RESULTS ON APPROXIMATE CONTROLLABILITY FOR QUASILINEAR DIFFUSION EQUATIONS

Angel Manuel RAMOS

Dept. de Sistemas Informáticos y Programación. Univ. Complutense de Madrid.

Introduction

The study of the approximate controllability property for linear parabolic problems was already treated in [9]. The study of this property for nonlinear parabolic equations seems to have its origins in [8]. Since then, many other results are today available in the literature but, to the best of our knowledge, always restricted to the semilinear case. This paper is a variation of the recent results in [4], [5] by considering the problem

$$
(1)\qquad \begin{cases} y_t - \Delta\varphi(y) = h & \text{in } Q := \Omega \times (0,T), \\ \varphi(y) = 0 & \text{on } \Sigma := \partial\Omega \times (0,T), \\ y(0) = v & \text{in } \Omega, \end{cases}
$$

where Ω is a bounded open subset of $I\!R^N$ of class C^4, $T > 0$, φ is a continuous nondecreasing real function, $h \in L^2(0,T;H^{-1}(\Omega))$ and v represents the control answering the following approximate controllability property: Fixed $\gamma > 0$, we find v such that $\| y(t;v) - y_d \|_{H^{-(1+\gamma)}(\Omega)} \leq \delta$ for a given $\delta > 0$ and for some *desired state* $y_d \in L^2(\Omega)$. With this regularity of the data, $y(v) \in \mathcal{C}([0,T];H^{-1}(\Omega))$ (see [2]). We prove that the approximate controllability holds for a certain class of functions φ. This class of functions includes the one associated to some type of *two phase Stefan problem* ($\varphi(s) = ks$ if $s < 0$ or $s > L$ and $\varphi(s) = 0$ in $[0,L]$, for some constants k, $L > 0$). The result is obtained through a variation of the main theorem of [6] for the vanishing viscosity problem

$$
(2)\qquad \begin{cases} y_t + \varepsilon\Delta^2 y - \Delta\varphi(y) = h & \text{in } Q, \\ y = \Delta y = 0 & \text{on } \Sigma, \\ y(0) = v & \text{in } \Omega. \end{cases}
$$

An approximate controllability result when φ is essentially linear at infinity Let us denote $H^r := H^r(\Omega)$, for every $r \in I\!R$ and $|.|_r$ its associated norm.

Theorem 1 *Let φ be a continuous nondecreasing function with $\varphi(0) = 0$. Assume that there exists some positive constans k, M_1, C_1, C_2 such that*

$$
(3)\qquad \varphi \in \mathcal{C}^1(I\!R\backslash[-M_1,M_1]) \text{ and } |\varphi'(s) - k| \leq \frac{C_1}{|s|} \quad \text{if } |s| > M_1,
$$

$$
(4)\qquad |\varphi(s) - ks| \leq C_2 \quad \forall\, s \in I\!R.
$$

Then, if $\varphi'(s) \geq c > 0$ a.e. $s \in I\!R$ or $h \in L^2(Q)$, then problem (1) satisfies the approximate controllability property in $H^{-(1+\gamma)}$ for any $\gamma > 0$.

Theorem 2 *Assume $\varphi \in \mathcal{C}^0(I\!R)$ satisfying $|\varphi(s)| \leq C(1+|s|)$ for $|s| > M_2$ $(C, M_2 > 0)$. Let $y_d \in H^{-(1+\gamma)}$ and $\delta > 0$. Then, for any $\varepsilon > 0$ there exists a control $v_\varepsilon \in H^0$ such that if $y(t;v)$ is the corresponding solution of (2) we have*

$$
(5)\qquad |y(T;v_\varepsilon) - y_d|_{-(1+\gamma)} < \delta.
$$

If in addition φ satisfies (3) and (4), then there exists a positive constant K, depending on k, C_1, C_2 and M_1 but independent of ε, such that the controls v_ε can be taken satisfying

