ring, called the Jacobian (or Milnor) algebra of f.

Let I_f be the saturation of the ideal J_f with respect to the maximal ideal (x, y, z) in S and let $N(f) = I_f/J_f$ be the corresponding graded quotient. Recall that the curve $C : f = 0$ is called a *free divisor* if $N((f) = I_f/J_f = 0$, see e.g. [24].

A. Dimca and G. Sticlaru introduced in [14] the notion of nearly free divisor which is a sligth modification of the notion of free divisor. The curve C is called *nearly free divisor* if $N(f) \neq 0$ and $\dim_{\mathbb{C}} N(f)_k \leq 1$ for any k.

The main results in [13, 14] and many series of examples motivate the following conjecture.

Conjecture 1.1. [14]

(i) *Any rational cuspidal curve C in the plane is either free or nearly free.*
(ii) *An irreducible plane curve C which is either free or nearly free, is rational.*

In [14], the authors provide some interesting results supporting the statement of Conjecture 1.1(i); in particular, Conjecture 1.1(i) holds for rational cuspidal curves of even degree [14, Theorem 4.1]. They need a topological assumption on the cusps which is not fulfilled all the time when the degree is odd, see [14, Theorem 4.1].

They proved also that this conjecture holds for a curve C with an abelian fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ or for those curves having as degree a prime power, see [14, Corollary 4.2] and the discussion in [2].

Using the classification given in [16] of unicuspidal rational curve with a unique Puiseux pair, Dimca and Sticlaru proved in [14, Corollary 4.5] that all of them are either free divisor or nearly free divisor, except the curves of odd degree in one case of the classification.

As for Conjecture 1.1(ii), note that reducible nearly free curves may have irreducible components which are not rational, see [14, Example 2.8]: a smooth cubic with three tangents at aligned inflection points is nearly free (note that, the condition of alignment can be removed, at least in some examples computed using [9]). For free curves, examples can be found using [30, Theorem 2.7] e.g. $(x^3 - y^3)(y^3 - z^3)(x^3 - z^3)(ax^3 + by^3 + cz^3)$ for generic $a, b, c \in \mathbb{C}$ such that $a + b + c = 0$. The conjectures in [30] give some candidate examples of smaller degree; it is possible to prove that $(y^2z - x^3)(y^2z - x^3 - z^3) = 0$ is free (also computed with [9]). Dimca and Sticlaru also proposed the following conjecture.

Conjecture 1.2. [14]

(i) *Any free irreducible plane curve C has only singularities with at most two branches.*
(ii) *Any nearly free irreducible plane curve C has only singularities with at most three branches.*

In this paper we give some examples of irreducible free and nearly free curves in the complex projective plane which are not rational curves giving counterexamples to Conjecture 1.1(ii). Using these counterexamples we have found an example of irreducible free curve whose two singular points have any odd number of branches, giving counterexamples to Conjecture 1.2(i). Moreover an irreducible nearly free curve with just one singular point which has 4 branches giving counterexamples to Conjecture 1.2(ii) are provided too.

Section 2 is devoted to collect well known results in the theory of free divisors and nearly free divisors mainly from their original papers of A. Dimca and G. Sticlaru in [13, 14]. Also a characterization for being nearly-free reduced plane curve from A. Dimca in [10] is recorded. This characterization is similar to the characterization of being free given by du Plessis and Wall in [21].

From Section 3.2 it can be deduced that, for every odd integer $k \geq 1$, the irreducible plane curve C_{5k} of degree $d = 5k$ defined by

$$C_{5k}: \ f_{5k} := (y^k z^k - x^{2k})^2 y^k - x^{5k} = 0,$$

has 1) geometric genus $g(C_{5k}) = \frac{(k-1)(k-2)}{2}$, 2) its singularities consists of two points and the number of branches of C_{5k} at each of them is exactly k, 3) C_{5k} is a free divisor, see Theorem 3.9. This is a counterexample to both the free divisor part of Conjecture 1.1(ii) and Conjecture 1.2(i).

From Section 3.3 it can also be deduced that, for any odd integer $k \geq 1$, the irreducible plane curve C_{4k} of degree $d = 4k$ defined by

$$C_{4k}: \ f_{4k} := (y^k z^k - x^{2k})^2 - x^{3k} y^k = 0,$$

has 1) genus $g(C_{4k}) = \frac{(k-1)(k-2)}{2}$, 2) its singular set consists of two singular points and the number of branches of C_{4k} at each of them is k, 3) C_{4k} is a nearly free divisor, see Theorem 3.11. This is a counterexample to both the nearly free divisor part of Conjecture 1.1(ii) and Conjecture 1.2(ii) too.

In the families studied above the number of singular points of the curves is exactly two. In Section 3.4, we are looking for curves giving a counterexample to the nearly free divisor part of Conjecture 1.1(ii) with unbounded genus and number of singularities. In particular, for every odd integer $k \geq 1$, the irreducible curve C_{2k} of degree $d = 2k$ defined by

$$C_{2k}: \ f_{2k} := x^{2k} + y^{2k} + z^{2k} - 2(x^k y^k + x^k z^k + y^k z^k) = 0,$$

has 1) genus $g(C_{2k}) = \frac{(k-1)(k-2)}{2}$, 2) its singular set $\operatorname{Sing}(C_{2k})$ consists of exactly $3k$ singular points, each of them of type $\mathbb{A}_{k-1}$, 3) C_{2k} is a nearly free divisor, see Theorem 3.12.

One of the main tools to find such examples is the use of Kummer covers. A Kummer cover is a map $\pi_k : \mathbb{P}^2 \to \mathbb{P}^2$ given by

$$\pi_k([x : y : z]) := [x^k : y^k : z^k].$$

Since Kummer covers are finite Galois unramified covers of $\mathbb{P}^2 \setminus \{xyz = 0\}$ with $\operatorname{Gal}(\pi_k) \cong \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z}$, Kummer covers are a very useful, using them one can construct complicated algebraic curves starting from simple ones. We meanly refer to [4, §5] for a systematic study of Kummer covers.

