RESULTS ON APPROXIMATE CONTROLLABILITY FOR QUASILINEAR DIFFUSION EQUATIONS

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Introduction

The study of the approximate controllability property for parabolic problems was treated firstly for the linear case in the book of Lions [15]. The study of this property for nonlinear parabolic equations seems to have its origins in the work of Henry [14]. Since then, many other results are today available in the literature (see some references in Díaz [8]) but, to the best of our knowledge, always restricted to the case of semilinear parabolic equations. This paper extends the recent results of the works of Díaz and Ramos [9], [10]. We consider this property for the, so called, *nonlinear diffusion equation*

$$(1)\qquad \begin{cases} y_t - \Delta\varphi(y) = h & \text{in } Q := \Omega \times (0,T), \\ \varphi(y) = 0 & \text{on } \Sigma := \partial\Omega \times (0,T), \\ y(0) = v & \text{in } \Omega, \end{cases}$$

where Ω is a bounded open subset of $I\!R^N$ of class C^4, $T > 0$, φ is a continuous non-decreasing real function, $h \in L^2(0,T : H^{-1}(\Omega))$ is a prescribed datum and v represents the searched output control answering the following approximate controllability property: Fixed $\gamma > 0$, we find v such that $\| y(t;v) - y_d \|_{H^{-(1+\gamma)}(\Omega)} \leq \delta$ for a given $\delta > 0$ and for some *desired state* $y_d \in L^2(\Omega)$. We recall that, with this regularity on the data, $y(v) \in \mathcal{C}([0,T] : H^{-1}(\Omega))$ (see Brezis [4]).
We prove that the approximate controllability holds for a certain class of functions φ which are *essentially linear* at infinity. This class of functions includes the one associated to some type of *two phase Stefan problem* ($\varphi(s) = ks$ for $s < 0$, $\varphi(s) = 0$ in $[0, L]$ and $\varphi(s) = ks$ for $s > L$, for some positive constants k and L). The result is obtained through the application of a variation of the main theorem of Díaz and Ramos [11], adaptated to the vanishing viscosity higher order problem

$$(2)\qquad \begin{cases} y_t + \varepsilon\Delta^2 y - \Delta\varphi(y) = h & \text{in } Q, \\ y = \Delta y = 0 & \text{on } \Sigma, \\ y(0) = v & \text{in } \Omega \end{cases}$$

($\varepsilon > 0$ arbitrary) and posterior passing to the limit $\varepsilon \to 0$. This argument seems to lead to approximate controllability results for a very large class on nonlinear parabolic equations even in non divergence form as $y_t - \mathcal{F}(t, x, y, \nabla y, D^2 y) = 0$.

An approximate controllability result when φ is essentially linear at infinity

The main result of this section is the following:

Theorem 1 *Let φ be a continuous nondecreasing function with $\varphi(0) = 0$. Assume that there exists $k > 0$ such that*

$$(3) \qquad \begin{cases} \varphi \in \mathcal{C}^1(I\!R \backslash [-M_1, M_1]) \text{ and } |\varphi'(s) - k| \leq \dfrac{C_1}{|s|} \text{ if } |s| > M_1, \\ \text{for some positive constants } C_1 \text{ and } M_1 \end{cases}$$

and

$$(4) \qquad |\varphi(s) - ks| \leq C_2 \quad \forall\, s \in I\!R.$$

Then, if $\varphi'(s) \geq c > 0$ a.e. $s \in I\!R$ or $h \in L^2(Q)$, then problem (1) satisfies the approximate controllability property in $H^{-(1+\gamma)}(\Omega)$ for any $\gamma > 0$, i.e., given $y_d \in H^{-(1+\gamma)}(\Omega)$ and $\delta > 0$ there exists $v \in L^2(\Omega)$ such that $\| y(T; v) - y_d \|_{H^{-(1+\gamma)}(\Omega)} < \delta$.

Remark 2 Corollaries 14 and 15 of the Appendix contain some sufficient conditions, easier to verify than (3) and (4).

As mentioned at the Introduction, the proof of Theorem 1 will be obtained through the study of the approximate controllability for the vanishing viscosity higher order problem (2).

Theorem 3 *Assume $\varphi \in \mathcal{C}^0(I\!R)$ (non necessarily nondecreasing) satisfying*

$$|\varphi(s)| \leq C(1 + |s|) \quad \text{for } |s| > M_2 \quad (C, M_2 > 0).$$

Let $y_d \in H^{-(1+\gamma)}(\Omega)$ and $\delta > 0$. Then, for any $\varepsilon > 0$ there exists a control $v_\varepsilon \in L^2(\Omega)$ such that if $y(t; v)$ is the corresponding solution of (2) we have

$$(5) \qquad \| y(T; v_\varepsilon) - y_d \|_{H^{-(1+\gamma)}(\Omega)} < \delta.$$

If in addition φ satisfies (3) and (4), then there exists a positive constant K, depending on k, C_1, C_2 and M_1 but independent of ε, such that the above controls v_ε can be taken satisfying

$$(6) \qquad \| v_\varepsilon \|_{L^2(\Omega)} \leq K, \quad \text{for any } \varepsilon > 0.$$

The proof of the first part of Theorem 3 is an special formulation of the main result (Theorem 1) of Díaz and Ramos [11] (and one can show that property even in the space $L^2(\Omega)$). The second part reproduces some of the steps of the proof of Theorem 1 of Díaz and Ramos [11] that here will be merely sketched but putting emphasis on the new arguments needed to arrive to the conclusion. The first step consists in proving the approximate controllability for a linearized problem (a posterior fixed point argument will extend the conclusion to the nonlinear problem). Since assumption (3) clearly implies that $\varphi'(s) \to k$ as $|s| \to \infty$, it is natural to define the function

$$(7) \qquad \varphi_0(s) := \varphi(s) - ks, \quad s \in I\!R$$

(so that $\varphi_0'(s) \to 0$ as $|s| \to \infty$). Then, it suffices to linearize function φ_0 which (by convenience) will be done near a point $s_\varepsilon \in I\!R$ depending on ε in a suitable way as shows the following result (proved in the appendix):

Lemma 4 *Let $\varphi \in \mathcal{C}^0(I\!R)$ (non necessarily nondecreasing) satisfying (3). For any $\varepsilon > 0$ there exists $s_\varepsilon \in I\!R$ such that the function*

$$g_\varepsilon(s) := \frac{\varphi_0(s) - \varphi_0(s_\varepsilon)}{s - s_\varepsilon} \tag{8}$$

satisfies $g_\varepsilon \in L^\infty(I\!R) \cap \mathcal{C}(I\!R)$ and

$$\| g_\varepsilon \|_{L^\infty(I\!R)} \leq \sqrt{\varepsilon}. \tag{9}$$

If in addition φ satisfies (4), then there exists a positive constant K_2, depending on C_1, C_2 and M_1 but independent of ε, such that

$$|g_\varepsilon(s)s_\varepsilon| \leq K_2, \quad \text{for any } \varepsilon > 0 \text{ and any } s \in I\!R. \tag{10}$$

