ON THE CANCELATION METHOD FOR THE APPROXIMATE CONTROLLABILITY OF SOME NONLINEAR DIFFUSION PROCESSES

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1 Introduction.

The main goal of this communication is to present some of the results of the work Díaz-Henry-Ramos [1994] related to the L^p-approximate controllabiliy of the Dirichlet semilinear problem

$$(\mathcal{P}_D)\begin{cases} y_t - \Delta y + f(y) = v & \text{in } Q = \Omega \times (0,T), \\ y = 0 & \text{on } \Sigma = \partial\Omega \times (0,T), \\ y(0) = y_0 & \text{on } \Omega, \end{cases}$$

and the nonlinear Neumann type problem

$$(\mathcal{P}_N)\begin{cases} y_t - \Delta y = 0 & \text{in } Q, \\ \dfrac{\partial y}{\partial \nu} + f(y) = v & \text{on } \Sigma, \\ y(0) = y_0 & \text{on } \Omega, \end{cases}$$

where in both cases v represents the control. Similar nonlinear problems arise very often in the study of environmental problems.

For problem $(\mathcal{P}_D)$ we show a stronger property than the usual approximate controllability: for suitable desired states we can control the problem by using merely nonnegative controls. In both cases we prove the L^p-approximate controllability for any p such that $1 < p < \infty$.

Our treatement of problems $(\mathcal{P}_D)$ and $(\mathcal{P}_N)$ relies on the same general programme: we first establish the conclusion for the linear associated problem and as a second step, we prove the result for the nonlinear case by means of a *cancelation technique* already introduced in Henry [1978]. This technique consists in modifying the control associated to the linear case by means of a perturbation which cancels the nonlinearity appearing at the equation.

2 Internal nonnegative controls.

In spite of the very large literature on the approximate controllability for nonlinear parabolic problems (see e.g. the list of references of the survey Díaz [1993]) the study of the approximate controllability property under nonnegativeness constraint on the controls seems to be unexplored until the work Díaz [1991] dealing with the parabolic obstacle problem.

We point out that, in constrast with the case of unconstrained control problems (see e.g. Henry [1978] and Díaz-Fursikov [1994]) the constraint on the controls introduces some important difficulties, even if the control v acts on the whole domain Q.

We start by considering the linear case, which we will use in the proof of the nonlinear case. In the rest of this paper we will always assume $1 < p < \infty$ (the limit cases $p = 1$ and $p = \infty$ can be also treated after some modifications: see Díaz-Henry-Ramos [1994]).

Theorem 1 *Let $h \in L^p(Q)$, $Y_0 \in L^p(\Omega)$ and $a \in L^\infty(Q)$. We denote by $Y(\cdot : v)$ the solution of*

$$(\mathcal{LP}_D)\begin{cases} Y_t - \Delta Y + aY = h + v & \text{in } Q \\ Y = 0 & \text{on } \Sigma \\ Y(0) = Y_0 & \text{on } \Omega. \end{cases}$$

Then, if $\mathcal{U}$ is a dense subset of $L^p_+(Q)$, the set $F := \{Y(T : v) :\ v \in \mathcal{U}\}$ is dense in $Y(T : 0) + L^p_+(\Omega)$, where $L^p_+(\Omega) = \{g \in L^p(\Omega) : g \geq 0\ a.e\}$.

Proof. By linearity we can assume $Y_0 \equiv 0$ and $h \equiv 0$. Suppose that there exists $y_d \in L^p_+(\Omega)$ such that $y_d \notin \overline{F}$ (notice that $\overline{F}$ is a closed and convex set). Then, by the Hahn-Banach Theorem (in its geometrical form), we can separate y_d from $\overline{F}$, i.e. there exists $\alpha \in I\!R$ and $g \in L^{p'}(\Omega)$ (with $\frac{1}{p} + \frac{1}{p'} = 1$) such that

$$\int_\Omega y(T : v) g dx < \alpha < \int_\Omega y_d g dx \qquad \text{for all } v \in \mathcal{U}.$$

Besides, if $v \in L^p_+(Q)$ and $\lambda \in I\!R_+$, then by linearity, $y(T, \lambda v) = \lambda y(T, v) \in \overline{F}$ and so

$$(1) \qquad \int_\Omega y(T : v) g dx \leq 0 < \alpha < \int_\Omega y_d g dx \qquad \text{for all } v \in \mathcal{U}.$$

