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NUMERICAL EXPERIMENTS REGARDING THE LOCALIZED CONTROL OF SEMILINEAR PARABOLIC PROBLEMS

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Abstract. *This work deals with some numerical experiences regarding the control of semilinear equations of the type* $y_t - y_{xx} + f(y) = u(t)\delta(x - 1/2)$ *in* $(0, T) \times (0, 1)$ *with Dirichlet and initial auxiliary conditions, where* f *is a* $\mathcal{C}^1$ *nondecreasing real function,* $u(t)$ *is the output control,* $\delta(x-1/2)$ *is the Dirac measure at* $x = 1/2$ *and* $T > 0$ *is (arbitrarily) fixed. Given a target state* y_T *we study the associated approximate controllability problem (given* $\epsilon > 0$*, find* $u \in L^2(0, T)$ *such that* $\|y(T; u) - y_T\|_{L^2(0,1)} \leq \epsilon$*) by passing to the limit (when* $k \to \infty$*) in the penalized optimal control problem (find* u_k *as the minimum of* $J_k(u) = \frac{1}{2}\ \|u\|^2_{L^2(0,T)} + \frac{k}{2}\ \|y(T; u) - y_T\|^2_{L^2(0,1)}$*). In the superlinear case (e.g.* $f(y) = |y|^{n-1}\, y$*,* $n > 1$*) the existence of two obstruction functions* $Y_{\pm\infty}$ *shows that the approximate controllability is only possible if* $Y_{-\infty}(x, T) \leq y_T(x) \leq Y_\infty(x, T)$ *for a.e.* $x \in (0, 1)$*. We carry out some numerical experiences showing that, for a fixed k, the "minimal cost"* $J_k(u)$ *(and the norm of the optimal control* u_k*) for a superlinear function* f *becomes much larger when this condition is not satisfied. We also compare the values of* $J_k(u)$ *(and the norm of the optimal control* u_k*) for a fixed* y_T *associated with two nonlinearities: one sublinear and the other superlinear.*

1 INTRODUCTION

This work deals with some numerical experiences regarding the control of semilinear equations of the type

$$P(u)\begin{cases} y_t - y_{xx} + f(y) = u(t)\delta(x-1/2) & \text{in } (0,T)\times(0,1), \\ y(0,t) = y(1,t) = 0 & \text{for } t\in(0,T), \\ y(x,0) = y_0(x) & \text{in } (0,1), \end{cases}$$

where f is a $\mathcal{C}^1$ nondecreasing real function, $u(t)$ is the output control, $\delta(x-1/2)$ is the Dirac measure at $x = 1/2$, $T > 0$ is (arbitrarily) fixed, and y_0 is a given function (for instance $y_0 \in C^0([0,1])$).

Given a target state y_T (we can assume, for simplicity, that $y_T \in C^0([0,1])$), the associated *approximate controllability problem* consists of, given $\epsilon > 0$, finding $u \in L^2(0,T)$ such that $\|y(T;u) - y_T\|_{L^2(0,1)} \le \epsilon$ where $y(T;u)$ denotes the solution of $P(u)$ at time T.

It is well known (see Fabre-Puel-Zuazua [1] and Díaz-Ramos [2]) that the answer is positive if "f is sublinear at infinity" ($|f(s)| \le M(|s|+1)$ for $|s|$ large). In the "superlinear at infinity" case ($|f(s)| \ge M(|s|^n + 1)$ for $|s|$ large and for some $n > 1$), the answer is negative. This type of negative results can be proved in different ways: via an energy argument (see, e.g. the case of control on the Neumann boundary condition, due to A. Bamberger, in Henry [3]) or via some pointwise obstruction phenomenon (see Díaz-Ramos [2] for problem $P(u)$). In fact, in some of the above references the control u is localized not at a point but in a bounded open set $\omega \subset (0,1)$. However, the adaptation to our setting is a routine matter.

It is also well known that in the sublinear case, the solution to the controllability problem can be obtained by passing to the limit (as $k \to \infty$) in the *penalized optimal control problem* in which the control u_k is found as the minimum of the functional

$$J_k(u) = \frac{1}{2}\|u\|^2_{L^2(0,T)} + \frac{k}{2}\|y(T;u) - y_T\|^2_{L^2(0,1)}$$

(see Lions [4] for the linear case and Fernández-Zuazua [5] for the semilinear case).

For the superlinear case the approximate controllability was obtained in Díaz [6] under the assumption

$$Y_{-\infty}(x,T) \le y_T(x) \le Y_{\infty}(x,T) \text{ for a.e. } x \in (0,1), \tag{1}$$

where $Y_{\pm\infty}$ are the "largest solutions". In our case, $Y_{\pm\infty}$ are the solutions of the problem

$$P(\pm\infty)\begin{cases} y_t - y_{xx} + f(y) = 0 & \text{in } (0,T)\times((0,1/2)\cup(1/2,1))), \\ y(0,t) = 0,\ y(1/2,t) = \pm\infty,\ y(1,t) = 0 & \text{for } t\in(0,T), \\ y(x,0) = y_0(x) & \text{in } (0,1) \end{cases}$$

(the existence of such large solutions requires f to be superlinear). Notice that the special case of $y_T \equiv 0$ is included (see, e.g., Fernández-Cara [7], for other results on null controllability).

Recently, some results on the approximate controllability of the projections on finite dimensional subspaces were obtained by Khapalov [8] (see also its references).

The main goal of this work is to carry out some numerical experiences on the *penalized optimal control problem* for difference target states y_T and different nonlinear terms $f(y)$. We illustrate that, for a fixed k, the "minimal cost" $J_k(u)$ (and the norm of the optimal control u_k) for a superlinear function f becomes much larger when (1) is not satisfied (see numerical test # 1 and # 2 below). We also compare the values of $J_k(u)$, the norm of the optimal control u_k, and $\|y(T;u) - y_T\|$, for a fixed y_T, associated with two nonlinearities: one sublinear ($f(y) = y^3$) and the other superlinear ($f(y) = arctg(y)$).