Now we return to our linearizing process. Since $\varphi_0(s) = \varphi_0(s_\varepsilon) + g_\varepsilon(s)s - g_\varepsilon(s)s_\varepsilon$, we shall start by considering the approximate controllability for a linear problem obtained by replacing the term $\varphi(y)$ by

$$ky + g_\varepsilon(z)y + \varphi_0(s_\varepsilon) - g_\varepsilon(z)s_\varepsilon,$$

where z is an arbitrary function in $L^2(Q)$. Notice that when $z = y$ this expression coincides with $\varphi(y)$ and that if we denote

$$h_\varepsilon(z) := \Delta\left(\varphi_0\ (s_\varepsilon) - g_\varepsilon(z)s_\varepsilon\right) = -\Delta\left(g_\varepsilon\ (z)s_\varepsilon\right),$$

then $h_\varepsilon(z) \in L^\infty(0, T : H^{-2}(\Omega))$ for all $z \in L^2(Q)$ and for all $\varepsilon > 0$. Now, we consider the approximate controllability property corresponding to the linear problem

$$\begin{cases} y_t + \varepsilon\Delta^2 y - k\Delta y - \Delta\left(g_\varepsilon(z)y\right) = h + h_\varepsilon(z) & \text{in } Q, \\ y = \Delta y = 0 & \text{on } \Sigma, \\ y(0) = u_\varepsilon & \text{in } \Omega. \end{cases} \tag{11}$$

Let us denote $E := H^2(\Omega) \cap H_0^1(\Omega)$. The existence and uniqueness of a solution $y \in L^2(0, T : E)$, with $y_t \in L^2(0, T : E')$ is similar to Proposition 4 of Díaz and Ramos [11]. In order to state an approximate controllability result for this problem, following Lions [17], we try to solve the optimal control problem

$$\inf_{v \in L^2(\Omega)} \left\{ \frac{1}{2} \int_\Omega v^2 dx, \quad y_{\varepsilon,z}(T, v) \in y_d + \delta B_{-(1+\gamma)} \right\},$$

where $B_{-(1+\gamma)}$ is the unit ball in $H^{-(1+\gamma)}(\Omega)$. Then, as in [17], by duality theory in Convex Analysis, it is easy to prove that the above optimal control problem is equivalent to the following one:

$$\inf_{p^0 \in H_0^{1+\gamma}(\Omega)} J_\varepsilon(p^0),$$

with $J_\varepsilon = J_\varepsilon(\cdot; z, y_d) : H_0^{1+\gamma}(\Omega) \to I\!R$ defined by

$$J_\varepsilon(p^0) = \frac{1}{2} \| p(x, 0) \|^2_{L^2(\Omega)} + \delta \| p^0 \|_{H_0^{1+\gamma}(\Omega)} - < y_d, p^0 >_{H^{-(1+\gamma)}(\Omega) \times H_0^{1+\gamma}(\Omega)}. \tag{12}$$

Here p denotes the solution of the parabolic backward problem

$$(13)\qquad \begin{cases} -p_t + \varepsilon\Delta^2 p - k\Delta p - g_\varepsilon(z)\Delta p = 0 & \text{in } Q, \\ p = \Delta p = 0 & \text{on } \Sigma, \\ p(T) = p^0 & \text{in } \Omega, \end{cases}$$

for any $p^0 \in H_0^{1+\gamma}(\Omega)$ given. The existence and uniqueness of a solution $p \in L^2(0,T:E)$, with $p_t \in L^2(0,T:E')$ was given in Proposition 1 of Díaz and Ramos [11]. The connection between both minimizing problems is that the solution $u \in L^2(\Omega)$ of the first one is $u = \widehat{p}(x,0)$, where $\widehat{p}$ is the solution of (13) with $\widehat{p}(T) = \widehat{p}^0$.

Now, some easy modifications of the arguments given in Fabre, Puel and Zuazua [12], [13] for a functional similar to this one and the backward uniqueness theorem of Bardos and Tartar [3] (when applied to the present situation, the hipothesis $B(t) \in L^2(0,T:\mathcal{L}(V,H))$ of Theorem II.1 in [3] yields $g_\varepsilon(z) \in L^2(0,T:L^\infty(\Omega))$ allow to show that the functional $J_\varepsilon(\cdot;z,y_d)$ is continuous, strictly convex on $H_0^{1+\gamma}(\Omega)$ and satisfies

$$(14)\qquad \liminf_{\|p^0\|_{H_0^{1+\gamma}(\Omega)}\to\infty} \frac{J_\varepsilon(p^0;z,y_d)}{\| p^0 \|_{H_0^{1+\gamma}(\Omega)}} \geq \delta.$$

Then $J_\varepsilon(\cdot;z,y_d)$ attains its minimum at a unique point $\widehat{p}_\varepsilon^0$ in $H_0^{1+\gamma}(\Omega)$. Furthermore, $\widehat{p}_\varepsilon^0 = 0$ iff $\| y_d \|_{H^{-(1+\gamma)}(\Omega)} \leq \delta$.

Now we shall give an approximate controllability result for an special case:

Lemma 5 *Let $z \in L^2(Q)$ and $y_d \in H^{-(1+\gamma)}(\Omega)$. Then, for any $\delta > 0$ there exists $v_\varepsilon \in L^2(\Omega)$ such that the solution y_ε of the problem*

$$(15)\qquad \begin{cases} y_t + \varepsilon\Delta^2 y - k\Delta y - \Delta\left(g_\varepsilon(z)y\right) = 0 & \text{in } Q, \\ y = \Delta y = 0 & \text{on } \Sigma, \\ y(0) = v_\varepsilon & \text{in } \Omega \end{cases}$$

satisfies

$$\| y_d - y_\varepsilon(T) \|_{H^{-(1+\gamma)}(\Omega)} \leq \delta.$$

Remark 6 *We could prove the approximate controllability property in $L^2(\Omega)$ for any $\varepsilon > 0$ but, in order to be able to pass to the limit when $\varepsilon \to 0$, we obtain this property merely in $H^{-(1+\gamma)}(\Omega)$.*

Proof of Lemma 5. If $q^0 \in H_0^{1+\gamma}(\Omega)$ and q, $\widehat{p}$ are the solutions of (13) satisfying $q(T) = q^0$ and $\widehat{p}_\varepsilon(T) = \widehat{p}_\varepsilon^0$ respectively, then, from the characterization of the minimum (see, for instace, Proposition 3 in page 187 and Theorem 16 in page 198 of Aubin-Ekeland [2]), we obtain that

$$-\int_\Omega \widehat{p}_\varepsilon(x,0)q(x,0)dx + < y_d, q^0 >_{H^{-(1+\gamma)}\times H^{1+\gamma}} \leq$$

$$\lim_{h\to 0^+} \frac{\delta \| \widehat{p}_\varepsilon^0 + hq^0 \|_{H_0^{1+\gamma}(\Omega)} - \delta \| \widehat{p}_\varepsilon^0 \|_{H_0^{1+\gamma}(\Omega)}}{h} \leq \delta \| q^0 \|_{H_0^{1+\gamma}(\Omega)} \qquad \forall q^0 \in H_0^{1+\gamma}(\Omega).$$

Now if we take $v_\varepsilon \equiv \widehat{p}(x,0)$ and multiply in the equation of (15) by q we obtain that

$$< y_\varepsilon(T,v_\varepsilon), q^0 >_{H^{-(1+\gamma)}\times H^{1+\gamma}} = \int_\Omega \widehat{p}_\varepsilon(x,0)q(x,0)dx$$

and therefore

$$< y_d - y_\varepsilon(T; v_\varepsilon), q^0 >_{H^{-(1+\gamma)}(\Omega)\times H_0^{1+\gamma}(\Omega)} \leq \delta \parallel q^0 \parallel_{H_0^{1+\gamma}(\Omega)} \quad \forall\, q^0 \in H_0^{1+\gamma}(\Omega),$$

which shows that

$$\parallel y_d - y_\varepsilon(T; v_\varepsilon) \parallel_{H^{-(1+\gamma)}(\Omega)} \leq \delta$$

and concludes the proof of Lemma 5.