In particular, these families of examples $\{C_{5k}\}$ (which are free divisors), $\{C_{4k}\}$ and $\{C_{2k}\}$ (which are nearly free divisors) are constructed as the pullback under the Kummer cover π_k of the corresponding rational cuspidal curves: the quintic C_5 which is a free divisor, and the corresponding nearly free divisors defined by either, the cuartic C_4, or by the conic C_2.

In the last section, Section 4, an irreducible curve C_{49} of degree 49 is given which has 1) genus $g(C_{49}) = 0$, 2) its singular set consists of just one singular point which has 4 branches, 3) C_{49} is a nearly free divisor. These example can be constructed as a general element of the unique pencil associated to any rational unicuspidal plane curve, see [8].

2. Free and nearly free plane curves after Dimca and Sticlaru

Let $S := \mathbb{C}[x, y, z]$ be the polynomial ring endowed with the natural graduation $S = \bigoplus_{m=0}^{\infty} S_m$ by homogeneous polynomials. Let $f \in S_d$ be a homogeneous polynomial of degree d in the polynomial ring Let C be the plane curve in $\mathbb{P}^2$

defined by $f = 0$ and assume that C is reduced. We have denoted by J_f the Jacobian ideal of f, which is the homogeneous ideal in S spanned by f_x, f_y, f_z. Let $M(f) = S/J_f$ be the corresponding graded ring, called the Jacobian (or Milnor) algebra of f.

The minimal degree of a Jacobian relation for f is the integer $\mathrm{mdr}(f)$ defined to be the smallest integer $m \geq 0$ such that there is a nontrivial relation

$$af_x + bf_y + cf_z = 0, \quad (a, b, c) \in S_m^3 \setminus (0, 0, 0). \tag{2.1}$$

When $\mathrm{mdr}(f) = 0$, then C is a union of lines passing through one point, a situation easy to analyse. We assume from now on that $\mathrm{mdr}(f) \geq 1$.

2.1. **Free plane curves.**

We have denoted by I_f the saturation of the ideal J_f with respect to the maximal ideal (x, y, z) in S. Let $N(f) = I_f/J_f$ be the corresponding homogeneous quotient ring.

Consider the graded S-submodule

$$\mathrm{AR}(f) = \{(a, b, c) \in S^3 \mid af_x + bf_y + cf_z = 0\} \subset S^3$$

of *all relations* involving the partial derivatives of f, and denote by $\mathrm{AR}(f)_m$ its homogeneous part of degree m.

Notation 2.1. We set $\mathrm{ar}(f)_k = \dim \mathrm{AR}(f)_k$, $m(f)_k = \dim M(f)_k$ and $n(f)_k = \dim N(f)_k$ for any integer k.

We use the definition of freeness given by Dimca [10].

Definition 2.2. The curve $C : f = 0$ is a *free divisor* if the following equivalent conditions hold.

(1) $N(f) = 0$, i.e. the Jacobian ideal is saturated.

(2) The minimal resolution of the Milnor algebra $M(f)$ has the following form

$$0 \to S(-d_1 - d + 1) \oplus S(-d_2 - d + 1) \to S^3(-d + 1) \xrightarrow{(f_x, f_y, f_z)} S$$

for some positive integers d_1, d_2.

(3) The graded S-module $\mathrm{AR}(f)$ is free of rank 2, i.e. there is an isomorphism

$$\mathrm{AR}(f) = S(-d_1) \oplus S(-d_2)$$

for some positive integers d_1, d_2.

When C is a free divisor, the integers $d_1 \leq d_2$ are called the exponents of C. They satisfy the relations

$$d_1 + d_2 = d - 1 \text{ and } \tau(C) = (d-1)^2 - d_1 d_2, \tag{2.2}$$

where $\tau(C)$ is the total Tjurina number of C, see for instance [11, 13]. Using deformation results in [24], Sticlaru [26] defines a curve $C \subset \mathbb{P}^2$ to be *projectively rigid* if $(I_f)_d = (J_f)_d$. In particular, if C is free then it is projectively rigid.

Remark 2.3. This notion of *projectively rigid* differs from the classical one, see e.g. [17], where a curve is projectively rigid if its equisingular moduli space is discrete. Note that four lines passing through a point define a free divisor but its equisingular moduli space is defined by the cross-ratio.

2.2. **Nearly free plane curves.**

Dimca and Sticlaru introduced a more subtle notion for a divisor to be nearly free, see [14].

Definition 2.4. [14] The curve $C : f = 0$ is a *nearly free divisor* if the following equivalent conditions hold.

(1) $N(f) \neq 0$ and $n(f)_k \leq 1$ for any k.
(2) The Milnor algebra $M(f)$ has a minimal resolution of the form

$$0 \to S(-d-d_2) \to S(-d-d_1+1) \oplus S^2(-d-d_2+1) \to S^3(-d+1) \xrightarrow{(f_0,f_1,f_2)} S \tag{2.3}$$

for some integers $1 \leq d_1 \leq d_2$, called the exponents of C.
(3) There are 3 syzygies ρ_1, ρ_2, ρ_3 of degrees d_1, $d_2 = d_3 = d - d_1$ which form a minimal system of generators for the first-syzygy module $\mathrm{AR}(f)$.

If $C : f = 0$ is nearly free, then the exponents $d_1 \leq d_2$ satisfy

$$d_1 + d_2 = d \text{ and } \tau(C) = (d-1)^2 - d_1(d_2 - 1) - 1, \tag{2.4}$$

see [14]. For both a free and a nearly free curve $C : f = 0$, it is clear that $\mathrm{mdr}(f) = d_1$.

Remark 2.5. In [14] it is shown that to construct a resolution (2.3) for a given polynomial f, the following conditions must be satisfied:

(i) the integer $b := d_2 - d + 2$,

(ii) three syzygies $r_i = (a_i, b_i, c_i) \in S^3_{d_i}$, $i = 1, 2, 3$, for (f_x, f_y, f_z), i.e.

$$a_i f_x + b_i f_y + c_i f_z = 0,$$

necessary to construct the morphism

$$\bigoplus_{i=1}^{3} S(-d_i - (d-1)) \to S^3(-d+1), \quad (u_1, u_2, u_3) \mapsto u_1 r_1 + u_2 r_2 + u_3 r_3.$$

(iii) One relation $R = (v_1, v_2, v_3) \in \bigoplus_{i=1}^{3} S(-d_i - (d-1))_{b+2(d-1)}$ among r_1, r_2, r_3, i.e. $v_1 r_1 + v_2 r_2 + v_3 r_3 = 0$, necessary to construct the morphism

$$S(-b - 2(d-1)) \to \bigoplus_{i=1,3} S(-d_i - (d-1))$$

by the formula $w \mapsto wR$. Note that $v_i \in S_{b-d_i+d-1}$.