2 PROBLEM FORMULATION.

Let us consider a given target function $y_T \in L^2(0,1)$. We define the control space as $\mathcal{U} = L^2(0,T)$. The goal is to find a control $u \in \mathcal{U}$ so that $y(T)$ is *close* to y_T at a minimal cost for the control, where $y(x,t)$ is (unique) solution of $P(u)$). We recall that a *variational* formulation of $P(u)$ is provided by $y(t) \in L^2(0,T;H_0^1(0,1)) \cap H^1(0,T;H^{-1}(0,1))$ satisfying

$$\begin{cases} f(y) \in L^2(0,T;L^2_\rho(0,1)), \text{ and } \forall z \in L^2(0,T;H_0^1) \\ \displaystyle\int_0^T < y_t, z >_{V_0' \times V_0} dt + \int_0^T \int_0^1 y_x z_x dx dt + \int_0^T \int_0^1 f(y) z dx dt = \int_0^T u(t) z(1/2,t) dt, \\ y(x,0) = y_0(x), \end{cases}$$

where $L^2_\rho(0,1) = \{w \text{ such that } \int_0^1 \rho(x) w(x)^2 dx < +\infty\}$, with $\rho(x) = dist(x, \{0\} \cup \{1\})$. Notice that $\int_0^T \int_0^1 f(y) z dx dt$ is well-defined for $z \in L^2(0,T;H_0^1(0,1))$.

To do this, for every $k \in I\!N$, we define the *cost function* J_k by

$$J_k(v) = \frac{1}{2} \parallel v \parallel^2_{\mathcal{U}} + \frac{k}{2} \parallel y(T) - y_T \parallel^2_{L^2(0,1)}, \ \forall\, v \in \mathcal{U}.$$

The control problem is then

$$(\mathcal{CP}_k) \begin{cases} \text{Find } u_k \in \mathcal{U}, \text{ such that} \\ J_k(u_k) \leq J_k(v), \quad \forall v \in \mathcal{U}. \end{cases}$$

A common way to solve this problem is to solve the problem

$$J_k'(u) = 0,$$

where J_k' denotes the Gateaux differential of J_k.

Now, it is easy to prove (see, e.g., [9], [10] and [11] for the case of the Burgers equation) that

$$J_k'(v) = v + p(1/2, \cdot),$$

i.e.,

$$(J_k'(v), w) = \int_0^T (v(t) + p(1/2, t))w(t)dt, \ \forall \ w \in \mathcal{U},$$

where p is the solution of the adjoint system

$$\begin{cases} -p_t - p_{xx} + f'(y)p = 0 & \text{in } Q, \\ p(0,t) = p(1,t) = 0 & \text{in } (0,T), \\ p(T) = k(y(T;v) - y_T) & \text{in } (0,1) \end{cases}$$

and $(\cdot, \cdot)$ denotes the scalar product in $L^2(0,1)$ defined by $(u,v) = \int_0^1 u(t)v(t)dt$.

3 TIME DISCRETIZATION.

We consider the time discretization step Δt, defined by $\Delta t = T/N$, where N is a positive integer. Then, if $t^n = n\Delta t$, we have $0 < t^1 < t^2 < \cdots < t^N = T$. We approximate then problem $(\mathcal{CP})$ by the following finite-dimensional minimization problem:

$$(\mathcal{CP}_k)^{\Delta t} \begin{cases} \text{Find } u^{\Delta t} = \{u^n\}_{n=1}^N \in \mathcal{U}^{\Delta t}, \text{ such that} \\ J_k^{\Delta t}(u) \leq J_k^{\Delta t}(v), \ \forall v = \{v^n\}_{n=1}^N \in \mathcal{U}^{\Delta t}, \end{cases}$$

with the *discrete control space* $\mathcal{U}^{\Delta t} = I\!R^N$ and

$$J_k^{\Delta t}(v) = \frac{\Delta t}{2} \sum_{n=1}^N |v^n|^2 + \frac{k}{2} \Big((1-\theta) \parallel y^{N-1} - y_T \parallel^2_{L^2(0,1)} + \theta \parallel y^N - y_T \parallel^2_{L^2(0,1)} \Big),$$

where $\theta \in (0,1]$ and $\{y^n\}_{n=1}^N$ is defined from the solution of the following *second order accurate time discretization* scheme of problem $(P(u))$:

$$y^0 = y_0,$$

$$\begin{cases} \dfrac{y^1 - y^0}{\Delta t} - \dfrac{\partial^2}{\partial x^2}(\dfrac{2}{3}y^1 + \dfrac{1}{3}y^0) + f(y^1) = \dfrac{2}{3}v^1 \delta(x - 1/2) \quad \text{in } (0,1), \\ y^1(0) = y^1(1) = 0, \end{cases}$$

and for $n \geq 2$,

$$\begin{cases} \dfrac{\frac{3}{2}y^n - 2y^{n-1} + \frac{1}{2}y^{n-2}}{\Delta t} - \dfrac{\partial^2}{\partial x^2} y^n + f(y^n) = v^n \delta(x - 1/2), \\ y^n(0) = y^n(1) = 0. \end{cases}$$

Remark. We have used an *implicit* scheme. We could also have used a *semi-implicit* scheme, treating *implicitly* the diffusion term and *explicitly* the reaction term (as done in [9], [10] and [11] for the case of the diffusion and advection terms of the Burgers equation), but this choice implies a problem-dependent limit on the size of Δt, in particular for reaction-dominated problem as the one we are treating.

