Approximate Controllability and Obstruction for Higher Order Parabolic Semilinear Equations.

J.I. Díaz[1] and A.M. Ramos[2]

Departamento de Matemática Aplicada, Facultad de Matemáticas,
Universidad Complutense de Madrid.
Avda. Complutense s/n, 28040, Madrid.

1 Introduction.

Let Ω be a bounded open subset of $I\!R^N$ of class C^{2m}, $T > 0$, ω a nonempty open subset of Ω, f a continuous real function and $k \in I\!N$ such that $0 \leq 2k < m$. The main goal of this communication is the study of the approximate controllability of the Dirichlet problem

$$(1) \qquad \begin{cases} y_t + (-\Delta)^m y + f(\Delta^k y) = h + v\chi_\omega & \text{in } Q := \Omega \times (0,T) \\ \dfrac{\partial^j y}{\partial \nu^j} = 0 \quad , \quad j = 0, 1, \cdots, m-1 & \text{on } \Sigma := \partial\Omega \times (0,T) \\ y(0) = y_0 & \text{in } \Omega, \end{cases}$$

where v is a suitable output control, χ_ω is the characteristic function of ω, ν is the unit outward normal vector, $h \in L^2(Q)$ and $y_0 \in L^2(\Omega)$. Due to the factor χ_ω the controls are supported on the set $\mathcal{O} := \omega \times (0,T)$.

Definition 1 *We say that Problem (1) has the approximate controllability property at time T with state space X and control space Y if the set of solutions of (1) at time T, when v span Y, is dense in X.*

We obtain the following result on approximate controllability.

Theorem 1 *Assume that f satisfies the following conditions: there exist some positive constants c_1 and c_2 such that*

$$(2) \qquad |f(s)| \leq c_1 + c_2|s| \quad \textit{for all } s \in I\!R$$

and

$$(3) \qquad \textit{there exists } f'(s_0) \textit{ for some } s_0 \in I\!R.$$

Then problem (1) has the approximate controllability property at time T with state space $L^2(\Omega)$ and control space $L^2(\mathcal{O})$.

Remark 1 *For the sake of simplicity of the notation we chose $L^2(\mathcal{O})$ as control space but following the proof it's easy to see that if we change the norm in (27) we can also choose $L^\infty(\mathcal{O})$ if $k = 0$ and $L^\infty(0,T; H^{2k}(\Omega) \cap H_0^k(\Omega))$ if $k \geq 1$.*

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Condition (2) is a sublinear hypothesis (for large values of s). Nevertheless, we shall prove that when f is superlinear the approximate controllability property does not hold in general, as explained in Section 6. Therefore, if for instance $f(s) = |s|^{p-1}s$, Theorem 1 gives a positive approximate controllability result for $0 < p \leq 1$ and the results of section 6 a negative approximate controllability answer for $1 < p < \infty$. A similar negative answer for second order parabolic problems was given in Díaz and Ramos [6].

Definition 2 *We say that a function*

$$y \in L^2(0,T; H_0^m(\Omega)) \ \cap \ C([0,T]; L^2(\Omega))$$

is a solution of problem (1) if y satisfies the differential equation in $\mathcal{D}'(Q)$ and $y(0) = y_0$.

Remark 2 *The existence of solutions is also obtained in the proof of Theorem 1 by using the Kakutani's fixed point theorem. The uniqueness can be easily proved if f is nondecreasing or Lipschitz, but that is not necessary in our arguments.*

Remark 3 *Notice that as $2k < m$ then if y is any solution of (1) $\Delta^k u \in L^2(\Omega)$ and so, by (2), $f(\Delta^k y) \in L^2(Q)$. Besides the boundary conditions are satisfied in the sense that $y(t) \in H_0^m(\Omega)$ for a.e. $t \in (0,T)$.*

2 Preliminaries.

We consider the spaces

$$V := L^2(0,T; H_0^m(\Omega)) \quad \text{and its dual} \quad V' = L^2(0,T; H^{-m}(\Omega))$$

and denote by $< \cdot, \cdot >$ the duality product between $H^{-m}(\Omega)$ and $H^m(\Omega)$ and by $(\cdot,\cdot)$ the scalar product in $L^2(\Omega)$. The norm of V is defined by

$$\|y\|_V^2 = \sum_{j=0}^{m} \int_Q |D^j y|^2 \, dx \, dt$$

where

$$|D^j y|^2 := \sum_{|\alpha|=j} (D^\alpha y)^2 \tag{4}$$

(the sum extending to all x-derivatives of order j). By Poincaré's inequality we have that

$$\|y\|_V^2 \leq C \int_Q |D^m y|^2 \, dx \, dt. \tag{5}$$

We summarize some well-known properties of these spaces in the following two lemmas. We refer to Lions [9] or Lions and Magenes [12] for Lemma 1, and to [9] or Simon [15] for Lemma 2.