Concerning the approximate controllability for the linearized problem (11) we have

Theorem 7 *Let $z \in L^2(Q)$ and $y_d \in H^{-(1+\gamma)}(\Omega)$. Assume g_ε satisfying (9) and (10). Let $\parallel y_d - y(T; z, 0) \parallel_{H^{-(1+\gamma)}(\Omega)} > \delta$ and let $\widehat{p}_\varepsilon$ be the solution of (13) corresponding to $\widehat{p}(T) = \widehat{p}^0_\varepsilon$, with $\widehat{p}^0_\varepsilon$ minimum of $J_\varepsilon(\cdot; z, y_d - y(T; z, 0))$, where in general $y(t; z, u)$ denotes the solution of (11) corresponding to the control u. Then the solution y_ε of*

$$\begin{cases} y_t + \varepsilon\Delta^2 y - k\Delta y - \Delta\left(g_\varepsilon(z) y\right) = h + h_\varepsilon(z) & \text{in } Q, \\ y = \Delta y = 0 & \text{on } \Sigma, \\ y(0) = \widehat{p}_\varepsilon(x, 0) & \text{in } \Omega, \end{cases}$$

satisfies

(16) $$\parallel y_\varepsilon(T) - y_d \parallel_{H^{-(1+\gamma)}(\Omega)} \leq \delta.$$

Moreover, if $\parallel y_d - y(T; z, 0) \parallel_{H^{-(1+\gamma)}(\Omega)} \leq \delta$, then property (16) holds for the control $v_\varepsilon \equiv 0$. Finally, there exists a positive constant K, depending on k, C_1, C_2 and M_1 but independent of ε, such that the above funtions $\widehat{p}_\varepsilon$ satisfy

(17) $$\parallel \widehat{p}_\varepsilon \parallel_{\mathcal{C}([0,T]:L^2(\Omega))} \leq K, \quad \textit{for any } \varepsilon > 0 \textit{ and any } z \in L^2(Q).$$

Remark 8 Theorem 7 solves the approximate controllability problem for (11) with control $u_\varepsilon := \widehat{p}_\varepsilon(x, 0)$. Therefore

(18) $$\parallel u_\varepsilon \parallel_{L^2(\Omega)} \leq K.$$

Proof of Theorem 7. We put $y_\varepsilon = L_\varepsilon + Y_\varepsilon$, where $L_\varepsilon = L_\varepsilon(z) \in \mathcal{C}([0,T] : L^2(\Omega))$ satisfies

(19) $$\begin{cases} L_t + \varepsilon\Delta^2 L - k\Delta L - \Delta\left(g_\varepsilon(z) L\right) = h + h_\varepsilon(z) & \text{in } Q, \\ L = \Delta L = 0 & \text{on } \Sigma, \\ L(0) = 0 & \text{in } \Omega \end{cases}$$

and $Y_\varepsilon = Y_\varepsilon(z)$ is taken associated to the approximate controllability problem

$$\begin{cases} Y_t + \varepsilon\Delta^2 Y - k\Delta Y - \Delta\left(g_\varepsilon(z) Y\right) = 0 & \text{in } Q, \\ Y = \Delta Y = 0 & \text{on } \Sigma, \\ Y(0) = u_\varepsilon(z) & \text{in } \Omega, \end{cases}$$

with desired state $y_d - L_\varepsilon(T)$, i.e. such that $\parallel Y_\varepsilon(T) - (y_d - L_\varepsilon(T)) \parallel_{H^{-(1+\gamma)}(\Omega)} \leq \delta$. We find the control u_ε in the same way as in Lemma 5. Therefore, if $\widehat{p}_\varepsilon$ is the solution of (13) with final data $\mathcal{M}(\varepsilon, z, y_d - L_\varepsilon(T))$, where

$$\begin{array}{rcccl} \mathcal{M} & : & (0, R] \times L^2(Q) \times H^{-(1+s)}(\Omega) & \longrightarrow & L^2(\Omega) \\ & & (\varepsilon, z, y_d) & \rightarrow & \widehat{p}^0_\varepsilon, \end{array}$$

then the control $u_\varepsilon(z) := \widehat{p}_\varepsilon(x, 0)$ leads to $\parallel Y(T) - \widehat{y}_d \parallel_{H^{-(1+\gamma)}(\Omega)} \leq \delta$, where $\widehat{y}_d := y_d - L_\varepsilon(T)$ (in the case $\parallel \widehat{y}_d \parallel_{H^{-(1+\gamma)}(\Omega)} \leq \delta$ it suffices to take $u_\varepsilon \equiv 0$). For the proof of (17) we need the following four lemmas:

Lemma 9 *Assume (9) and (10). Let $z \in L^2(Q)$. Let $p_0 \in L^2(\Omega)$ be given. Then, if p_ε is the solution of (13), we have*

$$\text{(20)} \qquad \| p_\varepsilon \|_{\mathcal{C}([0,T]:L^2(\Omega))} \leq e^T \| p^0 \|_{L^2(\Omega)} \qquad \textit{for any } \varepsilon > 0 \textit{ and any } z \in L^2(Q).$$

Proof. If we "multiply" in (13) by p_ε, for any $t \in (0,T]$ we obtain

$$\frac{1}{2} \| p_\varepsilon(t) \|^2_{L^2(\Omega)} + \varepsilon \| \Delta p_\varepsilon \|^2_{L^2((t,T)\times\Omega)} + k \| \nabla p_\varepsilon \|^2_{L^2((t,T)\times\Omega)} \leq$$

$$\frac{1}{2} \| p_\varepsilon(T) \|^2_{L^2(\Omega)} + \| g_\varepsilon(z(t,x)) \|_{L^\infty(Q)} \| \Delta p_\varepsilon \|_{L^2((t,T)\times\Omega)} \| p_\varepsilon \|_{L^2((t,T)\times\Omega)} .$$

Then, applying Young's inequality, we have that

$$\frac{1}{2} \| p_\varepsilon(t) \|^2_{L^2(\Omega)} + \frac{\varepsilon}{2} \| \Delta p_\varepsilon \|^2_{L^2((t,T)\times\Omega)} \leq \frac{1}{2} \| p_\varepsilon(T) \|^2_{L^2(\Omega)} + \frac{1}{2} \| p_\varepsilon \|^2_{L^2((t,T)\times\Omega)} .$$

Then we obtain that

$$\| p_\varepsilon(t) \|^2_{L^2(\Omega)} \leq \| p_\varepsilon(T) \|^2_{L^2(\Omega)} + \int_t^T \| p_\varepsilon(\tau) \|^2_{L^2(\Omega)} \, d\tau.$$

Applying Gronwall's inequality, we deduce the following inequality leading to (20)

$$\| p_\varepsilon(t) \|^2_{L^2(\Omega)} \leq \| p_\varepsilon(T) \|^2_{L^2(\Omega)} e^{T-t} \qquad \forall\, t \in [0,T].$$

Lemma 10 *The mappings*

$$\begin{array}{cccc} S_\varepsilon : & E & \longrightarrow & H^{2m,1}(Q) \\ & p^0 & \rightarrow & p \end{array}$$

and

$$\begin{array}{cccc} T_\varepsilon : & H_0^{1+\gamma}(\Omega) & \longrightarrow & L^2(0,T : H^{\frac{5}{2}-\alpha}(\Omega)) \\ & p^0 & \rightarrow & p, \end{array}$$

where p is the solution of (13) asociated to p^0, are linear and continuous for any $\varepsilon > 0$ and any $\gamma \geq -1/2$.