4 FULL DISCRETIZATION.

We consider the space discretization step h, defined by $h = 1/I$, where I is a positive integer. Then, if $x_i = (i-1)h$, we have $0 = x_1 < x_2 < \cdots < x_I < x_{I+1} = 1$. We approximate $H_0^1(0,1)$ by

$$V_{0h} = \{z \in \mathcal{C}^0[0,1] : z(0) = z(1) = 0, z|_{(x_i,x_{i+1})} \in P_1, i = 1, \cdots, I\},$$

where P_1 is the space of the polynomials of degree ≤ 1. We define a_h by

$$a_h(y,z) = \int_0^1 y_x z_x dx.$$

We approximate then problem $(\mathcal{CP}_k)$ by the following finite-dimensional minimization problem:

$$(\mathcal{CP}_k)_h^{\Delta t} \begin{cases} \text{Find } u_h^{\Delta t} = \{u^n\}_{n=1}^N \in \mathcal{U}^{\Delta t}, \text{ such that} \\ J_{k,h}^{\Delta t}(u_h^{\Delta t}) \leq J_{k,h}^{\Delta t}(v), \ \forall v = \{v^n\}_{n=1}^N \in \mathcal{U}^{\Delta t}; \end{cases}$$

with

$$J_{k,h}^{\Delta t}(v) = \frac{\Delta t}{2} \sum_{n=1}^{N} |v^n|^2 + \frac{k}{2} \Big((1-\theta) \parallel y_h^{N-1} - y_T \parallel_{L^2(0,1)}^2 + \theta \parallel y_h^N - y_T \parallel_{L^2(0,1)}^2 \Big),$$

where $\theta \in (0,1]$ and $\{y_h^n\}_{n=1}^N$ is defined from the solution of the following full discretization of problem $(P(u))$:

$$\begin{cases} y_h^0 \in V_{0h}, \\ (y_h^0, z) = (y_0, z), \quad \forall z \in V_{0h}; \end{cases}$$

$$\begin{cases} y_h^1 \in V_{0h}, \\ \left(\dfrac{y_h^1 - y_h^0}{\Delta t}, z \right) + a_h(\dfrac{2}{3} y_h^1 + \dfrac{1}{3} y_h^0, z) + (f(y_h^1), z) = \dfrac{2}{3} v^1 z(1/2), \qquad \forall z \in V_{0h}; \end{cases}$$

and for $n \geq 2$,

$$\begin{cases} y_h^n \in V_{0h}, \\ \left(\dfrac{\frac{3}{2} y_h^n - 2 y_h^{n-1} + \frac{1}{2} y_h^{n-2}}{\Delta t}, z \right) + a_h(y_h^n, z) + (f(y_h^n), z) = v^n z(1/2), \qquad \forall z \in V_{0h}. \end{cases}$$

As for the continuous case, to solve problem $(\mathcal{CP})_h^{\Delta t}$, we look for the solution $u_h^{\Delta t}$ of

$$\frac{\partial J_h^{\Delta t}}{\partial v}(u_h^{\Delta t}) = 0.$$

Computing $\frac{\partial J_h^{\Delta t}}{\partial v}(v)$ is more complicated than in the continuous case but, following the same approach, we can show that

$$< \frac{\partial J_{k,h}^{\Delta t}}{\partial v}(v), w >= \Delta t \sum_{n=1}^{N} (v^n + p^n(1/2))w^n,$$

where $\{p_h^n\}_{n=1}^{N+2}$ is the solution of

$$\begin{cases} p_h^{N+2} \in V_{0h}, \\ (p_h^{N+2}, z) = -8l(1-\theta)\int_0^1 (y_h^{N-1} - y_T)zdx - 2l\theta\int_0^1 (y_h^N - y_T)zdx, \quad \forall z \in V_{0h}; \end{cases}$$

$$\begin{cases} p_h^{N+1} \in V_{0h}, \\ (p_h^{N+1}, z) = -2l(1-\theta)\int_0^1 (y_h^{N-1} - y_T)zdx, \quad \forall z \in V_{0h}; \end{cases}$$

and for $n = N, \cdots, 1$,

$$\begin{cases} p_h^n \in V_{0h}, \\ \left(\frac{\frac{3}{2}p_h^n - 2p_h^{n+1} + \frac{1}{2}p_h^{n+2}}{\Delta t}, z \right) + a_h(p_h^n, z) + (f'(y_h^n)p_h^n, z) = 0, \quad \forall z \in V_{0h}. \end{cases}$$

Now, once we know how to compute $\frac{\partial J_h^{\Delta t}}{\partial v}(v)$, we use a *quasi-Newton method* à la BFGS (see, e.g., [12] for BFGS algorithms and their implementations) to compute the solution of the fully discrete control problem $(\mathcal{CP})_h^{\Delta t}$.

5 NUMERICAL EXPERIMENTS.

In all the tests considered we have taken $T = 1$, $I = 512$, $N = 1000$, $k = 12$, $a = 1/2$ and $y_0 = 0$ (notice that this implies $y(x,t;0) \equiv 0$). We use, for our algorithm, $\theta = 3/2$. Further, if v_p $(p = 1, 2, \cdots)$ is the sequence of controls we get from the BFGS algorithm, we use the following stopping criteria: We stop iterating after step p if either

$$\| \frac{\partial J_h^{\Delta t}}{\partial v}(u_p) \|_\infty \leq 10^{-5}$$

or

$$\frac{J_h^{\Delta t}(u_{p-1}) - J_h^{\Delta t}(u_p)}{\max\{|J_h^{\Delta t}(u_{p-1})|, |J_h^{\Delta t}(u_p)|, 1\}} \leq 2 \cdot 10^{-9}.$$

We have considered 5 different tests, depending on the target function.

5.1 Test 1: $y_T \equiv 3$.

On Figure 1 (resp., 2) we have shown the super-solution $Y_\infty(T)$ (...), the target function y_T (- - -), and the controlled state solution $y(T)$ (—) corresponding to the nonlinearity $f(y) = y^3$ (resp. $f(y) = arctg(y)$). The corresponding control functions have been represented on Figures 3 and 4.