Lemma 1 *The space $\{y \in V : \; y_t \in V'\}$ is continuously imbedded in $C([0,T];L^2(\Omega))$. If $y, z \in V$ and $y_t, z_t \in V'$ then*

$$\int_0^T < y_t + (-\Delta)^m y, z > \; dt - \int_0^T < -z_t + (-\Delta)^m z, y > \; dt = (y(T), z(T)) - (y(0), z(0)) \tag{6}$$

and

$$\int_0^T < y_t + (-\Delta)^m y, y > \; dt = \int_Q |D^m y|^2 \, dx \, dt + \frac{1}{2}\int_\Omega y(T,x)^2 \, dx - \frac{1}{2}\int_\Omega y(0,x)^2 \, dx. \tag{7}$$

Lemma 2 *The space $\{y \in V : \; y_t \in V'\}$ is compactly imbedded in $L^2(Q)$.*

Lemma 3 *If $0 \le 2k < m$, the space*

$$W = \{y \in L^2(0,T; H_0^{m+2k}(\Omega)); \; y_t \in L^2(0,T; H^{-m+2k}(\Omega))\}$$

is continuously imbedded in $\mathcal{C}([0,T]; H^{2k}(\Omega))$. Besides, if $y, \; z \in W$ then

$$\int_0^T < y_t + (-\Delta)^m y, (-\Delta)^k z > \; dt - \int_0^T < -z_t + (-\Delta)^m z, (-\Delta)^k y > \; dt = (y(T), (-\Delta)^k z(T)) - (y(0), (-\Delta)^k z(0)) \tag{8}$$

Proof. To see that W is continuously imbedded in $\mathcal{C}([0,T], H^{2k}(\Omega))$ is as in the previous lemma. The equality can be proved by taking $z \in C_c^\infty(\Omega)$ and by using that $C_c^\infty(\Omega)$ is dense in $H_0^{m+2k}(\Omega)$.

We proceed to study the problem

$$\begin{cases} y_t + (-\Delta)^m y + a(t,x)\Delta^k y = h & \text{in } Q \\ \dfrac{\partial^j y}{\partial \nu^j} = 0 \quad , \quad j = 0, 1, \cdots, m-1 & \text{on } \Sigma \\ y(0) = y_0 & \text{in } \Omega. \end{cases} \tag{9}$$

Besides of $h \in L^2(Q)$ and $y_0 \in L^2(\Omega)$ we assume that

$$a \in L^\infty(Q) \quad \text{and} \quad \|a\|_{L^\infty(Q)} \le M. \tag{10}$$

The following Proposition collects some basic results about problem (9).

Proposition 1 *There exists a unique function $y \in V \cap C([0,T]; L^2(\Omega))$ with $y_t \in V'$ which solves Problem (9) and satisfies the estimate*

$$\|y\|_V + \|y_t\|_{V'} \le C\left(\|h\|_{L^2(Q)} + \|y_0\|_{L^2(\Omega)}\right). \tag{11}$$

where the constant C depends only on M (provided that Ω, T and m are kept fixed). Besides, the solution y also satisfies that

$$y \in L^2(\delta, T; H^{2m}(\Omega)) \quad \text{and} \quad y_t \in L^2((\delta,T)\times\Omega) \quad \text{for all } \delta \in (0,T). \tag{12}$$

3 A functional associated to a backward problem

Following Lions [11] and Fabre, Puel and Zuazua [7] [8] we consider

$$\varepsilon > 0 \ , \ y_d \in L^2(\Omega) \ , \ a \in L^\infty(Q) \tag{13}$$

and introduce the functional $J = J(\cdot; a, y_d) : L^2(\Omega) \to I\!R$ defined by

$$J(\varphi^0) = \frac{1}{2}\left(\int_{\mathcal{O}} |\varphi(t,x)| dx dt\right)^2 + \varepsilon|\varphi^0|_{L^2(\Omega)} - \int_\Omega y_d \, \varphi^0 \, dx \tag{14}$$

where $\varphi(t,x)$ is the solution of the backward problem

$$\begin{cases} -\varphi_t + (-\Delta)^m \varphi + a(t,x)\Delta^k \varphi = 0 & \text{in } Q := \Omega \times (0,T) \\ \dfrac{\partial^j \varphi}{\partial \nu^j} = 0 \quad , \quad j = 0, 1, \cdots, m-1 & \text{on } \Sigma := \partial\Omega \times (0,T) \\ \varphi(T) = r(\varphi^0) & \text{in } \Omega \end{cases} \tag{15}$$

with $r(\varphi^0)$ given by $r(\varphi^0) = \varphi^0$ if $k = 0$ and by the solution of

$$\begin{cases} (-\Delta)^k r = \varphi^0 & \text{in } \Omega \\ \dfrac{\partial^j r}{\partial \nu^j} = 0 \quad j = 0, ..., k-1 & \text{on } \partial\Omega \end{cases}$$

if $k \geq 1$. We point out that $r \in H^{2k}(\Omega) \cap H_0^k(\Omega)$ and $\varphi \in W$.

As usual in controllability theory we shall need to use a property of *unique continuation* for solutions of a linear problem (in our case Problem (15)).

Lemma 4 *Let ω be a nonempty open subset of Ω. Assume that*

$$\varphi \in L^2(0,T; H_0^m(\Omega)) \ \cap \ C([0,T]; L^2(\Omega))$$

is a solution of Equation (15) in $\mathcal{D}'(Q)$ and that $\varphi \equiv 0$ in $\mathcal{O} = \omega \times (0,T)$. Then $\varphi \equiv 0$ in Q.

Proof. ¿From Proposition 1 (applied with the time inversed) we deduce that $\varphi \in L^2(0, T - \delta; H^{2m}(\Omega))$ for all $\delta \in (0,T)$. Then Lemma 4 follows from Theorem 3.2 of Saut and Scheurer [14].

The following two results are easy adaptation of the similar ones given in [7], [8] for second order parabolic problems.