$$
(6)\qquad |v_\varepsilon|_0 \leq K, \quad \text{for any } \varepsilon > 0.
$$

The proof of the first part of Theorem 2 is an special formulation of the main result (Theorem 1) of [6]. The second part reproduces some of the steps of the proof of Theorem 1 of [6] that here will be merely sketched. The first step consists in proving the approximate controllability for a linearized problem (a posterior fixed point argument will extend the conclusion to the nonlinear problem). We define function $\varphi_0(s) := \varphi(s) - ks$ and linearize φ_0 near a point $s_\varepsilon \in I\!R$ depending on ε. This point will be chosen in a suitable way as the following result shows (proved in [5]):

Lemma 3 *Let $\varphi \in \mathcal{C}^0(I\!R)$ satisfying (3). For any $\varepsilon > 0$ there exists $s_\varepsilon \in I\!R$ such that the function $g_\varepsilon(s) := \frac{\varphi_0(s)-\varphi_0(s_\varepsilon)}{s-s_\varepsilon}$ satisfies $g_\varepsilon \in L^\infty(I\!R) \cap \mathcal{C}(I\!R)$ and*

$$\| g_\varepsilon \|_{L^\infty(I\!R)} \leq \sqrt{\varepsilon}. \tag{7}$$

If φ satisfies (4), then there exists a positive constant K_2, independent of ε, such that

$$|g_\varepsilon(s)s_\varepsilon| \leq K_2, \qquad \text{for any } \varepsilon > 0 \text{ and any } s \in I\!R. \tag{8}$$

Now we return to our linearizing process. Since $\varphi_0(s) = \varphi_0(s_\varepsilon) + g_\varepsilon(s)s - g_\varepsilon(s)s_\varepsilon$, we shall start by replacing the term $\varphi(y)$ by $ky + g_\varepsilon(z)y + \varphi_0(s_\varepsilon) - g_\varepsilon(z)s_\varepsilon$, where z is an arbitrary function in $L^2(Q)$. If we denote $E := H^2 \cap H_0^1$ and $h_\varepsilon(z) := \Delta\left(\varphi_0\left(s_\varepsilon\right) - g_\varepsilon(z)s_\varepsilon\right) = -\Delta\left(g_\varepsilon\left(z\right)s_\varepsilon\right)$, then $h_\varepsilon(z) \in L^\infty(0,T;E')$. Now, we consider the approximate controllability property for the linear problem

$$\begin{cases} y_t + \varepsilon\Delta^2 y - k\Delta y - \Delta\left(g_\varepsilon(z)y\right) = h + h_\varepsilon(z) & \text{in } Q, \\ y = \Delta y = 0 & \text{on } \Sigma, \\ y(0) = u_\varepsilon & \text{in } \Omega. \end{cases} \tag{9}$$

The existence and uniqueness of a solution $y \in \{y \in L^2(0,T;E) : y_t \in L^2(0,T : E'\}$ was proved in [6]. In order to state an approximate controllability result for this problem, we look for the $\inf\limits_{v\in H^0} \left\{\frac{1}{2}\int_\Omega v^2 dx, \quad Y_{\varepsilon,z}(T,v) \in y_d + \delta B_{-(1+\gamma)}\right\}$, where $B_{-(1+\gamma)}$ is the unit ball in $H^{-(1+\gamma)}$ and $Y_{\varepsilon,z}$ is the solution of

$$\begin{cases} Y_t + \varepsilon\Delta^2 Y - k\Delta Y - \Delta\left(g_\varepsilon(z)Y\right) = 0 & \text{in } Q, \\ Y = \Delta Y = 0 & \text{on } \Sigma, \\ Y(0) = u_\varepsilon(z) & \text{in } \Omega. \end{cases} \tag{10}$$