Proof. The first case is a simple corollary of the results in Section 4.13.3 of Lions-Magenes [18]. To prove the second case we notice that $p^0 \in H^{1+\gamma}(\Omega) \subset H^{2(-\frac{3}{8}-\frac{\alpha}{4}+\frac{1}{2})2}(\Omega) = H^{\frac{1}{2}-\alpha}(\Omega)$ for any $\alpha > 0$, then applying the results of section 4.15.1 of Lions and Magenes [18] we obtain that T_ε is a continuous mapping (even from $H^{\frac{1}{2}-\alpha}(\Omega)$) on $L^2(0,T : H^{4(-\frac{3}{8}-\frac{\alpha}{4}+1)}(\Omega)) = L^2(0,T : H^{\frac{5}{2}-\alpha}(\Omega))$.

Remark 11 In the case of operator T_ε, it seems (very likely) that, since p^0 satisfies the compatibility relation $p^0(x) = 0$ in $\partial\Omega$, then the associated solution p of (13) belongs to $\{p \in L^2(0,T : H^3(\Omega)) : p, \Delta p \in L^2(0,T : H_0^1(\Omega))\}$ (see Remark 4.14.3 and Section 4.15.1 of Lions-Magenes [18]) but we don't know a rigurous proof of this fact

Lemma 12 *If K is a compact subset of $H^{-(1+\gamma)}(\Omega)$ then $\mathcal{M}((0,R] \times L^2(Q) \times K)$ is a bounded subset of $H_0^{1+\gamma}(\Omega)$.*

Proof. If Lemma 12 is not true there will be three sequences $\{z_n\}_{n\in\mathbb{N}} \subset L^2(Q)$, $\{y_d^n\}_{n\in\mathbb{N}} \subset K$ and $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,R]$ such that

$$\| p^0(\varepsilon_n, z_n, y_d^n) \|_{H_0^{1+\gamma}(\Omega)} = \| \mathcal{M}(\varepsilon_n, z_n, y_d^n) \|_{H_0^{1+\gamma}(\Omega)} \stackrel{n\to\infty}{\longrightarrow} \infty. \tag{21}$$

Then we can suppose (by renaming the sequences) that

$$g_{\varepsilon_n} \stackrel{n\to\infty}{\rightharpoonup} a \quad \text{in the weak-}* \text{ topology of } L^\infty(Q),$$

$$y_d^n \to y_d \quad \text{in the strong topology of } H^{-(1+\gamma)}(\Omega)$$

and

$$\varepsilon_n \to \widetilde{\varepsilon} \quad \text{in } \mathbb{R}$$

(notice that, due to (9), if $\widetilde{\varepsilon}=0$ then $a\equiv 0$).
Now, in order to obtain a contradiction, let us prove that for any sequence $\{p_n^0\}_{n\in\mathbb{N}} \subset H_0^{1+\gamma}(\Omega)$ such that $\| p_n^0 \|_{H_0^{1+\gamma}(\Omega)} \stackrel{n\to\infty}{\longrightarrow} \infty$ we have that

$$\liminf_{n\to\infty} \frac{J_{\varepsilon_n}(p_n^0; z_n, y_d^n)}{\| p_n^0 \|_{H_0^{1+\gamma}(\Omega)}} \geq \delta. \tag{22}$$

Let us suppose that (22) is not true. Then, there exists a sequence $\{p_n^0\}_{n\in\mathbb{N}} \subset H_0^{1+\gamma}(\Omega)$ such that $\| p_n^0 \|_{H_0^{1+\gamma}(\Omega)} \stackrel{n\to\infty}{\longrightarrow} \infty$ and

$$\liminf_{n\to\infty} \frac{J_{\varepsilon_n}(p_n^0; z_n, y_d^n)}{\| p_n^0 \|_{H_0^{1+\gamma}(\Omega)}} < \delta. \tag{23}$$

Let us denote $\widetilde{p}_n^0 = \dfrac{p_n^0}{\| p_n^0 \|_{H_0^{1+\gamma}(\Omega)}}$ and $\widetilde{p}_n$ the solution of (13) associated to z_n, ε_n and with $\widetilde{p}_n(T) = \widetilde{p}_n^0$. Then

$$\| \widetilde{p}_n(x,0) \|^2_{H_0^{1+\gamma}(\Omega)} \to 0 \quad \text{as } n\to\infty, \tag{24}$$

because in other case

$$\liminf_{n\to\infty} \frac{J_{\varepsilon_n}(p_n^0; z_n, y_d^n)}{\| p_n^0 \|_{H_0^{1+\gamma}(\Omega)}} \geq$$

$$\liminf_{n\to\infty} \left(\frac{1}{2} \| p_n^0 \|_{H_0^{1+\gamma}(\Omega)} \| \widetilde{p}_n(x,0) \|^2_{L^2(\Omega)} + \delta - \| y_d^n \|_{H^{-(1+\gamma)}(\Omega)} \right) = \infty,$$

which is a contradiction with (23).
Now, we can suppose (again by relabeling the sequence) that there exists $\widetilde{p}^0 \in H_0^{1+\gamma}(\Omega)$ such that

$$\widetilde{p}_n^0 \rightharpoonup \widetilde{p}^0 \quad \text{in the weak topology of } H_0^{1+\gamma}(\Omega).$$

Then, by Lemma 9, we obtain that $\widetilde{p}_n$ is uniformly bounded in $\mathcal{C}([0,T] : L^2(\Omega))$ and therefore there exists $\widetilde{p} \in L^\infty(0,T : L^2(\Omega))$ such that $\widetilde{p}_n \rightharpoonup \widetilde{p}$ in the weak topolgy of $L^2(Q)$. This is not sufficient to pass to the limit in the equation satisfied by $\widetilde{p}_n$ (because of the terms $g_{\varepsilon_n}(z_n)\Delta\widetilde{p}_n$).
In order to pass to the limit in the equation satisfied by $\widetilde{p}_n$ we distinguish three different cases: a) $\widetilde{\varepsilon} > 0$, b) $\widetilde{\varepsilon} = 0$ and $k>0$ and c) $\widetilde{\varepsilon}=0$ and $k=0$.
To pass to the limit in the three cases above we would like to be able to "multiply" in (13) by $-\Delta p$. Now, if $p^0 \in H_0^{1+\gamma}(\Omega)$, then it seems (very likely) that the associated solution p

of (13) belongs to $\{p \in L^2(0,T : H^3(\Omega)) : p, \Delta p \in L^2(0,T : H_0^1(\Omega))\}$ (see Remark 11) and therefore we could "multiply" in the equation of (13) by $-\Delta p$ by means of the duality product $< \cdot,\cdot >_{H^{-1}(\Omega)\times H_0^1(\Omega)}$. Nevertheless, we don't know a rigurous proof of this fact and we have to use a different strategy.
From Lemma 10 and the dense imbedding $E \subset H_0^1(\Omega)$, we know that for every $n \in I\!N$ we can choose $\overline{p}_0^n \in E$ such that $\| \widetilde{p}_n^0 - \overline{p}_n^0 \|_{H_0^1(\Omega)} \leq 1$ and

$$(25) \qquad \| \widetilde{p}_n - \overline{p}_n \|_{L^2(0,T:H^{\frac{5}{2}-\alpha}(\Omega))} \leq 1.$$