Proposition 2 *Under the assumption (13) the functional $J(\cdot; a, y_d)$ is continuous and strictly convex on $L^2(\Omega)$ and verifies*

$$\liminf_{|\varphi^0|_2 \to \infty} \frac{J(\varphi^0; a, y_d)}{|\varphi^0|_2} \geq \varepsilon. \tag{16}$$

Besides $J(\cdot; a, y_d)$ attains its minimum at a unique point $\widehat{\varphi}^0$ in $L^2(\Omega)$ and

$$\widehat{\varphi}^0 = 0 \quad \Leftrightarrow \quad |y_d|_2 \leq \varepsilon. \tag{17}$$

Proposition 3 *Let M be the mapping*

$$\begin{array}{rccc} M: & L^\infty(Q)\times L^2(\Omega) & \rightarrow & L^2(\Omega) \\ & (a(t,x),y_d) & \longrightarrow & \widehat{\varphi}^0. \end{array}$$

If B is a bounded subset of $L^\infty(Q)$ and K is a compact subset of $L^2(\Omega)$, then $M(B\times K)$ is a bounded subset of $L^2(\Omega)$.

Definition 3 *Given $V: X\to I\!R\cup\{+\infty\}$ a convex and prope function on the Banach space X, it is said that a element p_0 of V' belongs to the set $\partial V(x_0)$ (subdifferential of V at $x_0\in X$) if*

$$V(x_0)-V(x)\le (p_0,x_0-x) \quad \forall\ x\in X.$$

Remark 4 *In the conditions of Definition 3, x_0 minimizes V over X (or over a convex subset of X) if and only if*

$$0\in\partial V(x_0).$$

Proposition 4 *Under the above conditions, if V is a lower semicontinuous function, then $p_0\in\partial V(x_0)$ if and only if*

$$(p_0,x)\le \lim_{h\to 0^+}\frac{V(x_0+hx)-V(x_0)}{h}(<+\infty) \quad \forall\ x\in X.$$

For a proof see, for instance, Proposition 3 of page 187 and Theorem 16 of page 198 of Aubin-Ekeland [3].

Remark 5 If V is differentiable its differential coincides with its subdifferential.

4 Approximate Controllability for the linear associated problem.

Lemma 5 *For every $\varphi^0\in L^2(\Omega)$, $\varphi^0\neq 0$ if φ is the solution of (15) verifying $\varphi(T)=r(\varphi^0)$, we have that*

$$\partial J(\varphi^0;a,y_d)=\{\xi\in L^2(\Omega),\ \exists\ v\in sgn(\varphi)\chi_{\mathcal{O}} \ satisfying$$

$$\begin{aligned}\int_\Omega \xi(x)\theta^0(x)dx &= \left(\int_{\mathcal{O}}|\varphi(t,x)|d\Sigma\right)\left(\int_{\mathcal{O}}v(t,x)\theta(t,x)d\Sigma\right)\\ &\quad +\varepsilon\int_\Omega\frac{\varphi^0(x)}{|\ \varphi^0\ |_2}\theta^0(x)dx-\int_\Omega y_d(x)\theta^0(x)dx\ \forall\theta^0\in L^2(\Omega)\},\end{aligned}$$

where θ is the solution of (15) verifying $\theta(T)=r(\theta^0)$.

Proof. It is an easy modification of Proposition 2.4 of [8].

Before continue we need to introduce the control u_a given by $u_a=|\widehat{\varphi}|_{L^1(\mathcal{O})}v$ ($v\in sgn(\widehat{\varphi})\chi_{\mathcal{O}}$) if $k=0$ and by means of the solution of

$$\left\{\begin{array}{ll} (-\Delta_x)^k u_a(t_0,\cdot)=|\widehat{\varphi}|_{L^1(\mathcal{O})}v(t_0,\cdot)\chi_{\mathcal{O}} & \text{in } \mathcal{O}\cap\{t=t_0\} \\ \dfrac{\partial^j u_a}{\partial\nu^j}=0 \quad j=0,...,k-1 & \text{on } \partial[\mathcal{O}\cap\{t=t_0\}] \end{array}\right. \quad a.e\ \ t_0\in[0,T]$$

if $k \geq 1$. Here we point out that (since $\| v \|_{L^\infty(Q)} \leq 1$)

(18) $$u_a \in L^\infty(Q) \ \text{ and } \ \| u_a \|_{L^\infty(Q)} \leq \| \widehat{\varphi} \|_{L^1(\mathcal{O})} \quad \text{if } k = 0$$

and

(19) $$u_a \in L^\infty(0,T; H^{2k}(\Omega) \cap H_0^k(\Omega)), \ \| u_a \|_{L^\infty(0,T;H^{2k}(\Omega)\cap H_0^k(\Omega))} \leq C \| \widehat{\varphi} \|_{L^1(\mathcal{O})} \ \text{ if } k \geq 1.$$

Now we are ready to prove a linear version of Theorem 1.