Then, by duality theory, it is easy to prove that the above optimal control problem is equivalent to find the $\inf_{p^0\in H_0^{1+\gamma}} J_\varepsilon(p^0)$, with $J_\varepsilon = J_\varepsilon(\cdot; z, y_d) : H_0^{1+\gamma} \to I\!R$ defined by $J_\varepsilon(p^0) = \frac{1}{2}|p(x,0)|_0^2 + \delta|p^0|_{1+\gamma} - < y_d, p^0 >_{H^{-(1+\gamma)}\times H_0^{1+\gamma}}$. Here p denotes the solution of

$$\begin{cases} -p_t + \varepsilon\Delta^2 p - k\Delta p - g_\varepsilon(z)\Delta p = 0 & \text{in } Q, \\ p = \Delta p = 0 & \text{on } \Sigma, \\ p(T) = p^0 & \text{in } \Omega. \end{cases} \tag{11}$$

The existence and uniqueness of a solution $p \in L^2(0,T;E)$, was proved in [6]. The connection between both minimizing problems is that the solution $u \in H^0$ of the first one is $u = \widehat{p}(x,0)$, where $\widehat{p}$ is the solution of (11) with $\widehat{p}(T) = \widehat{p}_\varepsilon^0$ (minimizer of J_ε). Now, some easy modifications of the arguments given in [7] for a functional similar to this one and the backward uniqueness theorem of Bardos and Tartar [1] allow to show that the functional $J_\varepsilon(\cdot; z, y_d)$ is continuous, strictly convex on $H_0^{1+\gamma}$ and satisfies $\liminf\limits_{|p^0|_{1+\gamma}\to\infty} \frac{J_\varepsilon(p^0;z,y_d)}{|p^0|_{H_0^{1+\gamma}}} \geq \delta$. Then $J_\varepsilon(\cdot; z, y_d)$ attains its minimum at a unique point $\widehat{p}_\varepsilon^0$ in $H_0^{1+\gamma}$. Furthermore, $\widehat{p}_\varepsilon^0 = 0$ iff $|y_d|_{H^{-(1+\gamma)}} \leq \delta$. Now we shall give an approximate controllability result for an special case:

Lemma 4 *Let $z \in L^2(Q)$ and $y_d \in H^{-(1+\gamma)}$. Then, for any $\delta > 0$, the solution Y_ε of problem (10) with $v_\varepsilon \equiv \widehat{p}(x,0)$ satisfies $|y_d - Y_\varepsilon(T)|_{H^{-(1+\gamma)}} \leq \delta$.*

Theorem 5 *Let $z \in L^2(Q)$ and $y_d \in H^{-(1+\gamma)}$. Then there exists $K > 0$ and $u_\varepsilon \in H^0$ such that the associated solution y_ε of (9) satisfies*

$$|y_\varepsilon(T) - y_d|_{-(1+\gamma)} \leq \delta, \tag{12}$$

$$|u_\varepsilon|_0 \leq K, \quad \text{for any } \varepsilon > 0 \text{ and any } z \in L^2(Q). \tag{13}$$

Proof of Theorem 5. We put $y_\varepsilon = L_\varepsilon + Y_\varepsilon$, where $L_\varepsilon = L_\varepsilon(z) \in \mathcal{C}([0,T];H^0)$ satisfies

$$\begin{cases} L_t + \varepsilon\Delta^2 L - k\Delta L - \Delta\left(g_\varepsilon(z)L\right) = h + h_\varepsilon(z) & \text{in } Q, \\ L = \Delta L = 0 & \text{on } \Sigma, \\ L(0) = 0 & \text{in } \Omega \end{cases} \tag{14}$$

and $Y_\varepsilon = Y_\varepsilon(z)$ is taken associated to the approximate controllability problem (10), with desired state $y_d - L_\varepsilon(T)$. We find the control u_ε in the same way as in Lemma 4. Therefore, if $\widehat{p}_\varepsilon$ is the solution of (11) with final data $\mathcal{M}(\varepsilon, z, y_d - L_\varepsilon(T))$, where $\mathcal{M} : (0,R] \times L^2(Q) \times H^{-(1+s)} \longrightarrow H^0$ is defined by $\mathcal{M}(\varepsilon, z, y_d) = \widehat{p}^0_\varepsilon$, then the control $u_\varepsilon := \widehat{p}_\varepsilon(x,0)$ leads to $|Y(T) - \widehat{y}_d|_{-(1+\gamma)} \leq \delta$, where $\widehat{y}_d := y_d - L_\varepsilon(T)$ (if $|\widehat{y}_d|_{-(1+\gamma)} \leq \delta$ it suffices to take $u_\varepsilon \equiv 0$). For the proof of (13) we need the following four lemmas (some of them proved in [5]).