Now, since $\overline{p}_n \in \{p \in H^{2m,1}(Q) : p, \Delta p \in L^2(0,T : H_0^1(\Omega))\}$ we can "multiply" by $-\Delta\overline{p}_n$ in the equation satisfied by $\overline{p}_n$ and we obtain that there exists K independent of $n \in I\!N$ such that

$$(26) \qquad \| \overline{p}_n^0 \|_{H_0^1(\Omega)} + \widetilde{\varepsilon} \| \nabla\Delta\overline{p}_n \|_{L^2(Q)} + \| \sqrt{\varepsilon_n}\nabla\Delta\overline{p}_n \|_{L^2(Q)} + k \| \Delta\overline{p}_n \|_{L^2(Q)} \leq K.$$

Now, let us pass to the limit in the three different cases:
In case a), from estimates (25) and (26), we deduce that there exists $\widetilde{p} \in L^2(0,T : L^2(0,T : H^{\frac{5}{2}-\alpha}(\Omega))$ such that $\widetilde{p}_n \rightharpoonup \widetilde{p}$ in the weak topology of $L^2(0,T : L^2(0,T : H^{\frac{5}{2}-\alpha}(\Omega))$. Then, from the equation satisfied by $\widetilde{p}_n$, we deduce that $\frac{\partial\widetilde{p}_n}{\partial t}$ is uniformly bounded in $L^2(0,T : H^{-\frac{3}{2}-\alpha}(\Omega))$. Now, since $H^{\frac{5}{2}-\alpha}(\Omega) \subset H^2(\Omega) \subset H^{-\frac{3}{2}-\alpha}(\Omega)$ (for $\alpha > 0$ small enough) with compact imbeddings (see Theorem 1.16.1 of Lions [18]), we have (see Aubin [1] or Theorem 1.5.1 of Lions [16]) that $\{\widetilde{p}_n\}_{n\in I\!N}$ is relatively compact in $L^2(0,T : H^2(\Omega))$ (and $L^2(0,T : E)$) and so in $\{p \in L^2(0,T : E) : \ p_t \in L^2(0,T : E')\} \subset \mathcal{C}([0,T] : L^2(\Omega))$. Therefore, $g_{\varepsilon_n}(z_n)\Delta\widetilde{p}_n \to a\Delta p$ in the weak topology of $L^2(Q)$, which allows us to pass to the limit in the equation satisfied by $\widetilde{p}_n$ and deduce that $\widetilde{p}$ is solution of

$$\begin{cases} -\widetilde{p}_t + \widetilde{\varepsilon}\Delta^2\widetilde{p} - k\Delta\widetilde{p} - a\Delta\widetilde{p} = 0 & \text{in } Q, \\ \widetilde{p} = \Delta\widetilde{p} = 0 & \text{on } \Sigma, \\ \widetilde{p}(T) = \widetilde{p}^0 & \text{in } \Omega. \end{cases}$$

In case b), again from estimates (25) and (26), we deduce that there exists $\widetilde{p} \in L^2(0,T : E)$ such that $\widetilde{p}_n \rightharpoonup \widetilde{p}$ in the weak topology of $L^2(0,T : E)$ (and therefore $\Delta\widetilde{p}_n \rightharpoonup \Delta\widetilde{p}$ in the weak topology of $L^2(Q)$). Now in this case, since $\widetilde{\varepsilon} = 0$ and g_{ε_n} satisfies (9), $g_{\varepsilon_n}(z_n) \to 0$ in the strong topology of $L^\infty(Q)$. Therefore, $g_{\varepsilon_n}(z_n)\Delta\widetilde{p}_n \rightharpoonup a\Delta p \equiv 0$ in the weak topology of $L^2(Q)$, which allows us to pass in the limit in the equation satisfied by $\widetilde{p}_n$ and deduce that $\widetilde{p}$ is solution of

$$\begin{cases} -\widetilde{p}_t - k\Delta\widetilde{p} = 0 & \text{in } Q, \\ \widetilde{p} = 0 & \text{on } \Sigma. \end{cases}$$

Then, $\widetilde{p} \in \{p \in L^2(0,T : E) : \ p_t \in L^2(Q)\} \subset \mathcal{C}([0,T] : H_0^1(\Omega))$. Now, in order to see what is the final data $\widetilde{p}(T)$, for all $u \in L^2(Q)$ we consider $\varphi(u)$ solution of

$$\begin{cases} \varphi_t - \Delta\varphi = u & \text{in } Q, \\ \varphi = 0 & \text{on } \Sigma \\ \varphi(0) = 0 & \text{in } \Omega. \end{cases}$$

Then $\varphi(u) \in \{\varphi \in L^2(0,T : E) : \varphi_t \in L^2(Q)\}$ and we have that

$$-\int_\Omega (\widetilde{p}_n^0 - \widetilde{p}(T))\varphi(T)dx + \int_Q (\widetilde{p}_n - \widetilde{p})\varphi_t dxdt + \int_Q \varepsilon_n \Delta\widetilde{p}_n \Delta\varphi dxdt$$

$$-\int_Q k\Delta(\widetilde{p}_n-\widetilde{p})\varphi dxdt-\int_Q g_{\varepsilon_n}(z_n)\Delta\widetilde{p}_n\varphi dxdt=0 \quad \text{for any } u\in L^2(Q).$$

Now, passing to the limit as $n\to\infty$, we obtain

$$-\int_\Omega(\widetilde{p}^0-\widetilde{p}(T))\varphi(T)dx=0$$

for any $u\in L^2(Q)$, which shows that $\widetilde{p}(T)=\widetilde{p}^0$, since

$$\{\varphi(T;u):\ u\in L^2(Q)\} \text{ is a dense subset of } L^2(\Omega)$$

(see, for instance, Section 3.10.2 of Lions [15]). Therefore, $\widetilde{p}$ is solution of

$$\begin{cases} -\widetilde{p}_t-k\Delta\widetilde{p}=0 & \text{in } Q,\\ \widetilde{p}=0 & \text{on } \Sigma,\\ \widetilde{p}(T)=\widetilde{p}^0 & \text{in } \Omega.\end{cases}$$

In case c), again from estimates (25) and (26), we deduce that there exists $\widetilde{p}\in L^2(0,T:H_0^1(\Omega))$ such that $\widetilde{p}_n\rightharpoonup\widetilde{p}$ in the weak topology of $L^2(0,T:H_0^1(\Omega))$. Hence,

$$\sqrt{\varepsilon_n}\Delta\widetilde{p}_n\to 0 \quad \text{in the strong topology of } L^2(0,T:H^{-1}(\Omega)). \tag{27}$$