Theorem 2 *If $|y_d|_2 > \varepsilon$ and $\widehat{\varphi}$ is the solution of (15) verifying $\widehat{\varphi}(T) = \widehat{\varphi}^0$, then there exists $v \in sgn(\widehat{\varphi})\chi_{\mathcal{O}}$ such that the solution of*

(20) $$\begin{cases} y_t + (-\Delta)^m y + a(x,t)\Delta^k y = h + u_a \chi_{\mathcal{O}} & \text{in } Q \\ \dfrac{\partial^j y}{\partial \nu^j} = 0 \quad (j = 0 \cdots (m-1)) & \text{on } \Sigma \\ y(0) = y_0 & \text{on } \Omega \end{cases}$$

verifies

$$y(T) = y_d - \varepsilon \frac{\widehat{\varphi}^0}{| \widehat{\varphi}^0 |_2},$$

and then $| y(T) - y_d |_2 = \varepsilon$.

Remark 6 If $y_0 \equiv 0$, and $h \equiv 0$, the case $|y_d| \leq \varepsilon$ is trivially solved with the control $u_a \equiv 0$.

Proof of Theorem 2. By linearity we can assume $y_0 \equiv 0$ and $h \equiv 0$, since in other case we can take $y(T:0)$ the solution of the problem with null control and after we can take the new desired state $y_d' = y_d - y(T:0) \in L^2(\Omega)$ for the problem with $y_0 \equiv 0$ and $h \equiv 0$. Now, by using the subdifferentiability of $J(.; a, y_d)$ at $\widehat{\varphi}^0$ ($\neq 0$ by (17)), we know (see Remark 4) that

$$0 \in \partial J(\widehat{\varphi}^0),$$

which is equivalent, from Lemma 5, to the existence of $v \in sgn(\widehat{\varphi})\chi_{\mathcal{O}}$, such that

(21) $$- | \widehat{\varphi} |_{L^1(\mathcal{O})} \left(\int_{\mathcal{O}} v(x,t)\theta(x,t)dxdt \right) = \frac{\varepsilon}{| \widehat{\varphi}^0 |_2} \int_\Omega \widehat{\varphi}^0(x)\theta^0(x)dx$$
$$- \int_\Omega y_d(x)\theta^0(x)dx.$$

On the other hand, as $y \in W$, if we "multiply" by $(-\Delta)^k\theta$ in (20) we obtain by (8) and (15) that

(22) $$(y(T), \theta^0)_{L^2(\Omega)\times L^2(\Omega)} = |\widehat{\varphi}|_{L^1(\mathcal{O})} \left(\int_{\mathcal{O}} v(x,t)\theta(x,t)dxdt \right)$$

(Here we point out that, in order to be able to integrate by parts, we are taking into account that $0 \leq 2k < m$). Then, from (21) and (22), we obtain

$$(y(T), \theta^0)_{L^2(\Omega)\times L^2(\Omega)} = (y_d - \varepsilon \frac{\widehat{\varphi}^0}{|\widehat{\varphi}^0|_2}, \theta^0)_{L^2(\Omega)\times L^2(\Omega)} \quad \forall \ \theta^0 \in L^2(\Omega)$$

and we conclude that $y(T) = y_d - \varepsilon \dfrac{\widehat{\varphi}^0}{|\widehat{\varphi}^0|_2}$.

5 Controllability for the nonlinear problem.

For the nonlinear case we shall need to use a fixed point Theorem for multivalued operators:

Definition 4 *Let X, Y two Banach spaces and, $\Lambda : X \to \mathcal{P}(Y)$ a multivalued function. We say that Λ is* <u>upper hemicontinuous</u> *at $x_0 \in X$, if for every $p \in Y'$, the function*

$$x \to \sigma(\Lambda(x), p) = \sup_{y \in \Lambda(x)} < p, y >_{Y' \times Y}$$

is upper semicontinuous at x_0. We say that the multivalued funcion is upper hemicontinuous on a subset K of X, if it satisfies this properties for every point of K.

Theorem 3 (Kakutani's fixed point Theorem). *Let $K \subset X$ be a convex and compact subset and $\Lambda : K \to K$ an upper hemicontinuous application with convex, closed and nonempty values. Then, there exists a fixed point x_0, of Λ.*

For a proof see, for instance, Aubin [2] page 126.

Proof of Theorem 1. We fix $y_d \in L^2(\Omega)$, $\varepsilon > 0$ and we define

$$g(s) = \frac{f(s) - f(s_0)}{s - s_0}.$$

Then, from the assumptions, we have that $g \in L^\infty(I\!R) \cap \mathcal{C}(I\!R)$.

Now, by using Theorem 2, for each $z \in L^2(0, T; H_0^{2k}(\Omega))$ and $\varepsilon > 0$ it is possible to find two functions $\varphi(z) \in L^1(Q)$ and $v(z) \in sgn(\varphi(z))\chi_{\mathcal{O}}$ such that the solution $y = y^z$ of

$$(23) \qquad \begin{cases} y_t + (-\Delta)^m y + g(\Delta^k z)\Delta^k y = h - f(s_0) + g(\Delta^k z)s_0 + u\chi_{\mathcal{O}} & \text{in } Q \\ \dfrac{\partial^j y}{\partial \nu^j} = 0 \ , \ j = 0, 1, \cdots m-1 & \text{on } \Sigma \\ y(0) = y_0 & \text{on } \Omega, \end{cases}$$

(where $u = u_{g(\Delta^k z)}$) satisfies

$$(24) \qquad |y(T) - y_d|_{L^2(\Omega)} \leq \varepsilon.$$

Besides

$$(25) \qquad \{\| \varphi(z) \|_{L^1(\mathcal{O})} v(z), \ z \in L^2(0, T; H_0^{2k}(\Omega))\} \quad \text{is bounded in } L^\infty(Q)$$

since, following the proof of Theorem 2, $\varphi(z)$ is the solution of (15) with initial value $M(\ (g(\Delta^k z), y_d^z) \ )$ (see Proposition 3) and potential $g(\Delta^k z)$, where $y_d^z = y_d - y^z(T : 0)$, with $y^z(T : 0)$ the solution of (23) at time T for the control $u = 0$. Therefore, by applying Lemma 6, we obtain that y_d^z belongs to a compact set for all $z \in L^2(0, T; H_0^{2k}(\Omega))$ and so, by using Proposition 3 and Proposition 1, we obtain (25).