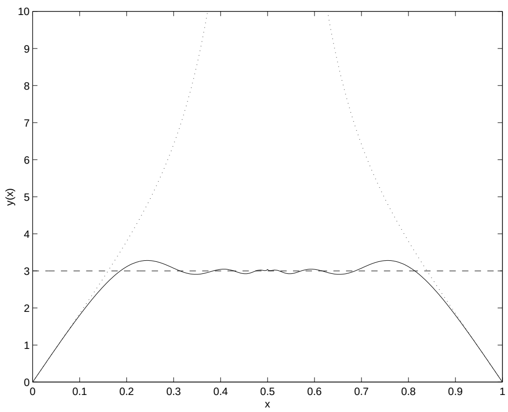

Figure 1: The target function (- -), the large solution (..) and controlled (–) states at time T, for $f(y) = y^3$.

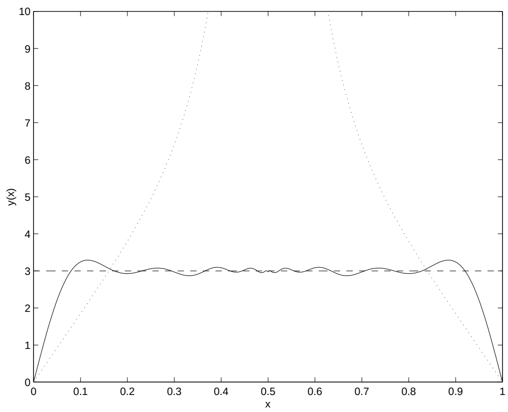

Figure 2: The target function (- -) and the controlled (–) state at time T, for $f(y) = arctg(y)$.

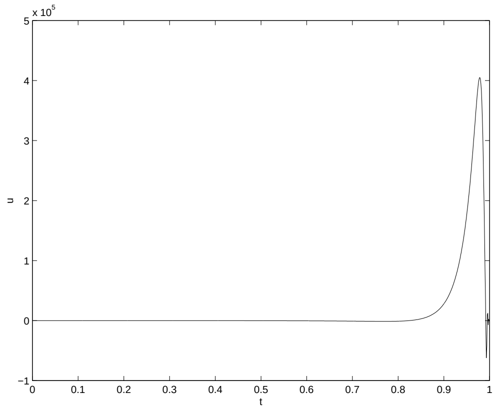

Figure 3: The computed optimal control for $f(y) = y^3$.

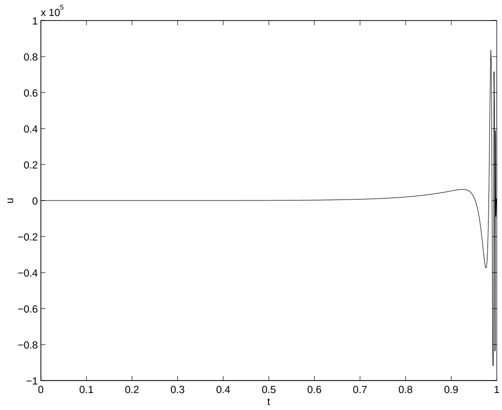

Figure 4: The computed optimal control for $f(y) = arctg(y)$.

On Figure 5 (resp., 6) we have shown the graphic of $\| y(t) - y_T \|_{L^2(0,1)}$, $t \in [0, 1]$, when $f(y) = y^3$ (resp. $f(y) = arctg(y)$).

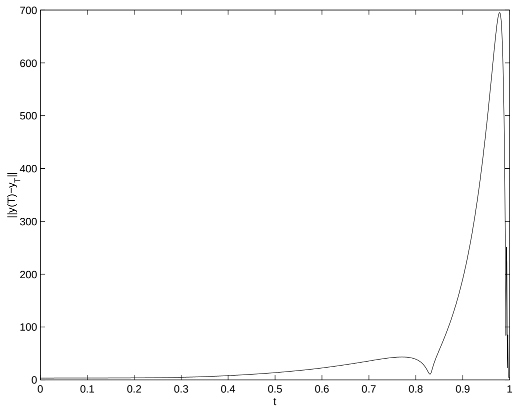

Figure 5: $\| y(t) - y_T \|$, for $f(y) = y^3$.

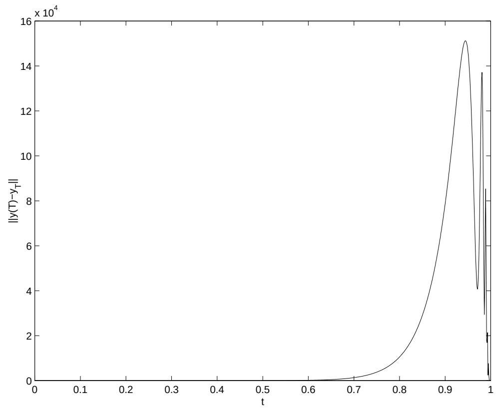

Figure 6: $\| y(t) - y_T \|$, for $f(y) = arctg(y)$.

On Figure 7 (resp., 8) we have shown a 3D graphic of $y(x,t)$ when $t \in [0.95, 1]$ and $f(y) = y^3$ (resp. $f(y) = arctg(y)$).

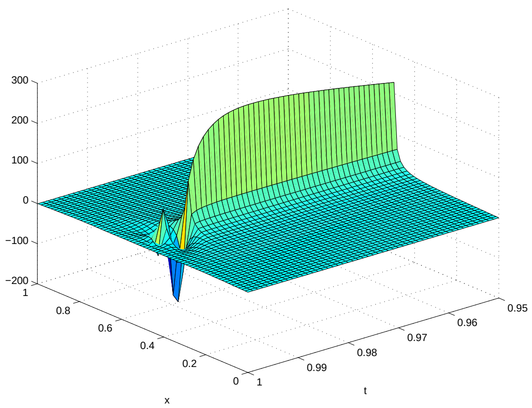

Figure 7: Graphic of $y(x,t)$ ($t \in [0.95, 1]$), for $f(y) = y^3$.