Corollary 2.6. [14] *Let $C : f = 0$ be a nearly free curve of degree d with exponents (d_1, d_2). Then $N(f)_k \neq 0$ for $d+d_1-3 \leq k \leq d+d_2-3$ and $N(f)_k = 0$ otherwise. The curve C is projectively rigid if and only if $d_1 \geq 4$.*

2.3. Characterization of free and nearly free reduced plane curves.

Just recently Dimca provides in [10] the following characterization of free and nearly free reduced plane curves. For a positive integer r,

$$\tau(r)_{\max} := (d-1)(d-r-1) + r^2,$$

is defined.

Theorem 2.7 ([10])**.** *Let $C \subset \mathbb{P}^2$ be a reduced curve of degree d defined by $f = 0$, and let $r := \mathrm{mdr}(f)$.*

(1) *If $r < \frac{d}{2}$, then $\tau(C) = \tau(r)_{\max}$ if and only if $C : f = 0$ is a free curve.*
(2) *If $r \leq \frac{d}{2}$, then $\tau(C) = \tau(r)_{\max} - 1$ if and only if C is a nearly free curve.*

As it is recalled in [10], Theorem 2.7(1) is Corollary of [21, Theorem 3.2] by du Plessis and Wall.

3. High-genus curves which are free or nearly free divisors

3.1. Transformations of curves by Kummer covers.

A Kummer cover is a map $\pi_k : \mathbb{P}^2 \to \mathbb{P}^2$ given by $\pi_k([x : y : z]) := [x^k : y^k : z^k]$. Kummer covers are a very useful tool in order to construct complicated algebraic curves starting from simple ones. Since Kummer covers are finite Galois unramified covers of $\mathbb{P}^2 \setminus \{xyz = 0\}$ with $\mathrm{Gal}(\pi_k) \cong \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z}$, topological properties of the new curves can be obtained,for instance: Alexander polynomial, fundamental group, characteristic varieties and so on (see [1, 3, 5, 29, 18, 6, 4, 19] for papers using these techniques).

Example 3.1. In [29], Uludağ constructs new examples of Zariski pairs using former ones and Kummer covers. He also uses the same techniques to construct infinite families of curves with finite non-abelian fundamental groups.

Example 3.2. In [18, 5], the Kummer covers allow to construct curves with *many cusps* and extremal properties for their Alexander invariants. These ideas are pushed further in [6] where the authors find Zariski triples of curves of degree 12 with 32 ordinary cusps (distinguished by their Alexander polynomial). Within the same ideas Niels Lindner [19] constructed an example of a cuspidal curve C' of degree 12 with 30 cusps and Alexander polynomial $t^2 - t + 1$. For this, he started with a sextic C_0 with 6 cusps, admitting a toric decomposition. He pulled back C_0 under a Kummer map $\pi_2 : \mathbb{P}^2 \to \mathbb{P}^2$ ramified above three inflectional tangents of C_0. Since the sextic is of torus type, then same holds for the pullback. Lindner showed that the Mordell-Weil lattice has rank 2 and that the Mordell-Weil group contains $A_2(2)$.

A systematic study of Kummer covers of projective plane curves has been done by J.I. Cogolludo , J. Ortigas and the first named author in [4, §5]. Some of their results are collected below.

Let C be a (reduced) projective curve of degree d of equation $F_d(x, y, z) = 0$ and let $\bar{C}_k$ be its transform by a Kummer cover π_k, $k \geq 1$. Note that $\bar{C}_k$ is a projective curve of degree dk of equation $F_d(x^k, y^k, z^k) = 0$.

Definition 3.3. [4] We define $P \in \mathbb{P}^2$ such that $P := [x_0 : y_0 : z_0]$. We say that P is a point of *type* $(\mathbb{C}^*)^2$ (or simply of *type* 2) if $x_0y_0z_0 \neq 0$. If $x_0 = 0$ but $y_0z_0 \neq 0$ the point is said to be of *type* $\mathbb{C}^*_x$ (types $\mathbb{C}^*_y$ and $\mathbb{C}^*_z$ are defined accordingly). Such points will also be referred to as *type* 1 points. The corresponding line (either $L_X := \{X = 0\}$, $L_Y := \{Y = 0\}$, or $L_Z := \{Z = 0\}$) the type-1 point lies on will be referred to as its *axis*. The remaining points $P_x := [1 : 0 : 0]$, $P_y := [0 : 1 : 0]$, and $P_z := [0 : 0 : 1]$ will be called *vertices* (or type 0 points) and their axes are the two lines (either L_X, L_Y, or L_Z) they lie on.

Remark 3.4. [4] Note that a point of type ℓ, $\ell = 0, 1, 2$ in $\mathbb{P}^2$ has exactly k^ℓ preimages under π_k. It is also clear that the local type of $\bar{C}_k$ at any two points on the same fiber are analytically equivalent. The singularities of $\bar{C}_k$ are described in the following proposition.

Proposition 3.5. [4] *Let $P \in \mathbb{P}^2$ be a point of type ℓ and $Q \in \pi_k^{-1}(P)$. The following conditions hold:*

(1) *If $\ell = 2$, then (C, P) and $(\bar{C}_k, Q)$ are analytically isomorphic.*
(2) *If $\ell = 1$, then $(\bar{C}_k, Q)$ is a singular point of type 1 if and only if $m > 1$, where $m := (C \cdot \bar{L})_P$ and $\bar{L}$ is the axis of P.*
(3) *If $\ell = 0$, then $(\bar{C}_k, Q)$ is a singular point.*

Remark 3.6. Using Proposition 3.5 (1), if $\mathrm{Sing}(C) \subset \{xyz = 0\}$ then $\mathrm{Sing}(\bar{C}_k) \subset \{xyz = 0\}$.