Lemma 6 *The set*

$$\{y_d^z, \ z \in L^2(0, T; H_0^{2k}(\Omega))\},$$

with y_d^z defined above is relatively compact in $L^2(\Omega)$.

Proof of Lemma 6. We can split the set of solutions $y^z(\cdot:0)$ of

$$\begin{cases} y_t + (-\Delta)^m y + g(\Delta^k z)\Delta^k y = h - f(s_0) + g(\Delta^k z)s_0 & \text{in } Q \\ \dfrac{\partial^j y}{\partial \nu^j} = 0 \ , \ j = 0, 1, \cdots m-1 & \text{on } \Sigma \\ y(0) = y_0 & \text{on } \Omega, \end{cases}$$

by $y^z(\cdot:0) = u + v$, where u is the solution of

$$\begin{cases} u_t + (-\Delta)^m u = h - f(s_0) & \text{in } Q \\ \dfrac{\partial^j u}{\partial \nu^j} = 0 \ , \ j = 0, 1, \cdots m-1 & \text{on } \Sigma \\ u(0) = y_0 & \text{on } \Omega \end{cases}$$

and v is the solution of

$$\begin{cases} v_t + (-\Delta)^m v + g(\Delta^k z)(\Delta^k u + \Delta^k v) = g(\Delta^k z)s_0 & \text{in } Q \\ \dfrac{\partial^j v}{\partial \nu^j} = 0 \ , \ j = 0, 1, \cdots m-1 & \text{on } \Sigma \\ v(0) = 0 & \text{on } \Omega. \end{cases}$$

Then, by applying Proposition 1 and the results of Lions-Magenes [13] (see page 78), we obtain that there exists $K > 0$ independent of z such that

$$\| v \|_{H^{1,2m}(Q)} \leq K(1 + \| y_0 \|_{L^2(\Omega)} + \| h \|_{L^2(Q)}).$$

Finally, we take into account that $H^{1,2m}(Q)$ is compactly imbedded in $\mathcal{C}([0,T]; L^2(\Omega))$ and we conclude the result.

End of the proof of Theorem 1. Thus

$$K_1 = \sup_{z \in L^2(0,T; H_0^{2k}(\Omega))} \| \varphi(z) \|_{L^1(\mathcal{O})} < \infty. \tag{26}$$

Obviously, as we had seen in (18) and (19) $u = u_{g(\Delta^k z)}$ satisfies

$$\| u \|_{L^2(Q)} \leq K_2. \tag{27}$$

Therefore, if we define the operator

$$\Lambda : L^2(0,T; H_0^{2k}(\Omega)) \to \mathcal{P}(L^2(0,T; H_0^{2k}(\Omega)))$$

by

$$\Lambda(z) = \{ y \text{ satisfies (23), (24) for some } u \text{ satisfying (27) } \},$$

we have seen that for each $z \in L^2(0,T; H_0^{2k}(\Omega))$, $\Lambda(z) \neq \emptyset$. In order to apply Kakutani's fixed point theorem, we have to chek that the next properties hold:

(i) There exists a compact subset U of $L^2(0,T; H_0^{2k}(\Omega))$, such that for every $z \in L^2(0,T; H_0^{2k}(\Omega))$, $\Lambda(z) \subset U$.

(ii) For every $z \in L^2(0,T;H_0^{2k}(\Omega))$, $\Lambda(z)$ is a convex, compact and nonempty subset of $L^2(0,T;H_0^{2k}(\Omega))$.

(iii) Λ is upper hemicontinuous.

The proof of these properties is as follows:

(i) From Proposition 1 we know that, there exists a bounded subset U of $\{y \in V \ : \ y_y \in V'\}$ such that for every $z \in L^2(0,T;H_0^{2k}(\Omega))$, $\Lambda(z) \subset U$. Now, to see that we can choose U compact we shall prove that the set

$$\mathcal{Y} = \{y \text{ satisfying (23) for some } z \in L^2(0,T;H_0^{2k}(\Omega)) \text{ and } u \text{ verifying (27)}\}$$

is a relatively compact subset of $L^2(0,T;H_0^{2k}(\Omega))$. But this is easy to prove by using that

$$\{y \in V \ : \ y_t \in V'\} \subset L^2(0,T;H_0^{2k}(\Omega)) \text{ with compact imbedding} \tag{28}$$

(see Aubin [1]).