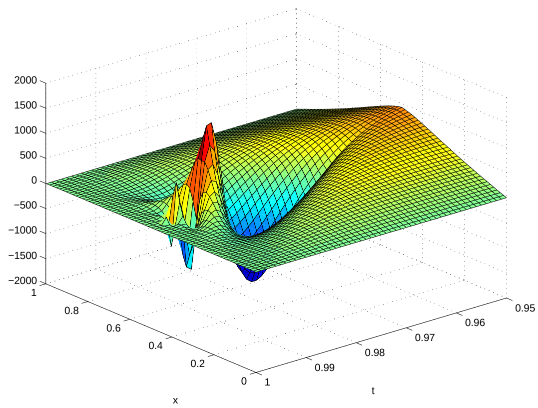

Figure 8: Graphic of $y(x,t)$ ($t \in [0.95, 1]$), for $f(y) = arctg(y)$.

In Table 1 we give some further results about our solutions. The norms considered in all the tables of the present article refer to the L^2-norm of the discrete entries. One of the entries of the table shows the number of discrete parabolic equations the BFGS algorithm has needed to solve (a half of this number corresponds to the nonlinear state system and the other half corresponds to the linear adjoint system). Further, $y(v;T)$ represents the solution at time T, associated with the control v ($y(0,T)$ represents the solution without control, at time T).

	$f(y) = y^3$	$f(y) = arctg(y)$
$\parallel y(u;T) - y_T \parallel$	0.5034	0.1994
$\parallel y(0;T) - y_T \parallel \ (=\parallel y_T \parallel)$	3	3
$\parallel u \parallel$	$6.4942 \cdot 10^4$	$8.5729 \cdot 10^3$
$J(0)$	$4.5 \cdot 10^{12}$	$4.5 \cdot 10^{12}$
$J(u)$	$5.085 \cdot 10^{11}$	$1.930 \cdot 10^{11}$

Table 1: Computational results.

5.2 Test 2: $y_T \equiv 10$.

On Figure 9 (resp., 10) we have shown the super-solution $Y_\infty(T)$ (...), the target function y_T (- - -), and the controlled state solution $y(T)$ (—) corresponding to the nonlinearity $f(y) = y^3$ (resp. $f(y) = arctg(y)$). The corresponding control functions have been represented on Figures 11 and 12.

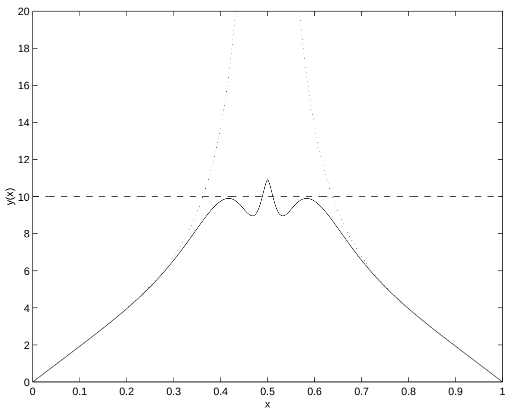

Figure 9: The target function (- -), the large solution (..) and controlled (–) states at time T, for $f(y) = y^3$.

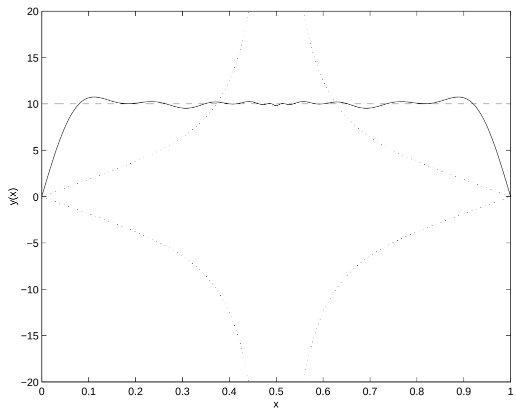

Figure 10: The target function (- -) and the controlled (–) state at time T, for $f(y) = arctg(y)$.

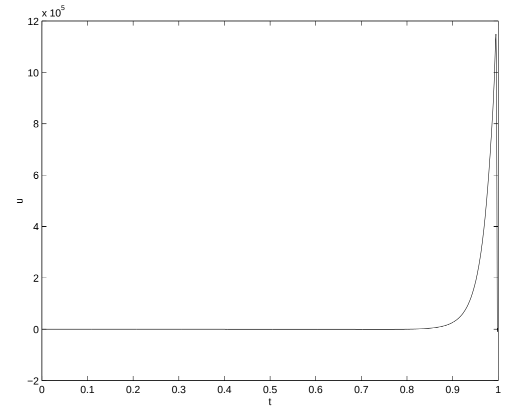

Figure 11: The computed optimal control for $f(y) = y^3$.

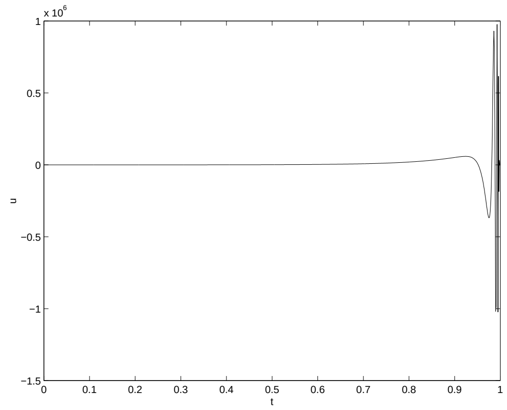

Figure 12: The computed optimal control for $f(y) = arctg(y)$.

On Figure 13 (resp., 14) we have shown the graphic of $\| \, y(t) - y_T \, \|_{L^2(0,1)}$, $t \in [0,1]$, when $f(y) = y^3$ (resp. $f(y) = arctg(y)$).

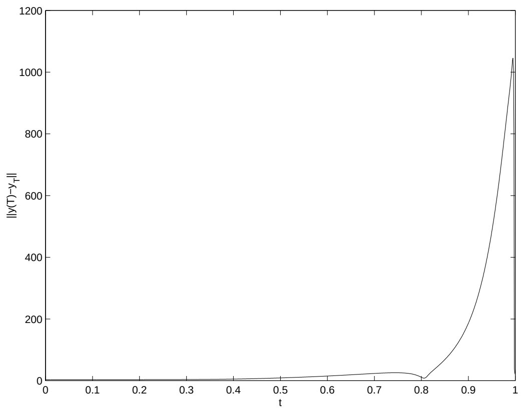

Figure 13: $\| \, y(t) - y_T \, \|$, for $f(y) = y^3$.