Example 3.7. [4] In some cases, we can be more explicit about the singularity type of $(\bar{C}_k, Q)$. If P is of type 1, (C, P) is smooth and $m := (C \cdot \bar{L})_P$ then $(\bar{C}_k, Q)$ has the same topological type as $u_0^k - v_0^m = 0$. In particular, if $m = 2$, then $(\bar{C}_k, Q)$ is of type $\mathbb{A}_{k-1}$.

In order to better describe singular points of type 0 and of type 1 of $\bar{C}_k$ we will introduce the following notation. Let $P \in \mathbb{P}^2$ be a point of type $\ell = 0, 1$ and $Q \in \pi_k^{-1}(P)$ a singular point of $\bar{C}_k$. Denote by μ_P (resp. μ_Q) the Milnor number of C at P (resp. $\bar{C}_k$ at Q). Since $\ell = 0, 1$, then P and Q belong to either exactly one or two axes. If P and Q belong to an axis $\bar{L}$, then $m_P^{\bar{L}} := (C \cdot \bar{L})_P$ (analogous notation for Q). More specific details about singular points of types 0 and 1 can be described as follows.

Proposition 3.8. [4] *Under the above conditions and notation, the following conditions hold:*

(1) *For $\ell = 1$, P belongs to a unique axis $\bar{L}$ and*
    (a) $\mu_Q = k\mu_P + (m_P^{\bar{L}} - 1)(k - 1)$,
    (b) *and, if (C, P) is locally irreducible and $r := \gcd(k, m_P^{\bar{L}})$, then (C, Q) has r irreducible components which are analytically isomorphic to each other.*
(2) *For $\ell = 0$, P belongs to exactly two axes $\bar{L}_1$ and $\bar{L}_2$*
    (a) $\mu_Q = k^2(\mu_P - 1) + k(k - 1)(m_P^{\bar{L}_1} + m_P^{\bar{L}_2}) + 1$ *(There is a typo in the printed formula in [4]: $k - k^2$ must be added).*

(b) *and, if* (C,P) *is locally irreducible and* $r := \gcd(k, m_P^{\bar{L}_1}, m_P^{\bar{L}_2})$*, then* (C,Q) *has* kr *irreducible components which are analytically isomorphic to each other.*

3.2. **Irreducible free curves with many branches and high genus.**

Let us consider the quintic curve C_5, see Figure 1, defined by $f_5 := (yz-x^2)^2y - x^5 = 0$. It has two singular points, $p_1 = [0:1:0]$ of type $\mathbb{A}_4$ and $p_2 = [0:0:1]$ of type $\mathbb{E}_8$. Therefore, it is a rational and cuspidal plane curve. This curve is free, see [13, Theorem 4.6]. Let us consider the Kummer cover $\pi_k : \mathbb{P}^2 \to \mathbb{P}^2$ given by $\pi_k([x:y:z]) := [x^k : y^k : z^k]$ and its Kummer transform C_{5k}, defined by $f_{5k} := (y^kz^k - x^{2k})^2y^k - x^{5k} = 0$.

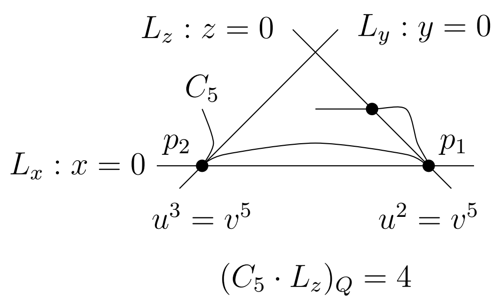

FIGURE 1. Curve C_5

Theorem 3.9. *For any* $k \geq 1$*, the curve* C_{5k} *of degree* $d = 5k$ *defined by*

$$C_{5k} : \ f_{5k} := (y^kz^k - x^{2k})^2y^k - x^{5k} = 0, \tag{3.1}$$

verifies the following properties:

(1) $\operatorname{Sing}(C_{5k}) = \{p_1, p_2\}$*. The number of branches of* C_{5k} *at* p_2 *is* k*, and at* p_1*, it equals* k *(if* k *is odd) or* $2k$ *(if* k *is even).*

(2) C_{5k} *is a free divisor with exponents* $d_1 = 2k$*,* $d_2 = 3k-1$ *and* $\tau(C_{5k}) = 19k^2 - 8k + 1$*.*

(3) C_{5k} *has two irreducible components of genus* $\frac{(k-2)^2}{4}$ *if* k *is even and irreducible of genus* $\frac{(k-1)(k-2)}{2}$ *otherwise.*

Proof. Part (1) is an easy consequence of [4, Lemma 5.3, Proposition 5.4 and Proposition 5.6]. The singularites $\operatorname{Sing}(C_5) = \{p_1, p_2\}$ are of type 0, in the sense of the Kummer cover π_k (see Definition 3.3) and C_5 has no singularities outside the intersection points of the axes. Moreover C_5 intersects the line L_z transversally at a point of type 1; then by Proposition 3.5 (2) and by Remark 3.6, the singularities of C_{5k} are exactly the points p_1 and p_2.

Since p_1 and p_2 are of type 0 we deduce the structure of C_{5k} at these points. using Proposition 3.8 (2) (b). At p_1 one has $(C_5, L_z)_{p_1} = 5$, $(C_5, L_x)_{p_1} = 2$ and $r_{p_1} = \gcd(k, 2, 5) = 1$ for all k , and so that the number of branches of C_{5k} at p_1 is equal to k. On the other hand, to study the number of branches at p_2, we compute the intersection numbers $(C_5, L_x)_{p_2} = 2$ and $(C_5, L_y)_{p_2} = 4$ and therefore $r_{p_2} = \gcd(k, 2, 4) = \gcd(k, 2)$. Ik k is odd, $r_{p_2} = 1$ and the number of branches of C_{5k} at p_2 is equal to k. Otherwise $r_{p_2} = 2$ and the number of branches of C_{5k} at p_2 is equal to $2k$.