(ii) We have already seen that for every $z \in L^2(0,T;H_0^{2k}(\Omega))$, $\Lambda(z)$ is a nonempty subset of $L^2(0,T;H_0^{2k}(\Omega))$. Besides $\Lambda(z)$ is obviously convex, because $B(y_d,\varepsilon)$ and $\{u \in L^2(Q) :$ satisfying (27)$\}$ are convex sets. Then, we have to see that $\Lambda(z)$ is a compact subset of $L^2(0,T;H_0^{2k}(\Omega))$. In (i) we have proved that $\Lambda(z) \subset U$ with U compact. Let $(y^n)_n$ be a sequence of elements of $\Lambda(z)$ which converges on $L^2(0,T;H_0^{2k}(\Omega))$ to $y \in U$. We have to prove that $y \in \Lambda(z)$. We know that there exist $u^n \in L^2(Q)$ satisfying (27) such that

$$\left\{\begin{array}{ll} y_t^n + (-\Delta)^m y^n + g(\Delta^k z)\Delta^k y^n = h - f(s_0) + g(\Delta^k z)s_0 + u^n\chi_{\mathcal{O}} & \text{in } Q \\ \dfrac{\partial^j y^n}{\partial \nu^j} = 0 \ , \ j = 0,1,\cdots,m-1 & \text{on } \Sigma \\ y^n(0) = y_0 & \text{on } \Omega \\ |y^n(T) - y_d|_2 \le \varepsilon. & \end{array}\right. \tag{29}$$

Now, by using that the controls u^n are uniformly bounded, we deduce that $u^n \to u$ in the weak topology of $L^2(Q)$ and u satisfies (27). Therefore, if we pass to the limit in (29) we obtain

$$\left\{\begin{array}{ll} y_t + (-\Delta)^m y + g(\Delta^k z)\Delta^k y = h - f(s_0) + g(\Delta^k z)s_0 + u\chi_{\mathcal{O}} & \text{in } Q \\ \dfrac{\partial^j y}{\partial \nu^j} = 0 \ , \ j = 0,1,\cdots,m-1 & \text{on } \Sigma \\ y(0) = y_0 & \text{on } \Omega. \end{array}\right.$$

Besides, $v^n = y - y^n$ is solution of

$$\left\{\begin{array}{ll} v_t^n + (-\Delta)^m v^n + g(\Delta^k z)\Delta^k v^n = (u - u^n)\chi_{\mathcal{O}} & \text{in } Q \\ \dfrac{\partial^j v^n}{\partial \nu^j} = 0 \ , \ j = 0,1,...,m-1 & \text{on } \Sigma \\ v^n(0) = 0 & \text{on } \Omega \end{array}\right.$$

and satisfies $v^n \in H^{1,2m}(Q)$ (see [13]). Therefore, v^n is a strong solution and if we "multiply" by v^n and integrate, we obtain that

$$\| v^n(T) \|_{L^2(\Omega)}^2 \le k \int_Q (u - u^n)\chi_{\mathcal{O}} v^n dxdt \to 0 \quad \text{as } n \to \infty.$$

Thus $y^n(T)$ converges to $y(T)$ in the topology of $L^2(\Omega)$ and $|y(T)-y_d|_2 \leq \varepsilon$. This prove that $y \in \Lambda(z)$ and concludes the proof of (ii).

(iii) We must prove that for every $z_0 \in L^2(0,T;H_0^{2k}(\Omega))$

$$\limsup_{z_n \overset{L^2(0,T;H_0^{2k}(\Omega))}{\longrightarrow} z_0} \sigma(\Lambda(z_n),k) \leq \sigma(\Lambda(z_0),k), \ \forall\ k \in L^2(0,T;H^{-2k}(\Omega)).$$

We have seen in (ii) that $\Lambda(z)$ is a compact set, which implies that for every $n \in I\!N$ there exists $y^n \in \Lambda(z_n)$ such that

$$\sigma(\Lambda(z_n),k) =< k(x,t), y^n(x,t) >_{L^2(0,T;H^{-2k}(\Omega))\times L^2(0,T;H_0^{2k}(\Omega))} .$$

Now, by (i), $(y^n)_n \subset U$ (compact set). Then, there exists $y \in L^2(0,T;H_0^{2k}(\Omega))$ such that (after extracting a subsequence) $y^n \to y$ on $L^2(0,T;H_0^{2k}(\Omega))$. We shall prove that $y \in \Lambda(z_0)$. We know that there exist $u^n \in L^2(Q)$ satisfying (27) such that

$$(30) \qquad \begin{cases} y_t^n + (-\Delta)^m y^n + g(\Delta^k z_n)\Delta^k y^n = h - f(s_0) + g(\Delta^k z_n)s_0 + u^n\chi_{\mathcal{O}} & \text{in } Q \\ \dfrac{\partial^j y^n}{\partial \nu^j} = 0 \ , \ j = 0,1,\cdots,m-1 & \text{on } \Sigma \\ y^n(0) = y_0 & \text{on } \Omega \\ |y^n(T) - y_d|_2 \leq \varepsilon. \end{cases}$$

Then there exists $u \in L^2(Q)$ satisfying (27) such that $u^n \to u$ in the weak topology of $L^2(\mathcal{O})$. On the other hand, by using the smoothing effect of the parabolic linear equation (in a similar way to the proof of (ii)) and that $g \in L^\infty(I\!R) \cap \mathcal{C}(I\!R)$, we deduce that y satisfies (23) and (24) with $z = z_0$ for some $u \in L^2(Q)$ satisfying (27), which implies that $y \in \Lambda(z_0)$. Then, for every $k \in L^2(0,T;H^{-2k}(\Omega))$,

$$\begin{aligned} \sigma(\Lambda(z_n),k) &=< k(x,t), y^n(x,t) >_{L^2(0,T;H^{-2k}(\Omega))\times L^2(0,T;H_0^{2k}(\Omega))} \\ &\to< k(x,t), y(x,t) >_{L^2(0,T;H^{-2k}(\Omega))\times L^2(0,T;H_0^{2k}(\Omega))} \\ &\leq \sup_{\overline{y}\in\Lambda(z_0)} < k(x,t), \overline{y}(x,t) >_{L^2(0,T;H^{-2k}(\Omega))\times L^2(0,T;H_0^{2k}(\Omega))} = \sigma(\Lambda(z_0),k), \end{aligned}$$

which proves that Λ is upper hemicontinuous and conclude the proof of (iii).