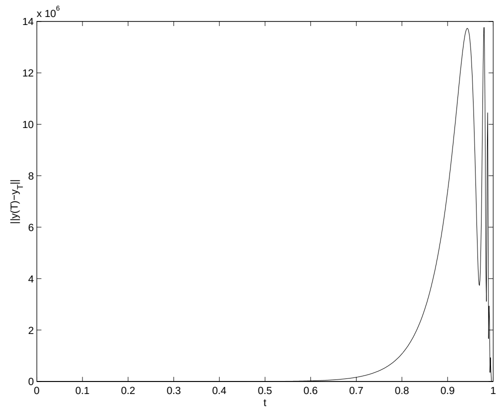

Figure 14: $\| \, y(t) - y_T \, \|$, for $f(y) = arctg(y)$.

On Figure 15 (resp., 16) we have shown a 3D graphic of $y(x,t)$ when $t \in [0.9, 1]$ and $f(y) = y^3$ (resp. $f(y) = arctg(y)$).

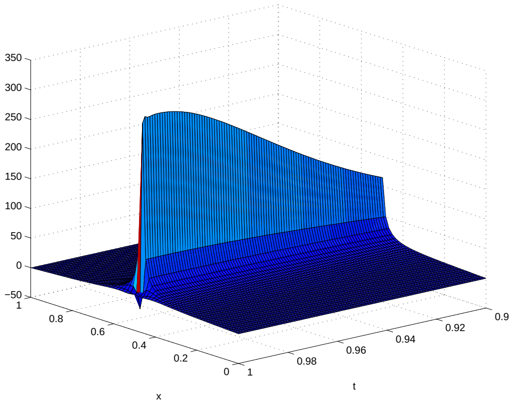

Figure 15: Graphic of $y(x,t)$ $(t \in [0.9, 1])$, for $f(y) = y^3$.

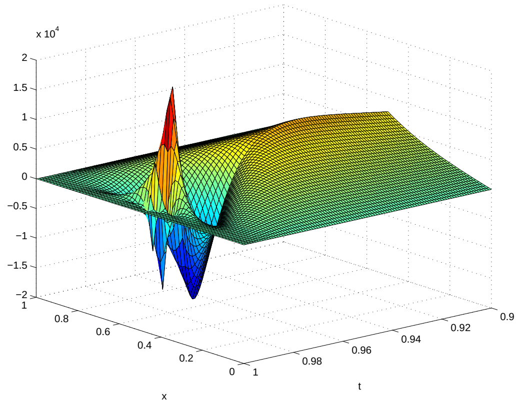

Figure 16: Graphic of $y(x,t)$ $(t \in [0.9, 1])$, for $f(y) = arctg(y)$.

In Table 2 we give some further results about our solutions.

	$f(y) = y^3$	$f(y) = arctg(y)$
$\parallel y(u;T) - y_T \parallel$	16.0711	2.1205
$\parallel y(0;T) - y_T \parallel \ (=\parallel y_T \parallel)$	10	10
$\parallel u \parallel$	$1.3468 \cdot 10^5$	$9.5798 \cdot 10^4$
$J(0)$	$5 \cdot 10^{13}$	$5 \cdot 10^{13}$
$J(u)$	$1.616 \cdot 10^{13}$	$2.145 \cdot 10^{12}$

Table 2: Computational results.

5.3 Test 3:

$$y_T(x) = \begin{cases} 0 & \text{if } x \in (0, 0.1) \cup (0.9, 1), \\ 1 & \text{if } x \in (0.1, 0.2) \cup (0.8, 0.9), \\ 2 & \text{if } x \in (0.2, 0.3) \cup (0.7, 0.8), \\ 6 & \text{if } x \in (0.3, 0.7). \end{cases}$$

On Figure 17 (resp., 18) we have shown the super-solutions $Y_{\pm\infty}(T)$ (...), the target function y_T (- - -), and the controlled state solution $y(T)$ (—) corresponding to the nonlinearity $f(y) = y^3$ (resp. $f(y) = arctg(y)$). The corresponding control functions have been represented on Figures 19 and 20.

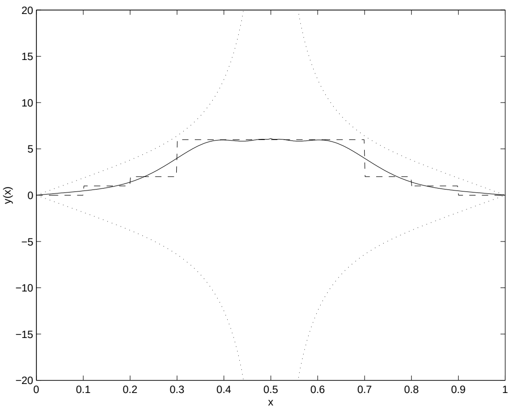

Figure 17: The target function (- -), the large solutions (..) and controlled (–) states at time T, for $f(y) = y^3$.

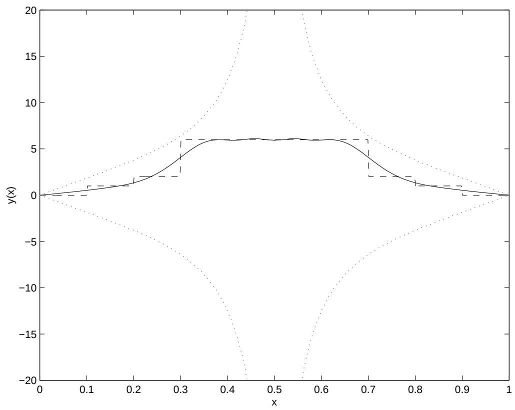

Figure 18: The target function (- -) and the controlled (–) state at time T, for $f(y) = arctg(y)$.