Now, let $q \in \mathcal{C}([0, T] : L^{p'}(\Omega))$ be the solution of the auxiliary backward problem

$$(2) \qquad \begin{cases} -q_t - \Delta q + aq = 0 & \text{in } Q \\ q = 0 & \text{on } \Sigma \\ q(T) = g & \text{on } \Omega. \end{cases}$$

Multiplying (2) by $Y(v)$, with $v \in \mathcal{U}$ arbitrary, we obtain

$$0 \geq \int_\Omega g(x) Y(T, x : v) dx = \int_Q q v dx dt \quad \forall\ v \in \mathcal{U}.$$

Then, $q \leq 0$ in Q. In particular $g \leq 0$, which is a contradiction with (1). ■

Now, we are ready to consider the nonlinear problem $(\mathcal{P}_S)$ under the assumption that f is a nondecreasing continuous real function. We also assume $y_0 \in L^\infty(\Omega)$ (for simplicity).

Theorem 2 *If $\mathcal{U}$ is a dense subset of $L^p_+(Q)$ then $F = \{y(T : v)$ solution of $(\mathcal{P}_D)$; $v \in \mathcal{U}\}$ is dense in $y(T:0) + L^p_+(\Omega)$.* □

Proof. As $y_0 \in L^\infty(\Omega)$, by the maximum principle $y(\cdot : 0) \in L^\infty(Q)$ and $h(\cdot) := -f(y(\cdot : 0)) \in L^\infty(Q)$. Then, applying Theorem 1 with $h = -f(y(\cdot : 0))$, there exists $w_\varepsilon \in L^\infty_+(Q)$ such that

$$\| Y(T : w_\varepsilon) - y_d \|_{L^p(\Omega)} < \varepsilon.$$

Besides, $f(Y(\omega_\varepsilon)) \in L^p(Q)$. Now, given $\delta > 0$, let $\tilde{y}$ be the unique solution of the auxiliary problem

$$(\mathcal{P}^*_D) \begin{cases} \tilde{y}_t - \Delta\tilde{y} + f(\tilde{y} + Y(\omega_\varepsilon)) = f(Y(\omega_\varepsilon)) + \delta & \text{in } Q \\ \tilde{y} = 0 \quad \text{on } \Sigma \\ \tilde{y}(0) = 0 \quad \text{on } \Omega. \end{cases}$$

Then, if we define $y = \tilde{y} + Y(\omega_\varepsilon)$, we easily check that y is solution of $(\mathcal{P}_D)$ with

$$v_\varepsilon = w_\varepsilon + f(Y(\omega_\varepsilon)) - f(y(\cdot : 0)) + \delta \in L^p(Q).$$

Besides, $v_\varepsilon \geq 0$ since f is nondecreasing and $Y(\cdot : \omega_\varepsilon) \geq Y(\cdot : 0) = y(\cdot : 0)$. Using the density of $\mathcal{U}$ and the continuous dependence of the data in problem $(\mathcal{P}^*_D)$, we can choose $v \in \mathcal{U}$ such that $\| v - v_\varepsilon \|_{L^p(Q)} \leq \varepsilon$. Finally applying Hölder and Young inequalities, we conclude (for $\delta > 0$ small enough) that

$$\| \tilde{y}(T) \|_{L^p(\Omega)} \leq C_1\varepsilon$$

and so

$$\| y(T : v) - y_d \|_{L^p(\Omega)} \leq C_2\varepsilon.$$

■

Remark 1. In the above theorem we can replace f by a β maximal monotone graph of $\mathbb{R}^2$. The existece of solution can be found, for instance, in Benilan [1978] and Theorem 2 remains true if we assume $\beta_+(r) < +\infty$ for all $r \in D(\beta)$, where

$$\beta_+(r) := \sup\{b \in \mathbb{R} : \quad b \in \beta(r)\}.$$

This assumption is verified in many cases: i) case of $D(\beta) = \mathbb{R}$ (as for instance β a continuous nondecreasing function or the Heaviside graph; ii) the condition is also satisfied in some cases for which $D(\beta) \neq \mathbb{R}$ as for instance

$$\beta(r) = \begin{cases} \emptyset & \text{if } r < 0 \\ (-\infty, 0] & \text{if } r = 0 \\ 0 & \text{if } r > 0. \end{cases}$$

Remark 2. It is easy to see that Theorem 1 and the decomposition $Y = Y_+ - Y_-$ allows to conclude the L^p-approximate controllability for the unconstrained linear problem. For the unconstrained nonlinear case the L^p-approximate controllability fo llows from obvious modifications of Theorem 2.