Finally, if we restrict Λ to $K = conv(U)$ (the convex enveloppe of U), which is a compact set in $L^2(0,T;H_0^{2k}(\Omega))$, it satisfies the assumptions of Kakutani's fixed point theorem. Then, Λ has a fixed point $y \in K$. Besides, by construction, there exists a control $u \in L^2(Q)$ satisfying (27) such that

$$(31) \qquad \begin{cases} y_t + (-\Delta)^m y + f(\Delta^k y) = h + u\chi_{\mathcal{O}} & \text{in } Q \\ \dfrac{\partial^j y}{\partial \nu^j} = 0 \ , \ j = 0,1,\cdots m-1 & \text{on } \Sigma \\ y(0) = y_0 & \text{on } \Omega \\ |y(T) - y_d|_2 \leq \varepsilon. \end{cases}$$

Therefore, y is the solution that we were looking for.

6 Non-controllability for superlinear problems.

In this section we assume $k = 0$. We shall prove a result of non-contrallability for a superlinear case with $\overline{\omega} \subset \Omega$.

Theorem 4 *If $p > 1$ and $y_0 \in L^2(\Omega)$ the problem*

$$\begin{cases} y_t + (-\Delta)^m y + |y|^{p-1}y = u\chi_\omega & \text{in } Q \\ y(0) = y_0 & \text{on } \Omega \end{cases}$$

with controls $u \in L^2(Q)$ (or more general with $u \in L^{r'}(Q)$ where $r = p + 1 > 2$ and so $r' \in (1, 2)$) and any boundary condition does not satisfy, in general, the approximate controllability property at time T.

In order to prove this theorem we need some previous results.

Young's inequality. If $a,\ B \geq 0$, $\varepsilon > 0$ and $q > 1$ then

$$AB \leq \varepsilon A^q + K(\varepsilon, q)B^{q'} \ \text{ with } \ \frac{1}{K(\varepsilon, q)} = q'(q\varepsilon)^{q'/q}. \tag{32}$$

Notation. If we take $R > 0$ we can define in $I\!R^N$ the functions

$$\xi_R(x) = (R^2 - |x|^2)/R \ \text{ if } \ |x| < R, \ \ \xi_R(x) = 0 \ \text{ if } |x| \geq R$$

and the powers ξ_R^s of the function ξ_R, where $s > 1$ is a real number. We can also define

$$d_R(x) = R - |x| \ \text{ if } \ |x| < R, \ \ d_R(x) = 0 \ \text{ if } |x| \geq R \tag{33}$$

and then, the following relation holds for all $x \in I\!R^N$.

$$d_R(x) \leq \xi_R(x) \leq 2d_R(x). \tag{34}$$

The following result was proved in Bernis [4].

Proposition 5 *Let $s \geq 2m$ and $R > 0$. Then, for each $\varepsilon > 0$ there exist a constant C depending only on N, m, s and ε (thus independent of R) such that the following inequality holds for all $y \in H^m_{loc}(I\!R^n)$:*

$$((-\Delta)^m y, \xi_R^s y)_{H^{-m}_{loc}(I\!R^N)\times H^m_c(I\!R^N)} \geq (1 - \varepsilon)\int_{I\!R^N} \xi_R^s |D^m y|^2 dx - C\int_{I\!R^N} \xi_R^{s-2m} y^2 dx.$$

Remark 7 *Since $s \geq 2m$, $\xi_R^s \in W_c^{2m,\infty}(I\!R^N)$. Hence $\xi_R^s \in \mathcal{C}_c^m(I\!R^N)$ (see e.g. Corollary IX.13 of [5]) and $\xi_R^s u \in H_c^m(I\!R^N)$ (see e.g. Note 4 of Chapter IX of [5]).*

Corollary 1 *Let $s \geq 2m$ and $R > 0$ such that $\overline{B_R} \subset \Omega$. Then, for each $\varepsilon > 0$ there exist a constant C depending only on N, m, s and ε (thus independent of R) such that the following inequality holds for all $y \in H^m(\Omega)$:*

$$((-\Delta)^m y, \xi_R^s y)_{H^{-m}(\Omega)\times H_0^m(\Omega)} \geq (1 - \varepsilon)\int_\Omega \xi_R^s |D^m y|^2 dx - C\int_\Omega \xi_R^{s-2m} y^2 dx.$$

Proof. We take $\overline{y} \in H^m(\Omega)$ such that $\overline{y} = y$ in Ω (we can see that this $\overline{y}$ exists in Theorem IX of Brezis [5]). Then we have the inequality for $\overline{y}$, but as $\overline{B_R} \subset \Omega$ we obtain the result.