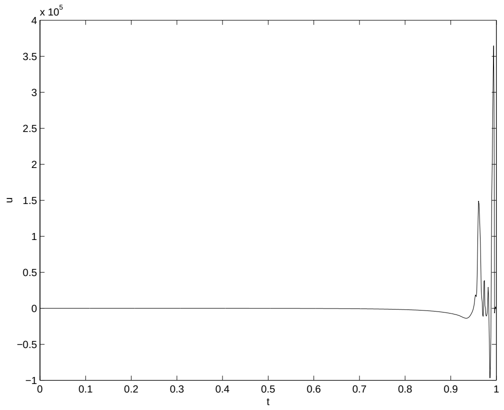

Figure 19: The computed optimal control for $f(y) = y^3$.

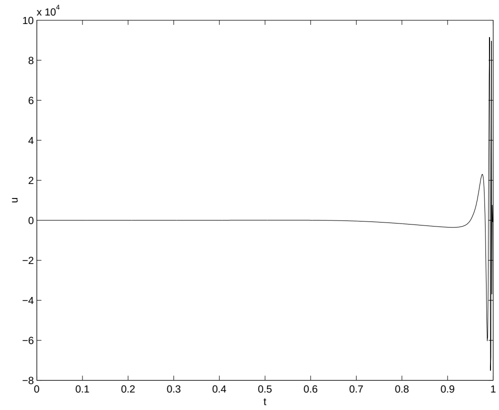

Figure 20: The computed optimal control for $f(y) = arctg(y)$.

On Figure 21 (resp., 22) we have shown the graphic of $\| y(t) - y_T \|_{L^2(0,1)}$, $t \in [0, 1]$, when $f(y) = y^3$ (resp. $f(y) = arctg(y)$).

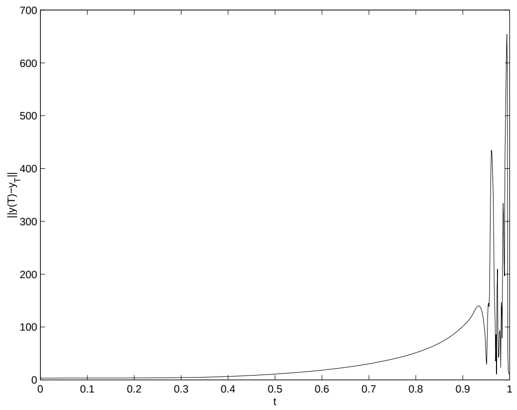

Figure 21: $\| y(t) - y_T \|$, for $f(y) = y^3$.

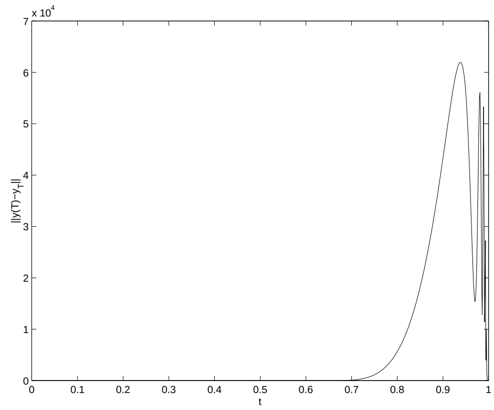

Figure 22: $\| y(t) - y_T \|$, for $f(y) = arctg(y)$.

On Figure 23 (resp., 24) we have shown a 3D graphic of $y(x,t)$ when $t \in [0.95, 1]$ and $f(y) = y^3$ (resp. $f(y) = arctg(y)$).

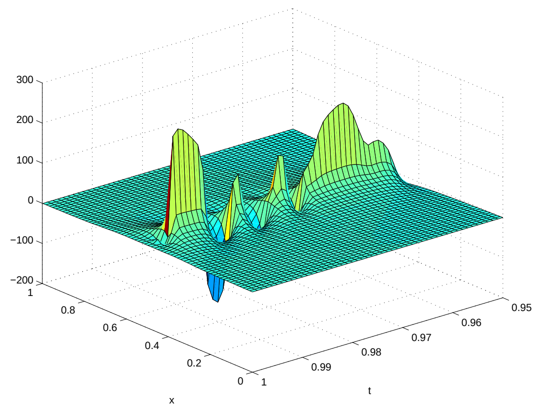

Figure 23: Graphic of $y(x,t)$ ($t \in [0.95, 1]$), for $f(y) = y^3$.

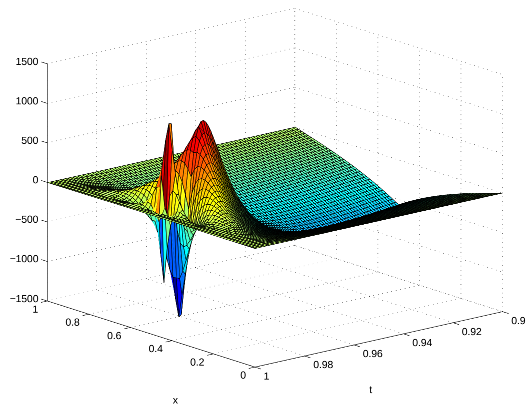

Figure 24: Graphic of $y(x,t)$ ($t \in [0.95, 1]$), for $f(y) = arctg(y)$.

In order to show a 3D better view of the final (in time) behavior of the solution, on Figure 25 (resp., 26) we have shown a 3D graphic of $y(x,t)$ when $t \in [t^{N-4}, 1]$ and $f(y) = y^3$ (resp. when $t \in [t^{N-2}, 1]$ and $f(y) = arctg(y)$).

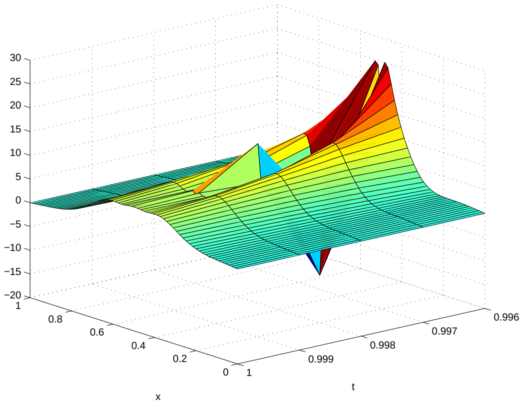

Figure 25: Graphic of $y(x,t)$ $(t \in [t^{N-4}, 1])$, for $f(y) = y^3$.