Lemma 6 *Assume (7) and (8). Let $z \in L^2(Q)$. Let $p_0 \in H^0$ be given. Then, if p_ε is the solution of (11), we have $\| p_\varepsilon \|_{\mathcal{C}([0,T];H^0)} \leq e^T |p^0|_0$ for any $\varepsilon > 0$ and any $z \in L^2(Q)$.*

Lemma 7 *Let $\alpha, \varepsilon > 0$ and $\gamma \geq -1/2$. Then the mappings*

$$\begin{array}{rccc} S_\varepsilon : & E & \longrightarrow & H^{2m,1}(Q) \\ & p^0 & \rightarrow & p \end{array} \quad \text{and} \quad \begin{array}{rccc} T_\varepsilon : & H_0^{1+\gamma} & \longrightarrow & L^2(0,T;H^{\frac{5}{2}-\alpha}) \\ & p^0 & \rightarrow & p, \end{array}$$

where p is the solution of (11) associated to p^0, are linear and continuous.

Lemma 8 *If K is a compact subset of $H^{-(1+\gamma)}$ then $\mathcal{M}((0,R] \times L^2(Q) \times K)$ is a bounded subset of $H_0^{1+\gamma}$.*

Proof. If Lemma 8 is not true there exists three sequences $\{z_n\} \subset L^2(Q)$, $\{y_d^n\} \subset K$ and $\{\varepsilon_n\} \subset (0,R]$ such that $|p^0(\varepsilon_n, z_n, y_d^n)|_{1+\gamma} = |\mathcal{M}(\varepsilon_n, z_n, y_d^n)|_{1+\gamma} \to \infty$. Then we can suppose that $g_{\varepsilon_n} \rightharpoonup a$ weak-$*$ in $L^\infty(Q)$, $y_d^n \to y_d$ in $H^{-(1+\gamma)}$ and $\varepsilon_n \to \widetilde{\varepsilon}$ in $I\!R$. To obtain a contradiction, let us prove that for any sequence $\{p_n^0\} \subset H_0^{1+\gamma}$ such that $|p_n^0|_{1+\gamma} \to \infty$

$$\liminf_{n\to\infty} \frac{J_{\varepsilon_n}(p_n^0; z_n, y_d^n)}{|p_n^0|_{1+\gamma}} \geq \delta. \tag{15}$$

If (15) is not true, then there exists $\{p_n^0\} \subset H_0^{1+\gamma}$ such that $|p_n^0|_{1+\gamma} \to \infty$ and

$$\liminf_{n\to\infty} \frac{J_{\varepsilon_n}(p_n^0; z_n, y_d^n)}{|p_n^0|_{1+\gamma}} < \delta. \tag{16}$$

If $\widetilde{p}_n^0 = \frac{p_n^0}{|p_n^0|_{1+\gamma}}$ and $\widetilde{p}_n$ is the solution of (11) associated to z_n, ε_n and $\widetilde{p}_n(T) = \widetilde{p}_n^0$, then

$$|\widetilde{p}_n(x,0)|^2_{1+\gamma} \to 0 \quad \text{as } n \to \infty, \tag{17}$$