Theorem 5 *Let $p > 1$, $r = p+1$, $y_0 \in L^2(\Omega)$ and $u \in L^{r'}(Q)$. Then any solution $y \in L^r(Q) \cap L^2(0,T;H^m(\Omega))$ of*

$$\left\{ \begin{array}{ll} y_t + (-\Delta)^m y + |y|^{p-1} y = u & \text{in } \mathcal{D}'(Q) \\ y(0) = y_0 & \text{on } \Omega, \end{array} \right. \tag{35}$$

with any boundary conditions, satisfies the local estimate

$$\sup_{0<t<T} \int_{B_R} y(x,t)^2 dx + \int_{B_R \times (0,T)} (|D^m y|^2 + |y|^r) dxdt$$
$$\leq K \left(1 + \int_{B_{R_1} \times (0,T)} |u|^{r'} dxdt + \int_{B_{R_1}} y_0^2 dx \right)$$

if $\overline{B_{R_1}} \subset \Omega$ and $0 < R \leq R_1$. Besides, the constant K depends only on N, m, p, R, R_1 and T.

Remark 8 *The set of solutions of the problem in Theorem 5 is not the empty set since, for instance with Dirichlet conditions on the boundary, we know that there exists a unique solution (see e.g. Lions [10]).*

Proof of Theorem 5. We take $X_r = L^r(Q) \cap L^2(0,T;H_0^m(\Omega))$. Then the equality of the equation of (35) is in $X_r' = L^{r'}(Q) + L^2(0,T;H^{-m}(\Omega))$. Then, if $s \geq 2m$, we can multiply in (35) by $\xi_R^s y$ with the duality product $(\cdot,\cdot)_{X_r' \times X_r}$ and we obtain

$$\frac{1}{2} \int_{B_R} \xi_R^s y(x,T)^2 dx + ((-\Delta)^m y, \xi_R^s y)_{L^2(0,T;H^{-m}(\Omega)) \times L^2(0,T;H_0^m(\Omega))} + (|y|^{p-1} y, \xi_R^s y)_{L^{r'}(Q) \times L^r(Q)}$$
$$= \frac{1}{2} \int_{B_R} \xi_R^s y_0(x)^2 dx + (u, \xi_R^s y)_{L^{r'}(Q) \times L^r(Q)}.$$

Now, from Corollary 1 it follows that

$$\begin{array}{l} \displaystyle \frac{1}{2} \int_{B_R} \xi_R^s y(x,T)^2 dx + \int_{B_R \times (0,T)} \xi_R^s (|D^m y|^2 + |y|^r) dxdt \\ \displaystyle \leq C \int_{B_R} \xi_R^s y_0(x)^2 dx + C \int_{B_R \times (0,T)} \xi_R^{s-2m} y^2 dxdt + C \int_{B_R \times (0,T)} \xi_R^s uy dxdt. \end{array} \tag{36}$$

By (33) and (34) we can replace in (36) $\xi_R(x)$ by $R - |x|$ (modifying the constants). Besides, writing $s - 2m = 2s/r + (s(r-2)/r) - 2m$, we can apply Hölder's or Young's inequality (32) with exponents $q = r/2$ and $q' = r/r-2$ and we obtain

$$\int_{B_R \times (0,T)} (R - |x|)^{s-2m} y^2 dxdt$$
$$\leq \varepsilon \int_{B_R \times (0,T)} (R - |x|)^s |y|^r dxdt + K(\varepsilon, r/2) \int_{B_R \times (0,T)} (R - |x|)^{s-\gamma} dxdt$$

with

$$\gamma = \frac{2mr}{r-2} \quad .$$

Hence, if we choose $s > \gamma - 1$, the last integral is finite and equal to $\widetilde{C}R^{s+N-\gamma}$. On the other hand, we can apply again (32) and we have

$$\int_{B_R\times(0,T)}(R-|x|)^s uy dxdt \leq \varepsilon \int_{B_R\times(0,T)}(R-|x|)^s|y|^r dxdt + k(\varepsilon,r)\int_{B_R\times(0,T)}(R-|x|)^s|u|^{r'}dxdt.$$

Thus, by changing the constants, we deduce that

$$\frac{1}{2}\int_{B_R}(R-|x|)^s y(x,T)^2 dx + \int_{B_R\times(0,T)}(R-|x|)^s(|D^m y|^2 + |y|^r)dxdt$$

$$\leq C\left(\int_{B_R}(R-|x|)^s y_0(x)^2 dx + R^{s+N-\gamma} + \int_{B_R\times(0,T)}(R-|x|)^s|u|^{r'}dxdt\right).$$

Finally, by replacing R by R_1 and by taking into account that $R_1 - |x| \geq R_1 - R$ and $R_1 - |x| \leq R_1$ if $|x| \leq R$ we deduce the result with

$$K = \max\left\{C(\frac{R_1}{R_1 - R})^s,\ \frac{CR_1^{s+N-\gamma}}{(R_1-R)^s}\right\}..$$

Proof of Theorem 4. The proof of Theorem 4 is a consequence of Theorem 5 since, if R_1 satisfies $\overline{B_{R_1}} \subset \Omega\backslash\omega$, then

$$\| y(u;T) \|^2_{L^2(\Omega)} \leq K(1 + \| y_0 \|^2_{L^2(\Omega)}) \quad \forall u \in L^{r'}(Q)$$

and if we take y_d such that $\| y_d \|_{L^2(\Omega)}$ is large enough we cannot find a satisfactory control.

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