In order to prove (2), we follow the ideas of [13, Theorem 4.6]. Let us study first the syzygies of the free curve C_5. Let us denote by $D_{u,v,w}$, the diagonal matrix with entries u, v, w, and define the vectors

$$R_1 = \big(0,\, 2y,\, x^2 - 3yz\big)\,, \quad R_2 = \big(2(x^2 - yz), 2(5x^2 - 4xy + 15yz), 8x - 45z\big)\,.$$

Let us denote by J the Jacobian ideal J of f_5. Let us denote by J_x the ideal generated by $(xf_{5x}, f_{5y}, f_{5z})$. In the same way, we consider the ideals J_y, J_z, J_{xy}, J_{xz}, J_{yz}, J_{xyz}. The Table 1 shows bases for the syzygies of these ideals, computed with `Singular` [9]. Note that

Ideal	First generator	Second generator
J	$R_1 \cdot D_{1,y,1}$	$R_2 \cdot D_{1,1,z}$
J_x	$R_1 \cdot D_{1,y,1}$	$R_2 \cdot D_{1,x,xz}$
J_y	R_1	$R_2 \cdot D_{y,1,yz}$
J_z	$R_1 \cdot D_{1,yz,1}$	R_2
J_{xy}	R_1	$R_2 \cdot D_{y,x,xyz}$
J_{xz}	$R_1 \cdot D_{1,yz,1}$	$R_2 \cdot D_{1,x,x}$
J_{yz}	$R_1 \cdot D_{1,z,1}$	$R_2 \cdot D_{y,1,y}$
J_{xyz}	$R_1 \cdot D_{1,z,1}$	$R_2 \cdot D_{y,x,xy}$

Table 1. Bases of syzygies

$$f_{5k_x} = kx^{k-1} f_{5x}(x^k, y^k, z^k),\ f_{5k_y} = ky^{k-1} f_{5y}(x^k, y^k, z^k),\ f_{5k_z} = kz^{k-1} f_{5z}(x^k, y^k, z^k).$$

Let $S_k := \mathbb{C}[x^k, y^k, z^k]$. We have a decomposition

$$S = \bigoplus_{(i,j,l)\in\{0,\dots,k-1\}} x^i y^j z^l S_k. \tag{3.2}$$

By construction, $f_{5k_x} \in x^{k-1}S_k$, $f_{5k_y} \in y^{k-1}S_k$ and $f_{5k_z} \in z^{k-1}S_k$. Hence, in order to compute the syzygies (a,b,c) among the partial derivatives of f_{5k}, we need to characterize the triples (a,b,c) such that each entry belongs to a factor of the decomposition (3.2).

Let us assume that $a \in x^{i_x}y^{j_x}z^{j_x}S_k$, $b \in x^{i_y}y^{j_y}z^{j_y}S_k$ and $c \in x^{i_z}y^{j_z}z^{j_z}S_k$. We deduce that

$$i_x + k - 1 \equiv i_y \equiv i_z \bmod k \Longrightarrow i = i_y = i_z \text{ and } i_x = \begin{cases} i+1 & \text{if } i < k-1 \\ 0 & \text{if } i = k-1. \end{cases}$$

Analogous relations hold for the other indices. We distinguish four cases:

Case 1. $i = j = l = k-1$.

In this case $a(x,y,z) = y^{k-1}z^{k-1}\alpha(x^k,y^k,z^k)$, $b(x,y,z) = x^{k-1}z^{k-1}\beta(x^k,y^k,z^k)$ and $c(x,y,z) = x^{k-1}y^{k-1}\gamma(x^k,y^k,z^k)$. Hence (α,β,γ) is a syzygy for the partial derivatives of f_5. We conclude that (a,b,c) is a combination of:

$$R_1(x^k,y^k,z^k)\cdot D_{1,x^{k-1}y^kz^{k-1},x^{k-1}y^{k-1}} = x^{k-1}y^{k-1}R_1(x^k,y^k,z^k)\cdot D_{1,yz^{k-1},1}$$

and

$$R_2(x^k,y^k,z^k)\cdot D_{y^{k-1}z^{k-1},x^{k-1}z^{k-1},x^{k-1}y^{k-1}z^k} = z^{k-1}R_2(x^k,y^k,z^k)\cdot D_{y^{k-1},x^{k-1},x^{k-1}y^{k-1}z}.$$

Divided by common factors we obtain syzygies of degree $2k$ and $3k-1$.

Case 2. $i < k-1$, $j = l = k-1$.

In this case $a(x,y,z) = x^{i+1}y^{k-1}z^{k-1}\alpha(x^k,y^k,z^k)$, $b(x,y,z) = x^i z^{k-1}\beta(x^k,y^k,z^k)$ and $c(x,y,z) = x^i y^{k-1}\gamma(x^k,y^k,z^k)$. Hence (α,β,γ) is a syzygy for the generators of the ideal J_x. It is easily seen that we obtain combination of generators of the above syzygies. The other cases are treated in the same way.

We conclude that C_{5k} is free with $d_1 = \operatorname{mdr}(f_{5k}) = 2k$ and $d_2 = d - 1 - d_1 = 5k - 1 - 2k = 3k - 1$. By equation (2.2) $\tau(C_{5k}) = 19k^2 - 8k + 1$ for all k.

In order to prove (3), we study the branched cover $\tilde{\pi}_k : \tilde{C}_{5k} \to \tilde{C}_5$ between the normalizations of the curves. The monodromy of this map as an unramified cover of $\mathbb{P}^2 \setminus \{xyz = 0\}$ is determined by an epimorphism

$$H_1(\mathbb{P}^2 \setminus \{xyz = 0\}; \mathbb{Z}) \to \mathbb{Z}_k \times \mathbb{Z}_k =: G_k$$

such that the meridians of the lines are sent to a_x, a_y, a_z, a system of generators of G_k such that $a_x + a_y + a_z = 0$. Since the singularities of C_5 are locally irreducible, then C_5 and $\tilde{C}_5$ are homeomorphic, and the covering $\tilde{\pi}_k$ is determined by the monodromy map

$$H_1(\tilde{C}_5 \setminus \{xyz = 0\}; \mathbb{Z}) \to \mathbb{Z}_k \times \mathbb{Z}_k =: G_k$$