because in other case $\liminf_{n\to\infty}\frac{J_{\varepsilon_n}(p_n^0;z_n,y_d^n)}{|p_n^0|_{1+\gamma}} \geq \liminf_{n\to\infty}(\frac{1}{2}|p_n^0|_{1+\gamma}|\widetilde{p}_n(x,0)|_0^2 + \delta - |y_d^n|_{-(1+\gamma)}) = \infty$. We can suppose that there exists $\widetilde{p}^0 \in H_0^{1+\gamma}$ such that $\widetilde{p}_n^0 \rightharpoonup \widetilde{p}^0$ weakly in $H_0^{1+\gamma}$. Then, by Lemma 6, we obtain that $\widetilde{p}_n$ is uniformly bounded in $\mathcal{C}([0,T];H^0)$ and therefore there exists $\widetilde{p} \in L^\infty(0,T;H^0)$ such that $\widetilde{p}_n \rightharpoonup \widetilde{p}$ weakly in $L^2(Q)$. To pass to the limit in the equation of $\widetilde{p}_n$ we distinguish different cases: a) $\widetilde{\varepsilon} > 0$, b) $\widetilde{\varepsilon} = 0$ and $k > 0$ and c) $\widetilde{\varepsilon} = 0$ and $k = 0$. From Lemma 7, we know that we can choose $\overline{p}_0^n \in E$ such that $|\widetilde{p}_n^0 - \overline{p}_n^0|_1 \leq 1$ and

$$\| \widetilde{p}_n - \overline{p}_n \|_{L^2(0,T;H^{\frac{5}{2}-\alpha})} \leq 1. \tag{18}$$

Since $\overline{p}_n \in \{p \in H^{2m,1}(Q) : p, \Delta p \in L^2(0,T;H_0^1)\}$ we can "multiply" by $-\Delta\overline{p}_n$ in the equation of $\overline{p}_n$ and we obtain that there exists K independent of $n \in \mathbb{N}$ such that

$$|\overline{p}_n^0|_1 + \widetilde{\varepsilon} \| \nabla\Delta\overline{p}_n \|_{L^2(Q)} + \| \sqrt{\varepsilon_n}\nabla\Delta\overline{p}_n \|_{L^2(Q)} + k \| \Delta\overline{p}_n \|_{L^2(Q)} \leq K. \tag{19}$$

Now, let us pass to the limit in the three different cases: In case a), from (18) and (19), we deduce that there exists $\widetilde{p} \in L^2(0,T;H^{\frac{5}{2}-\alpha})$ such that $\widetilde{p}_n \rightharpoonup \widetilde{p}$ weakly in $L^2(0,T;H^{\frac{5}{2}-\alpha})$. Then, we deduce that $\frac{\partial\widetilde{p}_n}{\partial t}$ is uniformly bounded in $L^2(0,T;H^{-\frac{3}{2}-\alpha})$. Now, since $H^{\frac{5}{2}-\alpha} \subset H^2 \subset H^{-\frac{3}{2}-\alpha}$ with compact imbeddings, $\{\widetilde{p}_n\}$ is relatively compact in $L^2(0,T;E)$ and so in $\{p \in L^2(0,T;E) :\ p_t \in L^2(0,T;E')\} \subset \mathcal{C}([0,T];H^0)$. Therefore, $g_{\varepsilon_n}(z_n)\Delta\widetilde{p}_n \rightharpoonup a\Delta p$ weakly in $L^2(Q)$, which allows us to pass to the limit and deduce that $\widetilde{p}$ is solution of

$$\begin{cases} -\widetilde{p}_t + \widetilde{\varepsilon}\Delta^2\widetilde{p} - k\Delta\widetilde{p} - a\Delta\widetilde{p} = 0 & \text{in } Q, \\ \widetilde{p} = \Delta\widetilde{p} = 0 & \text{on } \Sigma, \\ \widetilde{p}(T) = \widetilde{p}^0 & \text{in } \Omega. \end{cases}$$