Further, also from estimates (25) and (26), we know that $\sqrt{\varepsilon_n}\widetilde{p}_n$ is uniformly bounded in the topology of $L^2(0,T:H^{\frac{5}{2}-\alpha}(\Omega))$. Then, in a way similar to that of the case a), we obtain that $\sqrt{\varepsilon_n}\frac{\partial\widetilde{p}_n}{\partial t}$ is uniformly bounded in $L^2(0,T:H^{-\frac{3}{2}-\alpha}(\Omega))$ and, therefore $\sqrt{\varepsilon_n}\widetilde{p}_n$ is relatively compact in $L^2(0,T:E)$. Then, from (27), we deduce that $\sqrt{\varepsilon_n}\Delta\widetilde{p}_n\to 0$ in the strong topology of $L^2(Q)$. Thus,

$$g_{\varepsilon_n}(z_n)\Delta\widetilde{p}_n=\frac{g_{\varepsilon_n}(z_n)}{\sqrt{\varepsilon_n}}\sqrt{\varepsilon_n}\Delta\widetilde{p}_n\rightharpoonup 0 \text{ in the weak topology of } L^2(Q),$$

which allows us to pass in the limit in the equation satisfied by $\widetilde{p}_n$ and deduce that $\widetilde{p}$ is solution of

$$-\widetilde{p}_t=0 \text{ in } Q.$$

Then, $\widetilde{p}\in L^2(0,T:H_0^1(\Omega))$ and $\widetilde{p}(x,t)=\widetilde{p}(x,T)$ for all $t\in[0,T]$. Further, if we take $\varphi(u)$ as in case b), for any $u\in L^2(Q)$ we have that

$$-\int_\Omega(\widetilde{p}_n^0-\widetilde{p}(T))\varphi(T)dx+\int_Q(\widetilde{p}_n-\widetilde{p})\varphi_t dxdt+\int_Q\varepsilon_n\Delta\widetilde{p}_n\Delta\varphi dxdt$$

$$-\int_Q g_{\varepsilon_n}(z_n)\Delta\widetilde{p}_n\varphi dxdt=0 \quad \text{for any } u\in L^2(Q).$$

Now, passing to the limit as $n\to\infty$, we obtain

$$-\int_\Omega(\widetilde{p}^0-\widetilde{p}(T))\varphi(T)dx=0$$

for any $u\in L^2(Q)$, which shows (as in case b)) that $\widetilde{p}(T)=\widetilde{p}^0$. Hence, $\widetilde{p}$ is solution of

$$\begin{cases} -\widetilde{p}_t=0 & \text{in } Q,\\ \widetilde{p}=0 & \text{on } \Sigma,\\ \widetilde{p}(T)=\widetilde{p}^0 & \text{in } \Omega.\end{cases}$$

Let us see that $\widetilde{p}(x,0) \equiv 0$ in the three different cases:
In case a) we have proved that $\widetilde{p}_n \to \widetilde{p}$ in $\mathcal{C}([0,T]:L^2(\Omega))$ and therefore $\widetilde{p}_n(x,0) \to \widetilde{p}(x,0)$. Then, from (24), we obtain that $\widetilde{p} \equiv 0$.
In cases b) and c) we have that

$$\int_\Omega \widetilde{p}_n(x,0) - \widetilde{p}(0))\varphi dx - \int_\Omega (\widetilde{p}_n^0 - \widetilde{p}(T))\varphi dx + \int_Q \varepsilon_n \Delta \widetilde{p}_n \Delta \varphi dx dt$$

$$+ \int_Q k \nabla(\widetilde{p}_n - \widetilde{p}) \nabla \varphi dx dt - \int_Q g_{\varepsilon_n}(z_n) \Delta \widetilde{p}_n \varphi dx dt = 0 \quad \text{for any } \varphi \in E.$$

Finally, passing to the limit as $n \to \infty$, we obtain that $\widetilde{p}_n(x,0) \rightharpoonup \widetilde{p}(0)$ in the weak topology of $L^2(\Omega)$. Then, from (24), we obtain that $\widetilde{p}(x,0) \equiv 0$. Now, since $\widetilde{p}$ satisfies a suitable linear parabolic equation for any of the cases a), b) or c), we can apply a backward uniqueness result (see Theorem II.1 of Bardos and Tartar [3]) and deduce that $\widetilde{p} \equiv 0$ in Q. Therefore $\widetilde{p}^0 \equiv 0$ in Ω.
Thus,

$$\liminf_{n\to\infty} \frac{J_{\varepsilon_n}(p_n^0; z_n, y_d^n)}{\| p_n^0 \|_{H_0^{1+\gamma}(\Omega)}} \geq \liminf_{n\to\infty} \left(\delta - < y_d^n, \widetilde{p}_n^0 >_{H^{-(1+\gamma)}(\Omega)\times H_0^{1+\gamma}(\Omega)}\right) = \delta,$$

which contradicts (23) and proves (22).
Finally we point out that $J_{\varepsilon_n}(\widehat{p}^0(\varepsilon_n, z_n, y_d^n); z_n, y_d^n) \leq J_{\varepsilon_n}(0; z_n, y_d^n) = 0$, which is a contradiction with (22) and (21) and concludes the result.

Lemma 13 *The solutions $L_\varepsilon(z)$ of (19), with arbitrary $\varepsilon > 0$ (small emough) and $z \in L^2(Q)$, are uniformly bounded in $\mathcal{C}([0,T]:H^{-1}(\Omega)) \cap L^2(Q)$.*

Proof. For all $\varepsilon > 0$ and $z \in L^2(Q)$ we denote by $\psi = \psi_\varepsilon(z)$ to the solution of

$$\begin{cases} -\Delta\psi(t) = L(t) & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega \end{cases} \qquad \text{for all } t \in [0,T].$$

Then, since $L = L_\varepsilon(z) \in \mathcal{C}([0,T]:L^2(\Omega))$ for all $\varepsilon > 0$ and $z \in L^2(Q)$ (recall $\{L \in L^2(0,T:E): \ L_t \in L^2(0,T:E')\} \subset \mathcal{C}([0,T]:L^2(\Omega))$; see e.g. Lions-Magenes [18]), we have that $\psi_\varepsilon(z) \in \mathcal{C}([0,T]:E)$. Now if we take $t \in (0,T]$ and "multiply" in (19) by ψ, by using the duality product $< \cdot, \cdot >_{L^2(0,t:E')\times L^2(0,t:E)}$, we obtain

$$\frac{1}{2} \| \nabla\psi_\varepsilon(t) \|^2_{L^2(\Omega)} + \varepsilon \| \nabla L_\varepsilon \|^2_{L^2(0,t:L^2(\Omega))} + k \| L_\varepsilon \|^2_{L^2((0,t)\times\Omega)}$$

$$+ \int_0^t\int_\Omega g_\varepsilon(z(x,t)) L_\varepsilon^2(x,t) dx dt \leq$$

$$\| h \|_{L^2(0,t:H^{-1}(\Omega))} \| \nabla\psi_\varepsilon \|_{L^2((0,t)\times\Omega)} + \| g_\varepsilon(z(t,x)) s_\varepsilon \|_{L^\infty(Q)} \| L \|_{L^2((0,t)\times\Omega)} .$$

Here we point out that

$$\| \nabla\psi_\varepsilon(t) \|^2_{L^2(\Omega)} = \| L_\varepsilon(t) \|^2_{H^{-1}(\Omega)} .$$

Then, if we take ε small enough and use Young and Gronwall's inequalities, we easily deduce (taking into account that in the case $k=0$, $\varphi = \varphi_0$ is a nondecreasing function and therefore $g_\varepsilon(z) \geq 0$) that there exists a constant $K_3 > 0$ independent of ε, z and t such that

$$\| L_\varepsilon(t) \|^2_{H^{-1}(\Omega)} + \| L_\varepsilon \|^2_{L^2((0,t)\times\Omega)} \leq K_3,$$

which concludes the result.