obtained by composing using the map defined by the inclusion. Hence $\tilde{C}_5 \setminus \{xyz = 0\}$ is isomorphic to $\mathbb{P}^1 \setminus \{\text{three points}\}$). The image of a meridian corresponding to a point P in the axes is given by

$$m_P^{L_x} a_x + m_P^{L_y} a_y + m_P^{L_z} a_z.$$

Hence, we obtain a_z (the smooth point), $3a_x + 5a_y$ (the $\mathbb{E}_8$-point) and $2a_x + 4a_z$ (the $\mathbb{A}_4$-point). In terms of the basis a_y, a_z they read as $a_z, 2a_y - 3a_z, -2a_y + 2a_z$, i.e., the monodromy group is generated by $2a_y, a_z$. If k is even, the monodromy group is of index 2 in G_k, and hence $\tilde{C}_{5k}$ has two connected components. Otherwise, if it is equal to G_k when k is odd and $\tilde{C}_{5k}$ is connected. These properties give us the statement about the number of irreducible components.

The genus can be computed using the singularities of C_{5k} or via Riemann-Hurwitz's formula. Note that the covering $\tilde{\pi}_k$ is of degree k^2 with three ramification points; at p_2 and the smooth point in the axis where we find k preimages, while at p_1 we find k preimages if k is odd and $2k$ preimages if it is even, because of (1). Hence, for k odd, the Euler characteristic of the normalization is

$$\chi(\tilde{C}_{5k}) = -k^2 + 3k \Longrightarrow g(\tilde{C}_{5k}) = \frac{(k-1)(k-2)}{2}.$$

And for k even, where $\tilde{C}_{5k} = \tilde{C}_{5k}^1 \cup \tilde{C}_{5k}^2$, the Euler characteristic is

$$\chi(\tilde{C}_{5k}) = -k^2 + 4k \Longrightarrow g(\tilde{C}_{5k}^i) = \frac{2 - \frac{\chi(\tilde{C}_{5k})}{2}}{2} = \frac{(k-2)^2}{4}. \qquad \square$$

So, for odd $k \geq 3$, the curve C_{5k} is an irreducible free curve of positive genus whose singularities have k branches each. This is a counterexample to both the free divisor part of Conjecture 1.1(ii) and Conjecture 1.2(i).

Remark 3.10. Up to projective transformation, there are two quintic curves with two singular points of type $\mathbb{A}_4$ and $\mathbb{E}_8$. One is $C_5 : (yz - x^2)^2 y - x^5 = 0$, which is free; the other one is defined by $D_5 : g = y^3 z^2 - x^5 = 0$ (the contact of the tangent line to the $\mathbb{A}_4$-point distinguishes both curves). Moreover, the curve D_5 is nearly free; it can be computed that $\operatorname{mdr}(g) = 1$. Since both singular points are quasihomogeneous, $12 = \tau(C_5) = \mu(C_5) = \mu(D_5) = \tau(D_5)$, and we may apply Theorem 2.7(2); the pair (C_5,D_5) is a kind of counterexample to Terao's conjecture [20, Conjecture 4.138] for irreducible divisors (with constant Tjurina number), compare with [23].

3.3. **Irreducible nearly free curves with many branches and high genus.**

The quartic curve C_4 defined by $f_4 := (yz - x^2)^2 - x^3 y = 0$ has two singular points, $p_1 = [0 : 1 : 0]$ of type $\mathbb{A}_2$ and $p_2 = [0 : 0 : 1]$ of type $\mathbb{A}_4$. Therefore it is rational and cuspidal. We will consider the Kummer transform C_{4k}, defined by $f_{4k} := (y^k z^k - x^{2k})^2 - x^{3k} y^k = 0$, of the curve C_4.

Theorem 3.11. *For any $k \geq 1$, the curve C_{4k} of degree $d = 4k$ defined by*

$$C_{4k} : \; f_{4k} := (y^k z^k - x^{2k})^2 - x^{3k} y^k = 0,$$

verifies the following properties

1. $\operatorname{Sing}(C_{4k}) = \{p_1, p_2\}$. *The number of branches of C_{4k} at each p_2 is k, and at p_1, it equals k (if k is odd) or $2k$ (if k is even).*
2. *C_{4k} is a nearly free divisor with exponents $d_1 = d_2 = d_3 = 2k$ and $\tau(C_{4k}) = 6k(2k-1)$.*
3. *C_{4k} has two irreducible components of genus $\frac{(k-2)^2}{4}$ if k is even and it is irreducible of genus $\frac{(k-1)(k-2)}{2}$ otherwise.*

Proof. Since $\operatorname{Sing}(C_4)(= \{p_1, p_2\})$ are points of type 0, C_4 meets $\{xyz = 0\}$ at three points p_1, p_2 and transversally at p_3 which is of type 1, therefore $\operatorname{Sing}(C_{4k}) = \{p_1, p_2\}$. To prove Part (1) its enough to find the number of branches of C_{4k} at these points using Proposition 3.8 (2) (b). At p_1 one has $(C_4, L_z)_{p_1} = 3$, $(C_4, L_x)_{p_1} = 2$ and $r_{p_1} = \gcd(k, 2, 3) = 1$ for all k , and so that the number of branches of C_{4k} at p_1 is equal to k. In the same way, at p_2, the intersection $(C_4, L_x)_{p_2} = 2$, $(C_4, L_y)_{p_2} = 4$ and $r_{p_2} = \gcd(k, 2, 4) = \gcd(k, 2)$. If k is odd, $r_{p_2} = 1$ and the number of branches of C_{4k} at p_2 is equal to k. Otherwise $r_{p_2} = 2$ and the number of branches of C_{4k} at p_2 is equal to $2k$.

The proof of Part (2) follows the same guidelines as Theorem 3.9. With the notations of that proof, a generator system for the syzygies of J (Jacobian ideal of f_4) is given by:

$$\begin{aligned} R_1 &:= (y(3x-4z),\, 3y(4x-3y),\, z(9y-20x))\,, \\ R_2 &:= \left(-x(x+2z),\, -4x^2+3xy+10yz,\, -z(3x+10z)\right), \\ R_3 &:= \left(xy,\, -3y^2,\, 2x^2+3yz\right). \end{aligned} \tag{3.3}$$

These syzygies satisfy the relation $xR_1+3yR_2+10zR_3=0$. And by Dimca Steclaru Remark (2.5) C_4 is a nearly free divisor with exponents $d_1=d_2=d_3=2$.