In case b), again from estimates (18) and (19), we deduce that there exists $\widetilde{p} \in L^2(0,T;E)$ such that $\widetilde{p}_n \rightharpoonup \widetilde{p}$ weakly in $L^2(0,T;E)$. Now, since $\widetilde{\varepsilon} = 0$ and g_{ε_n} satisfies (7), $g_{\varepsilon_n}(z_n) \to 0$ in $L^\infty(Q)$. Therefore, $g_{\varepsilon_n}(z_n)\Delta\widetilde{p}_n \rightharpoonup a\Delta p \equiv 0$ weakly in $L^2(Q)$, which allows us to pass to the limit and deduce that $\widetilde{p}$ satisfies $-\widetilde{p}_t - k\Delta\widetilde{p} = 0$ in Q. Then, $\widetilde{p} \in \{p \in L^2(0,T;E) :\ p_t \in L^2(Q)\} \subset \mathcal{C}([0,T];H_0^1)$. Now, to obtain the final data $\widetilde{p}(T)$, for all $u \in L^2(Q)$ we consider $\varphi(u) \in L^2(0,T;E)$ such that $\varphi_t - \Delta\varphi = u$ in Q and $\varphi(0) = 0$ in Ω . Then

$$-\int_\Omega(\widetilde{p}_n^0 - \widetilde{p}(T))\varphi(T)dx + \int_Q(\widetilde{p}_n - \widetilde{p})\varphi_t dxdt + \int_Q \varepsilon_n\Delta\widetilde{p}_n\Delta\varphi dxdt$$

$$-\int_Q k\Delta(\widetilde{p}_n - \widetilde{p})\varphi dxdt - \int_Q g_{\varepsilon_n}(z_n)\Delta\widetilde{p}_n\varphi dxdt = 0 \quad \text{for any } u \in L^2(Q).$$

Passing to the limit, we obtain $\int_\Omega(\widetilde{p}^0 - \widetilde{p}(T))\varphi(T)dx = 0$ for any $u \in L^2(Q)$. Then $\widetilde{p}(T) = \widetilde{p}^0$, since $\{\varphi(T;u) :\ u \in L^2(Q)\}$ is a dense subset of H^0. Thus, $\widetilde{p} \in L^2(0,T;E)$ satisfies

$$\begin{cases} -\widetilde{p}_t - k\Delta\widetilde{p} = 0 & \text{in } Q, \\ \widetilde{p}(T) = \widetilde{p}^0 & \text{in } \Omega. \end{cases}$$

In case c), again from (18) and (19), we deduce that there exists $\widetilde{p} \in L^2(0,T;H_0^1)$ such that $\widetilde{p}_n \rightharpoonup \widetilde{p}$ weakly in $L^2(0,T;H_0^1)$. Hence, $\sqrt{\varepsilon_n}\Delta\widetilde{p}_n \to 0$ in $L^2(0,T;H^{-1})$. Further, also from (18) and (19), we know that $\sqrt{\varepsilon_n}\widetilde{p}_n$ is uniformly bounded in the topology of $L^2(0,T;H^{\frac{5}{2}-\alpha})$. Then, in a way similar to that of the case a), we obtain that $\sqrt{\varepsilon_n}\frac{\partial\widetilde{p}_n}{\partial t}$ is uniformly bounded in $L^2(0,T;H^{-\frac{3}{2}-\alpha})$ and so $\sqrt{\varepsilon_n}\widetilde{p}_n$ is relatively compact in $L^2(0,T;E)$. Then, $\sqrt{\varepsilon_n}\Delta\widetilde{p}_n \to 0$

in $L^2(Q)$ and $g_{\varepsilon_n}(z_n)\Delta\widetilde{p}_n = \frac{g_{\varepsilon_n}(z_n)}{\sqrt{\varepsilon_n}}\sqrt{\varepsilon_n}\Delta\widetilde{p}_n \rightharpoonup 0$ weakly in $L^2(Q)$, which allows us to pass to the limit in the equation satisfied by $\widetilde{p}_n$ and deduce that $\widetilde{p}$ satisfies $-\widetilde{p}_t = 0$ in Q. Then, $\widetilde{p} \in L^2(0,T;H_0^1)$ and $\widetilde{p}(x,t) = \widetilde{p}(x,T)$ for all $t \in [0,T]$. Further, as in case b), we deduce that $\widetilde{p}(T) = \widetilde{p}^0$. Hence, $\widetilde{p} \in L^2(0,T;H_0^1)$ is solution of

$$\begin{cases} -\widetilde{p}_t = 0 & \text{in } Q, \\ \widetilde{p}(T) = \widetilde{p}^0 & \text{in } \Omega. \end{cases}$$