3 Neumann type boundary controls.

In this section, we study the problem $(\mathcal{P}_N)$. The similar result to the internal nonnegative controls is in this case an open problem for us. However, we can apply the cancelation technique in order to prove the L^p-approximate controllability.

Theorem 3 *Let $y_0 \in L^\infty(\Omega)$ and $v \in L^p(\Sigma)$. Let f be a nondecreasing continuous real function and denotes by $y(v)$ the unique solution of*

$$(\mathcal{P}_N)\begin{cases} y_t - \Delta y = 0 & \textit{in } Q \\ \dfrac{\partial y}{\partial \nu} + f(y) = v & \textit{on } \Sigma \\ y(0) = y_0 & \textit{on } \Omega. \end{cases}$$

Then, if $\mathcal{U}$ is dense in $L^p(\Sigma)$, the set $F = \{y(T : v);\ v \in \mathcal{U}\}$ is dense in $L^p(\Omega)$.

Idea of the proof: For $y_d \in L^p(\Omega)$ and $\varepsilon > 0$ fix, we use the decomposition $y = \tilde{y}_\varepsilon + Y$ with Y solution of the associated linear problem

$$(\mathcal{LP}_N)\begin{cases} Y_t - \Delta Y = 0 & \text{in } Q \\ \dfrac{\partial Y}{\partial \nu} = -f(y(\cdot : 0)) + v_\varepsilon & \text{on } \Sigma \\ Y(0) = y_0 & \text{on } \Omega, \end{cases}$$

for a suitable v_ε such that $\| y(T : v_\varepsilon) - y_d \|_{L^p(\Omega)} < \varepsilon$ (this holds again by means of the Hahn-Banach Theorem; see Lions [1968]). For $\delta > 0$ let $\tilde{y}$ be the solution of

$$(\mathcal{P}_N^*)\begin{cases} \tilde{y}_t - \Delta \tilde{y} = 0 & \text{in } Q \\ \dfrac{\partial \tilde{y}}{\partial \nu} + f(\tilde{y} + Y(\omega_\varepsilon)) = f(Y(\omega_\varepsilon)) + \delta & \text{on } \Sigma \\ \tilde{y}(0) = 0 & \text{on } \Omega. \end{cases}$$

Then, if $\delta > 0$ is small enough, there exists $C > 0$ such that

$$\| \tilde{y}(T) \|_{L^p(\Omega)} \leq C\varepsilon,$$

and so we have the result by using the triangle inequality.

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References

Benilan,Ph. [1978]: *Operateurs accretifs et semi-groupes dans les espaces L^p ($1 \leq p \leq \infty$).* Functional Analysis and Numerical Analysis. H. Fujita, ed. Japan Society for the Promotion of Science, pp. 15-53.

Díaz,J.I. [1991]: *Sur la contrôlabilité approche des inequations varationnelles et d'autres problèmes paraboliques non linéaires.* C. R. Acad. Sci. Paris, 312, serie I, pp. 519-522.

Díaz,J.I. [1993]: *Approximate controllability for some nonlinear parabolic problems.* To appear in the Proceedings of the 16^{th} IFIP-TC7 Conference on "System Modelling and Optimization", Compiegne (France). Lecture Notes in Control and Information Sciences. Springer-Verlag.

Díaz,J.I.-Fursikov,A.V. [1994]: *A simple proof of the controllability from the interior for nonlinear parabolic problems.* To appear in Applied Math. Letters.

Díaz,J.I.-Henry,J.-Ramos,A.M. [1994]: Article in preparation.

Henry,J. [1978]: *Etude de la contrôlabilité de certaines équations paraboliques.* Thèse d'Etat, Université Paris VI.

Lions,J.L. [1968]: *Contrôle optimal de systemes gouvernés par des equations aux derivées partielles.* Dunod.

Mizohata,S. [1958]: *Unicité du prolongement des solutions pour quelques opérateurs différentiels paraboliques.* Mem. Coll. Sci. Univ. Kyoto, Ser. A31 (3), pp. 219-239.

Saut,J.C.-Sheurer,B. [1987]: *Unique continuation for some evolution equations.* Journal of Differential Equations, 66, pp. 118-139.