For the ideal J_z, we have a similar situation. For the other ideals, their syzygy space is free of rank 2. Using these results it is not hard to prove that the syzygies of f_{4k} are generated by

$$\begin{aligned} R_{k,1} &:= \left(y^k(3x^k-4z^k),\, 3x^{k-1}y(4x^k-3y^k),\, x^{k-1}z(9y^k-20x^k)\right), \\ R_{k,2} &:= \left(-xy^{k-1}(x^k+2z^k),\, -4x^{2k}+3x^ky^k+10y^kz^k,\, -y^{k-1}z(3x^k+10z^k)\right), \\ R_{k,3} &:= \left(xy^kz^{k-1},\, -3y^{k+1}z^{k-1},\, 2x^{2k}+3y^kz^k\right). \end{aligned}$$

The results follow as in the proof of Theorem 3.9.

These syzygies satisfy the relation $xR_{k,1}+3yR_{k,2}+10zR_{k,3}=0$ and by Dimca Steclaru Remark (2.5) C_{4k} is a nearly free divisor with exponents $d_1=d_2=d_3=2k$ and by equation (2.4) $\tau(C_{4k})=6k(2k-1)$.

The proof of Part (3) follows the same ideas as Theorem 3.9 (3). □

So, for odd $k\geq 3$, the curve C_{4k} is an irreducible nearly free curve of positive genus whose singularities have k branches each. This is a counterexample to both the nearly free divisor part of Conjecture 1.1(ii) and Conjecture 1.2(ii).

3.4. **Positive genus nearly-free curves with many singularities.**

Let us consider the conic C_2 given by $f_2=x^2+y^2+z^2-2(xy+xz+yz)=0$. This conic is tangent to three axes and it is very useful to produce interesting curves using Kummer covers.

Theorem 3.12. *For any $k\geq 1$, the curve C_{2k} of degree $d=2k$ defined by*

$$C_{2k}:\ f_{2k}:=x^{2k}+y^{2k}+z^{2k}-2(x^ky^k+x^kz^k+y^kz^k)=0,$$

verifies the following properties

(1) $\mathrm{Sing}(C_{2k})$ *are $3k$ singular points of type $\mathbb{A}_{k-1}$.*

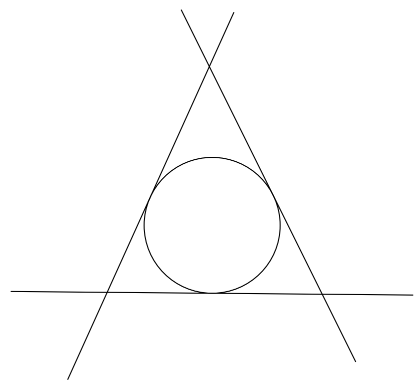

FIGURE 2. Conic C_2.

(2) *C_{2k} is a nearly free divisor with exponents $d_1 = d_2 = d_3 = k$ and $\tau(C_{2k}) = 3k(k-1)$.*
(3) *C_{2k} is irreducible of genus $\frac{(k-1)(k-2)}{2}$ if k is odd and it has four irreducible smooth components of degree $\frac{k}{2}$ if k is even.*

Proof. To prove (1) it is enough to take into the account that C_2 is nonsingular and by Remark 3.6 the singularites of C_{2k} satisfy $\operatorname{Sing}(C_{2k}) \subset \{xyz = 0\}$. Moreover C_2 is tangent to the three axes at 3 points $\{p_1, p_2, p_3\}$ of type 1 with $(C_2, L_x)_{p_1} = (C_2, L_y)_{p_2} = (C_2, L_z)_{p_3} = 2$ at these points. For $i = 1, \dots, 3$, the points p_i are of type 1 and by Remark 3.4 all the k preimages under π_k are analytically equivalent. By Example 3.7, over each p_i, one has k singular points of type $\mathbb{A}_{k-1}$.

Let us study (2). A generator system for the syzygies of J (Jacobian ideal of f_2) is given by:

$$\begin{aligned} R_1 &:= (y - z,\, y,\, -z)\,, \\ R_2 &:= (-x,\, z - x,\, z)\,, \\ R_3 &:= (x,\, -y,\, x - y)\,. \end{aligned} \tag{3.4}$$

These syzygies satisfy the relation $xR_1 + yR_2 + zR_3 = 0$. The other ideals have free 2-rank syzygy modules. A simple computation gives the following syzygies for f_{2k}:

$$\begin{aligned} R_{k,1} &:= \left(y^k - z^k,\, x^{k-1}y,\, -x^{k-1}z\right)\,, \\ R_{k,2} &:= \left(-xy^{k-1},\, z^k - x^k,\, y^{k-1}z\right)\,, \\ R_{k,3} &:= \left(xz^{k-1},\, -yz^{k-1},\, x^k - y^k\right)\,. \end{aligned}$$

These syzygies satisfy the relation $xR_{k,1} + yR_{k,2} + zR_{k,3} = 0$ and therefore, C_{2k} is a nearly free divisor with exponents $d_1 = d_2 = d_3 = k$ and $\tau(C_{2k}) = 3k(k-1)$.

To prove (3) we follow as in the proof of Theorem 3.9 (3); the main difference is that π_2 has no ramification over C_2 and in fact C_4 is the union of four lines in general position; their preimages. If $k = 2\ell$, since $\pi_k = \pi_\ell \circ \pi_2$, each irreducible component is a smooth Fermat curve.

□

For odd $k \geq 3$, these curves have positive genus and give a counterexample to the nearly free divisor part of Conjecture 1.1(ii) (with unbounded genus and number of singularities). Furthermore, if $k \geq 5$, since $d_1 = 5 \geq 4$ then by Corollary 2.6 C_{2k} is projectively rigid. Note that it is not the case for C_6, where we find it is the dual of a smooth cubic which is a nearly free divisor. A simple computation shows that the dual of a generic smooth cubic is also a nearly free divisor.