Let us see that $\widetilde{p}(x,0) \equiv 0$: In case a) we have proved that $\widetilde{p}_n \to \widetilde{p}$ in $\mathcal{C}([0,T];H^0)$ and so $\widetilde{p}_n(x,0) \to \widetilde{p}(x,0)$. Then, from (17), we obtain $\widetilde{p} \equiv 0$. In cases b) and c) we have that

$$\int_\Omega \widetilde{p}_n(x,0) - \widetilde{p}(0))\varphi dx - \int_\Omega (\widetilde{p}_n^0 - \widetilde{p}(T))\varphi dx + \int_Q \varepsilon_n \Delta\widetilde{p}_n \Delta\varphi dxdt$$

$$+ \int_Q k\nabla(\widetilde{p}_n - \widetilde{p})\nabla\varphi dxdt - \int_Q g_{\varepsilon_n}(z_n)\Delta\widetilde{p}_n\varphi dxdt = 0 \quad \text{for any } \varphi \in E.$$

Finally, passing to the limit, we obtain that $\widetilde{p}_n(x,0) \rightharpoonup \widetilde{p}(0)$ in the weak topology of H^0. Then, from (17), we obtain that $\widetilde{p}(x,0) \equiv 0$. Now, since $\widetilde{p}$ satisfies a suitable linear parabolic equation for any of the cases a), b) or c), we can apply a backward uniqueness result (see Theorem II.1 of [1]) and deduce that $\widetilde{p} \equiv 0$ in Q. Therefore $\widetilde{p}^0 \equiv 0$ in Ω. Thus, $\liminf_{n\to\infty} \frac{J_{\varepsilon_n}(p_n^0;z_n,y_d^n)}{|p_n^0|_{1+\gamma}} \geq \liminf_{n\to\infty} \left(\delta - < y_d^n, \widetilde{p}_n^0 >_{H^{-(1+\gamma)}\times H_0^{1+\gamma}}\right) = \delta$, which contradicts (16) and proves (15). Finally we point out that $J_{\varepsilon_n}(\widehat{p}^0(\varepsilon_n,z_n,y_d^n);z_n,y_d^n) \leq J_{\varepsilon_n}(0;z_n,y_d^n) = 0$, which is a contradiction with (15) and concludes the result.

Lemma 9 *The solutions $L_\varepsilon(z)$ of (14), with arbitrary $\varepsilon > 0$ (small emough) and $z \in L^2(Q)$, are uniformly bounded in $\mathcal{C}([0,T];H^{-1}) \cap L^2(Q)$.*

Completion of proof of Theorem 5. From Lemma 9 we can deduce that there exists a constant K_3, independent of ε, such that $\| L_\varepsilon(z) \|_{\mathcal{C}([0,T];H^{-1})} \leq K_3$ for any $\varepsilon > 0$ and any $z \in L^2(Q)$. Then $\{L_\varepsilon(z;T)$, for any $\varepsilon > 0$ and any $z \in L^2(Q)\}$ is a relatively compact subset of $H^{-(1+\gamma)}$ for all $\gamma > 0$. Then, applying Lemma 8, there exists a constant K_4, independent of ε, such that, if $\widehat{p}_\varepsilon^0$ is the minimum of $J_\varepsilon(\cdot;z,y_d - L_\varepsilon(T))$, we have $|\widehat{p}_\varepsilon^0|_0 \leq K_4$ for any $\varepsilon > 0$ and any $z \in L^2(Q)$. Lemma 6 implies (13) with $K = e^T K_4$.