Completion of proof of Theorem 7. From Lemma 13 we can deduce that there exists a constant K_3, depending on C_1, C_2 and M_1 but independent of ε, such that

$$\| L_\varepsilon(z) \|_{\mathcal{C}([0,T]:H^{-1}(\Omega))} \leq K_3 \quad \text{for any } \varepsilon > 0 \text{ and any } z \in L^2(Q).$$

Then $\{L_\varepsilon(z;T)$, for any $\varepsilon > 0$ and any $z \in L^2(Q)\}$ is a relatively compact subset of $H^{-(1+\gamma)}(\Omega)$ for all $\gamma > 0$. Then, applying Lemma 12, there exists a constant K_4, depending on C_1, C_2 and M_1 but independent of ε, such that, if $\widehat{p}^0_\varepsilon$ is the minimum of $J_\varepsilon(\cdot; z, y_d - L_\varepsilon(T))$, we have $\| \widehat{p}^0_\varepsilon \|_{L^2(\Omega)} \leq K_4$ for any $\varepsilon > 0$ and any $z \in L^2(Q)$. Lemma 9 implies (17) with $K = e^T K_4$.

Proof of Theorem 3. The first part is similar to that proved in Theorem 1 of Díaz and Ramos [11] by applying Kakutani's fixed point theorem to the operator $\Lambda_\varepsilon : L^2(Q) \to \mathcal{P}(L^2(Q))$ defined by $\Lambda_\varepsilon(z) := \{y_\varepsilon$ satisfying (11), (16), with a control u_ε satisfying (18)$\}$, where the constant K of (18) depends on ε. Finally, if φ satisfies (3) and (4), then Theorem 7 shows that (17) holds (i.e. K does not depend on ε), which leads to (6).

Proof of Theorem 1. *First step.* Assume additionally that $\varphi \in \mathcal{C}^1(I\!R)$. For any $\varepsilon > 0$, let v_ε and y_ε be the functions given in Theorem 3. Since the equation of (2) holds on $L^2(0,T : E')$, multiplying by $y_\varepsilon \in L^2(0,T : E)$ and applying Young and Gronwall inequalities we obtain, thanks to the uniform estimate (6) and the assumptions on φ' or h, that there exists a constant $C > 0$ independent of ε such that

$$\| y_\varepsilon \|_{L^\infty(0,T:L^2(\Omega))} + \int_Q \varphi'(y_\varepsilon)|\nabla y_\varepsilon|^2 dxdt \leq C. \tag{28}$$

Therefore, from (28) we obtain that y_ε is uniformly bounded in $L^\infty(0,T : L^2(\Omega))$ and by the equation of (2), $(y_\varepsilon)_t$ is uniformly bounded in $L^\infty(0,T : H^{-4}(\Omega))$. Then, since $L^2(\Omega) \subset H^{-1}(\Omega) \subset H^{-4}(\Omega)$ with compact imbeddings, we have (see Aubin [1] or Theorem 1.5.1 of Lions [16]) that y_ε is relatively compact in $\mathcal{C}([0,T] : H^{-1}(\Omega))$. Further, from (28) and the boundedness of function φ' (notice that $\varphi' \in L^\infty(I\!R)$ by (3)), we deduce that there exists a constant $K > 0$ independent of ε such that

$$\int_0^T \| \nabla\varphi(y_\varepsilon) \|^2_{L^2(\Omega)} dt = \int_Q \varphi'(y_\varepsilon(x,t))\ \varphi'(y_\varepsilon(x,t))|\nabla(y_\varepsilon(x,t))|^2 dxdt < K.$$

Thus, there exist $y \in L^\infty(0,T : L^2(\Omega))$ and $\zeta \in L^2(0,T : H^1_0(\Omega))$ (recall that $\varphi(0) = 0$) such that $y_\varepsilon \to y$ strongly in $L^2(0,T : H^{-1}(\Omega))$ and $\varphi(y_\varepsilon) \rightharpoonup \zeta$ weakly in $L^2(0,T : H^1_0(\Omega))$. But the operator $Au := -\Delta\varphi(u)$, $D(A) := \{u \in H^{-1}(\Omega) :\ \varphi(u) \in H^1_0(\Omega)\}$ is a maximal monotone operator on the space $H^{-1}(\Omega)$ (see Brézis [4]). Thus, the extension operator $\mathcal{A}$ of A is also a maximal monotone operator on $L^2(0,T : H^{-1}(\Omega))$ (see Brézis [5]), Example 2.33). Finally, as any maximal monotone operator is strongly-weakly closed (see Brézis [5], Proposition 2.5), we obtain that $\zeta = \varphi(y)$ in $L^2(0,T : H^1_0(\Omega))$. Moreover, from estimate (6) we have that $v_\varepsilon \rightharpoonup v$ weakly in $L^2(\Omega)$, with

$$\| v \|_{L^2(\Omega)} \leq K. \tag{29}$$

Then we deduce that $y \in \mathcal{C}([0,T] : H^{-1}(\Omega))$ is solution of (1). Further, since $y_\varepsilon(T) \to y(T)$ strongly in $H^{-1}(\Omega)$, we deduce that

$$\| y(T) - y_d \|_{H^{-(1+\gamma)}(\Omega)} = \lim_{\varepsilon \to 0} \| y_\varepsilon(T) - y_d \|_{H^{-(1+\gamma)}(\Omega)} \leq \delta.$$

Second step. Let φ as in the statement of Theorem 1. It is clear that we can approximate φ by $\varphi_n \in \mathcal{C}^1(I\!R)$, φ_n nondecreasing, satisfying (3) and (4) with the same constants k, C_1, C_2 and M_1 that the ones for φ. Then the respective controls v_n build as in step 1 are uniformly bounded (recall (29)) and therefore the conclusion comes from the well-known result expressing the continuous dependence in $\mathcal{C}([0,T]:H^{-1}(\Omega))$ on φ of solutions of (1) (see e.g. Damlamian [7], Theorem 2.3).

Appendix

Proof of Lemma 4. We can choose $s_\varepsilon \in I\!R$ large enough such that $|s_\varepsilon| > 2M_1$ and

$$(30) \qquad |\varphi_0'(s)| < \frac{\sqrt{\varepsilon}}{2} \quad \text{for any} \quad s \in I\!R \quad \text{with} \quad |s| \geq \frac{|s_\varepsilon|}{2}.$$

Indeed, (30) is implied by the assumption (3). Moreover, if $\varphi_0 \in L^\infty(I\!R)$ then we can choose s_ε satisfying (30) such that

$$\frac{\| \varphi_0 \|_{L^\infty(I\!R)}}{|s_\varepsilon|} \leq \frac{\sqrt{\varepsilon}}{8}$$

and if φ_0 is a not bounded function then we claim that we can choose s_ε satisfying (30) such that

$$(31) \qquad |\varphi_0(s)| \leq |\varphi_0(s_\varepsilon)| \quad \forall\, s \in I\!R \;\text{ such that }\; |s| \leq \frac{|s_\varepsilon|}{2}$$

and

$$(32) \qquad \frac{|\varphi_0(\stackrel{+}{-} s_\varepsilon)|}{|s_\varepsilon|} < \frac{\sqrt{\varepsilon}}{8}.$$