4. Pencil associated to unicuspidal rational plane curves

In this section we are going to show that it is possible to construct a rational nearly free curve whose singular points has more than three branches, that is the condition to have high genus is not needed.

Given a curve $C \subset \mathbb{P}^2$, let $\pi : \widetilde{\mathbb{P}}^2 \to \mathbb{P}^2$ be the minimal, (not the "embedded" minimal) resolution of singularities of C. Let $\widetilde{C} \subset \widetilde{\mathbb{P}}^2$ be the strict transform of C, and let $\tilde{\nu}(C) = \widetilde{C} \cdot \widetilde{C}$ denote the self-intersection number of $\widetilde{C}$ on $\widetilde{\mathbb{P}}^2$.

A *unicuspidal rational curve* is a pair (C, P) where C is a curve and $P \in C$ satisfies $C \backslash \{P\} \cong \mathbb{A}^1$. We call P the distinguished point of C. Given a unicuspidal rational curve (C, P), D. Daigle and the last named author proved the existence of a unique pencil Λ_C on $\mathbb{P}^2$ satisfying $C \in \Lambda_C$ and $\mathrm{Bs}(\Lambda_C) = \{P\}$ where $\mathrm{Bs}(\Lambda_C)$ denotes the base locus of Λ_C on $\mathbb{P}^2$, see [7, 8].

Let $\pi_m : \widetilde{\mathbb{P}}^2_m \to \mathbb{P}^2$ be the minimal resolution of the base points of the pencil. By Bertini theorem, the singularities of the general member C_{gen} of Λ_C are contained in $\mathrm{Bs}(\Lambda_C) = \{P\}$.

For a unicuspidal rational curve $C \subset \mathbb{P}^2$, we show (cf. [8, Theorem 4.1]) that the general member of Λ_C is a rational curve if and only if $\tilde{\nu}(C) \geq 0$. In this case

(1) the general element C_{gen} of Λ_C satisfies that the weighted cluster of infinitely near points of C_{gen} and C are equal (see [7, Proposition 2.7]).
(2) Λ_C has either 1 or 2 dicriticals, and at least one of them has degree 1.

In view of these results, it is worth noting that *all currently known unicuspidal rational curves* $C \subset \mathbb{P}^2$ *satisfy* $\tilde{\nu}(C) \geq 0$, see [8, Remark 4.3] for details.

Let $C \subset \mathbb{P}^2$ be a unicuspidal rational curve of degree d and with distinguished point P. In [8, Proposition 1] it is proved that Λ_C is in fact the set of effective divisors D of $\mathbb{P}^2$ such that $\deg(D) = d$ and $i_P(C, D) \geq d^2$. Since $i_P(C, C) = \infty > d^2$, then the curve $C \in \Lambda_C$.

The main idea here is to take the general member C_{gen} of the pencil Λ_C for a nonnegative curve, i.e $\tilde{\nu}(C) \geq 0$. Doing this one gets a rational cuve C_{gen} whose singularities is $\text{Sing}(C_{\text{gen}}) = \{P\}$ and the branches of C_{gen} at P equals to the sum of the degrees of the dicriticals divisors.

The clasification of unicuspidal rational plane curve with $\bar{\kappa}(\mathbb{P}^2 \setminus C) = 1$ was started by Sh. Tsunoda [28] and finished by K. Tono [27] (see also p. 125 in [15]).

Our next example starts with C_{49} with $\bar{\kappa}(\mathbb{P}^2 \setminus C_{49}) = 1$. Secondly we take the pencil $\Lambda_{C_{49}}$, and finally its general member $C_{49,gen}$ has degree 49 and is rational nearly-free with just one singular point which has 4 branches.

The curve C_{49} is given by

$$f_{49} = ((f_1^s y + \sum_{i=2}^{s+1} a_i f_1^{s+1-i} x^{ia-a+1})^a - f_1^{as+1})/x^{a-1} = 0,$$

where $f_1 = x^{4-1}z + y^4$, $a = 4$, $s = 3$, $a_2 = \ldots = a_s \in \mathbb{C}$ and $a_{s+1} \in \mathbb{C} \setminus \{0\}$. We can take for instance $a_2 = \ldots = a_s = 0 \in \mathbb{C}$ and $a_{s+1} = 1$. In this case, $d = a^2 s + 1 = 49$, and the multiplicity sequence of (C_{49}, P) of the singular point $P := [0, 0, 1]$ is $[36, 12_7, 4_6]$. It is no-negative with $\tilde{\nu}(C_{49}) = 1$.

If we consider the rational curves C_4 defined by $f_1 = 0$ (resp. C_{13} defined by $f_{13} : \ (f_1)^3 y + x^{13} = 0$) then $i_P(C_{49}, C_4) = 4 \cdot 49$ (resp. $i_P(C_{49}, C_{13}) = 13 \cdot 49$). Thus the curve $C_{13} C_4^{s(a-1)}$ belongs to the pencil $\Lambda_{C_{49}}$ if $s(a-1) = 9$.

If we take the curve $C_{49,gen}$ defined by $f_{49,gen} := f_{49} + 13 f_{13} f_4^9 = 0$. This curve is irreducible, rational and $\text{Sing}(C_{49,gen}) = \{P\}$ and the number of branches of $C_{49,gen}$ at P is 4.

It is a nearly free divisor, using the computations with `Singular` [9]. A minimal resolution (2.3) for $f_{49,gen}$ is determined by three syzygies of degrees $d_1 = 24$ and $d_2 = d_3 = 25$. Therefore, $\text{mdr}(f_{49,gen}) = 24$. The computations yield a relation between these syzygies of multidegree $(2, 1, 1)$. Then $C_{49,gen}$ is a rational nearly free curve. Let us note that a direct computation using `Singular` [9] of the Tjurina number of the singular point of the curve fails, but the *nearly-free* condition makes the computation possible via Theorem 2.7(2): $\tau(C_{49,gen}) = (49-1)(49-24-1)+24^2-1 = 1727$ which is the result in `Singular` using characteristic $p = 1666666649$.

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