Proof of Theorem 2. The first part is similar to that proved in Theorem 1 of [6] by applying Kakutani's fixed point theorem to the operator $\Lambda_\varepsilon : L^2(Q) \to \mathcal{P}(L^2(Q))$ defined by $\Lambda_\varepsilon(z) := \{y_\varepsilon$ satisfying (9), (12), with a control u_ε satisfying $|u_\varepsilon|_0 \leq K\}$, where the constant K depends on ε. Finally, if φ satisfies (3) and (4), then Theorem 5 shows that (13) holds (i.e. K does not depend on ε), which leads to (6).

Proof of Theorem 1. *First step.* Assume $\varphi \in \mathcal{C}^1(I\!\!R)$. For any $\varepsilon > 0$, let v_ε and y_ε be the functions given in Theorem 2. Since the equation of (2) holds in $L^2(0,T;E')$, multiplying by $y_\varepsilon \in L^2(0,T;E)$ we obtain the existence of a constant $C > 0$ independent of ε such that

$$\| y_\varepsilon \|_{L^\infty(0,T;H^0)} + \int_Q \varphi'(y_\varepsilon)|\nabla y_\varepsilon|^2 dxdt \leq C.$$

Therefore we obtain that y_ε is uniformly bounded in $L^\infty(0,T;H^0)$ and by the equation of (2), $(y_\varepsilon)_t$ is uniformly bounded in $L^\infty(0,T;H^{-4})$. Then, since $H^0 \subset H^{-1} \subset H^{-4}$ with compact imbeddings, we have that y_ε is relatively compact in $\mathcal{C}([0,T];H^{-1})$. Further, since φ' is a bounded function, we deduce that there exists a constant $K > 0$ independent of ε such that

$$\int_0^T |\nabla\varphi(y_\varepsilon)|_0^2 dt = \int_Q \varphi'(y_\varepsilon(x,t))\ \varphi'(y_\varepsilon(x,t))|\nabla(y_\varepsilon(x,t))|^2 dxdt < K.$$

Thus, there exist $y \in L^\infty(0,T;H^0)$ and $\zeta \in L^2(0,T;H_0^1)$ (recall that $\varphi(0)=0$) such that $y_\varepsilon \to y$ strongly in $L^2(0,T;H^{-1})$ and $\varphi(y_\varepsilon) \rightharpoonup \zeta$ weakly in $L^2(0,T;H_0^1)$. But the operator $Au := -\Delta\varphi(u)$, $D(A) := \{u \in H^{-1} : \ \varphi(u) \in H_0^1\}$ is a maximal monotone operator on the space H^{-1} (see [2]). Thus, the extension operator $\mathcal{A}$ of A is also a maximal monotone operator on $L^2(0,T;H^{-1})$ (see [3]), Example 2.33). Finally, as any maximal monotone operator is strongly-weakly closed (see [3], Proposition 2.5), we obtain that $\zeta = \varphi(y)$ in $L^2(0,T;H_0^1)$. Moreover, from (6) we have that $v_\varepsilon \rightharpoonup v$ weakly in H^0, with $|v|_0 \le K$. Then we deduce that $y \in \mathcal{C}([0,T];H^{-1})$ is solution of (1). Further, since $y_\varepsilon(T) \to y(T)$ strongly in H^{-1}, we deduce that $|y(T) - y_d|_{-(1+\gamma)} = \lim_{\varepsilon\to 0} |y_\varepsilon(T) - y_d|_{-(1+\gamma)} \le \delta$.
Second step. Let φ as in Theorem 1. We approximate φ by $\varphi_n \in \mathcal{C}^1(I\!\!R)$, φ_n nondecreasing, satisfying (3) and (4) with the same constants k, C_1, C_2 and M_1. Then the respective controls v_n built as in step 1 are uniformly bounded and the conclusion comes from the well-known result expressing the continuous dependence in $\mathcal{C}([0,T];H^{-1})$, on φ, of solutions of (1).

References

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