In fact, (32) is implied by assumption (3). In order to verify (31), if we define $s_N \in [-N,N]$ such that $|\varphi_0(s_N)| = \max\{|\varphi_0(s)| : \; s \in [-N,N]\}$, then, since $\varphi_0 \in \mathcal{C}^0(I\!R)$ and it is a not bounded function, it is clear that $\{s_N\} \to +\infty$ as $N \to +\infty$. Then, taking $s_\varepsilon = s_N$, with N large enough, properties (30) and (31) are simultaneously verified.
Let us supoose that $s_\varepsilon > 0$ (the other case is similar to this one). Then it is easy to check property (9) by taking into account last properties with s in the separate intervals $[\frac{s_\varepsilon}{2}, \infty)$, $(-\frac{s_\varepsilon}{2}, \frac{s_\varepsilon}{2})$ and $(-\infty, -\frac{s_\varepsilon}{2}]$.
In the case $s \in [\frac{s_\varepsilon}{2}, \infty)$ we have

$$|g(s)| \leq \sup_{\xi \in [\frac{s_\varepsilon}{2}, \infty)} |\varphi_0'(\xi)| \leq \frac{\sqrt{\varepsilon}}{2} < \sqrt{\varepsilon}.$$

When $s \in (-\frac{s_\varepsilon}{2}, \frac{s_\varepsilon}{2})$ we have that

$$|g(s)| \leq \frac{|\varphi_0(s) - \varphi_0(s_\varepsilon)|}{s_\varepsilon/2} \leq 2\left(\frac{\sqrt{\varepsilon}}{8} + \frac{\sqrt{\varepsilon}}{8}\right) < \sqrt{\varepsilon}.$$

Finally, in the case $s \in (-\infty, -\frac{s_\varepsilon}{2}]$ there exists $\theta(s) \in (-\infty, -\frac{s_\varepsilon}{2}]$ such that

$$|g_\varepsilon(s)| \;\leq\; \frac{|\varphi_0(s) - \varphi_0(-s_\varepsilon)|}{|s-(-s_\varepsilon)|} + \frac{|\varphi_0(-s_\varepsilon)|}{|s-s_\varepsilon|} + \frac{|\varphi_0(s_\varepsilon)|}{|s-s_\varepsilon|}$$

$$
\begin{aligned}
& \qquad (\text{since } |s-s_\varepsilon| \geq |s-(-s_\varepsilon)|) \\
\leq \ & |\varphi_0'(\theta(s))| + \frac{|\varphi_0(-s_\varepsilon)|}{|s_\varepsilon|} + \frac{|\varphi_0(s_\varepsilon)|}{|s_\varepsilon|} \quad (\text{since } |s-s_\varepsilon| \geq |s_\varepsilon|) \\
\leq \ & \frac{\sqrt{\varepsilon}}{2} + \frac{\sqrt{\varepsilon}}{8} + \frac{\sqrt{\varepsilon}}{8} < \sqrt{\varepsilon}.
\end{aligned}
$$

Let us check that (10) holds under the additional condition (4). Assume $s \in [\frac{s_\varepsilon}{2}, \infty)$: Then, by the mean value theorem, there exists $\theta(s) \in [\frac{s_\varepsilon}{2}, \infty)$ such that

$$
|g_\varepsilon(s)s_\varepsilon| = |\varphi_0'(\theta(s))\theta(s)||\frac{s_\varepsilon}{\theta(s)}| \leq 2|\varphi_0'(\theta(s))\theta(s)| \leq 2C_1
$$

(recall (3)). When $s \in (-\frac{s_\varepsilon}{2}, \frac{s_\varepsilon}{2})$

$$
|g_\varepsilon(s)s_\varepsilon| \leq \frac{|\varphi_0(s)|}{|s-s_\varepsilon|}|s_\varepsilon| + \frac{|\varphi_0(s_\varepsilon)|}{|s-s_\varepsilon|}|s_\varepsilon| \leq 2|\varphi_0(s)| + 2|\varphi_0(s_\varepsilon)| \leq 4C_2
$$

(since $|s-s_\varepsilon| \geq \frac{|s_\varepsilon|}{2}$). Finally, if $s \in (-\infty, -\frac{s_\varepsilon}{2}]$, then there exists $\theta(s) \in (-\infty, -\frac{s_\varepsilon}{2}]$ such that

$$
\begin{aligned}
|g_\varepsilon(s)s_\varepsilon| \ \leq \ & \frac{|\varphi_0(s)-\varphi_0(-s_\varepsilon)|}{|s-(-s_\varepsilon)|}|s_\varepsilon| + \frac{|\varphi_0(-s_\varepsilon)|}{|s-s_\varepsilon|}|s_\varepsilon| + \frac{|\varphi_0(s_\varepsilon)|}{|s-s_\varepsilon|}|s_\varepsilon| \\
& \qquad (\text{since } |s-s_\varepsilon| \geq |s-(-s_\varepsilon)|) \\
\leq \ & |\varphi_0'(\theta(s))\theta(s)|\frac{|s_\varepsilon|}{|\theta(s)|} + |\varphi_0(-s_\varepsilon)| + |\varphi_0(s_\varepsilon)| \quad (\text{since } |s-s_\varepsilon| \geq |s_\varepsilon|) \\
\leq \ & 2C_1 + 2C_2.
\end{aligned}
$$

The following two corollaries give two different sufficient conditions in order to obtain (9), (10):

Corollary 14 *Let us suppose that φ_0 satisfies:*

- *φ_0 is a bounded function (with $\varphi_0(0) = 0$),*
- *there exists $\overline{s} > 0$ such that* $\begin{cases} \varphi_0''(s) \leq 0 & \forall\ s \geq \overline{s}, \\ \varphi_0''(s) \geq 0 & \forall\ s \leq -\overline{s}, \end{cases}$
- *φ_0 is a non-decreasing function in $(-\infty, -\overline{s}] \cup [\overline{s}, +\infty)$.*

Then (10) is satisfied.

Proof. From the assumptions, for all $s \in (\overline{s}, +\infty)$ there exists $\gamma(s) \geq s$ such that

$$
\begin{aligned}
|\varphi_0'(s)s| \leq |\varphi_0'(s)\gamma(s)| &\leq |\varphi_0'(s)(\gamma(s)-\overline{s})| + |\varphi_0'(s)\overline{s}| \\
&= |\varphi_0(\gamma(s)) - \varphi_0(\overline{s})| + |\varphi_0'(s)\overline{s}| \\
&\leq 2 \parallel \varphi_0 \parallel_{L^\infty(I\!R)} + \parallel \varphi_0' \parallel_{L^\infty(\overline{s},+\infty)} |\overline{s}|.
\end{aligned}
$$

In a way similar to this one, for all $s \in (-\infty, -\overline{s})$ there exists $\gamma(s) \leq -s$ such that

$$
|\varphi_0'(s)s| \leq 2 \parallel \varphi_0 \parallel_{L^\infty(I\!R)} + \parallel \varphi_0' \parallel_{L^\infty(-\infty,\overline{s})} |\overline{s}|.
$$

The result is concluded by applying Lemma 4.

Corollary 15 *Let us suppose that φ_0 satisfies:*

- *φ_0 is a bounded function (with $\varphi_0(0) = 0$),*
- *there exists $\overline{s} > 0$ such that* $\begin{cases} \varphi_0''(s) \leq 0 & \forall\ s \geq \overline{s}, \\ \varphi_0''(s) \geq 0 & \forall\ s \leq -\overline{s}. \end{cases}$

Then (10) is satisfied.

Proof. The proof is easily deduced from the above corollary since necessarily φ_0 is a non-decreasing function in $(-\infty, -\overline{s}] \cup [\overline{s}, +\infty)$.

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