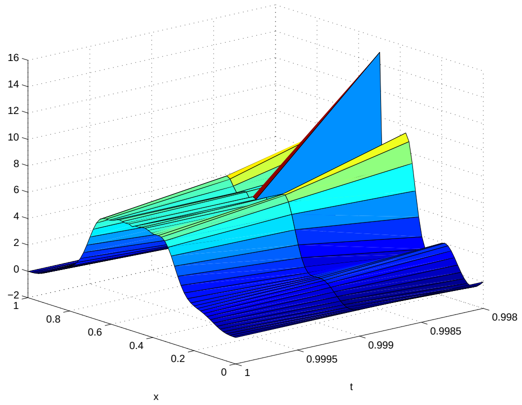

Figure 26: Graphic of $y(x,t)$ $(t \in [t^{N-2}, 1])$, for $f(y) = arctg(y)$.

In Table 3 we give some further results about our solutions.

	$f(y) = y^3$	$f(y) = arctg(y)$
$\parallel y(u;T) - y_T \parallel$	0.1898	0.1435
$\parallel y(0;T) - y_T \parallel \ \ (=\parallel y_T \parallel)$	3.9240	3.9240
$\parallel u \parallel$	$2.4651 \cdot 10^4$	$7.3360 \cdot 10^3$
$J(0)$	$7.699 \cdot 10^{12}$	$7.699 \cdot 10^{12}$
$J(u)$	$1.930 \cdot 10^{11}$	$1.434 \cdot 10^{11}$

Table 3: Computational results.

6 CONCLUSIONS AND CONJECTURES.

Our numerical results show that, as we had theoretically showed in [2], when we take a superlinear at infinity nonlinearity (e.g. $f(y) = y^3$) and the target function y_T does not satisfy (1), then the approximate controllability property can not be obtained.

We also (numerically) show this obstruction phenomenon does not appear when f is sublinear at infinity (e.g. $f(y) = arctg(y)$), which is consistent with the theoretical approximate controllability results obtained in [1] and [2].

For the superlinear case, our experiments also show that, as theoretically proved in [6] (the proof of that paper is for a control localized in an open subset of $(0,1)$, but it can be adapted to the pointwise control case), when the target function satisfies (1), the controllability property holds. The above mentioned proof in [6] is not constructive and follows a scheme different of the successive *penalized optimal control problems* used in this paper.

A remarkable fact is that, in both superlinear and sublinear cases, the solution y oscillates very fast for times $t \in (T - \delta, T)$, getting away from the target state y_T and finally approaching y_T at time T. This is an unstable phenomenon typical of optimal control problems of controllability type, in contrast with the non-oscillating behavior of the solution of optimal control problems of stabilization type (see, e.g. Glowinski-Ramos [13]).

Finally, we point out that the optimal controls obtained in our experiments follow the typical pattern of remaining *close* to zero until the last part of the time interval.

The above numerical experiences lead us to formulate the following conjectures:

A. A theoretical proof of the approximate controllability property for problems with superlinear at infinity nonlinearities and target states satisfying (1) can be also obtained in a constructive way, by means of the *penalized optimal control problems* $(\mathcal{CP}_k)$ used in this paper.

B. Fixed a target function y_T satisfying (1), the *cost* (in terms of the norm of the controls) to approximate this function is, in general, much bigger for superlinear cases than for sublinear cases. However, this result can be false if y_T is *small* enough. For instance, when $f(y) = |y|^{p-1}y$, the cost to approximate y_T is much bigger when $p > 1$, except for target functions satisfying $|y_T(x)| \leq 1$. This conjecture is exactly the opposite of the results obtained in Díaz-Lions [14] for the case of initial value control problems with nonlinearities of the type $f(y) = -y^3$.

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REFERENCES

[1] C. Fabre, J.P. Puel and E. Zuazua, "Contrôlabilité approchée de l'équation de la chaleur semi-linéaire", *C.R. Acad. Sci. Paris*, t. 315, Serie I, 807–812 (1992).

[2] J.I. Díaz and A.M. Ramos, "Positive and Negative approximate controllability results for semilinear parabolic equations", *Rev. Real Academia de Ciencias Exactas, Físicas y Naturales, de Madrid*, Tomo LXXXIX, 11–30 (1995).

[3] J. Henry, *Etude de la contrôlabilité de certaines équations paraboliques*, Tèse d'Etat, Université Paris VI, (1978).

[4] J.L. Lions, "Remarques sur la contrôlabilité approchée". In *Proceedings of Jornadas Hispano-Francesas sobre control de sistemas distribuidos*, Univ. de Málaga Eds., 77–87 (1991).

[5] L.A. Fernández and E. Zuazua, "Approximate Controllability of the Semilinear Heat Equation via Optimal Control", Preprint de la Univ. de Cantabria (1996).

[6] J.I. Díaz, "Controllability and Obstruction for Some Nonlinear Parabolic Problems in Climatology". In *Modelado de sistemas en Oceanografía, Climatología y Ciencias Medio-Ambientales*, Univ. de Málaga Eds., 43–58 (1994).

[7] E. Fernández-Cara, "Null Controllability for Semilinear Heat Equation", *ESSAIM: Control, Optimization and Calculus of Variations*, 87–103 (1997).

[8] A.Y. Khapalov, "Global controllability properties for the semilinear heat equation with superlinear terms", *Revista Matemática Complutense*, **12**, 511–535 (1999).

[9] M. Berggren, R. Glowinski and J.L. Lions, "A Computational Approach to Controllability Issues for Flow-Related Models. (I): Pointwise Control of the Viscous Burgers Equation", *International Journal of Computational Fluid Dynamics*, **7**, 237–252 (1996).

[10] R. Glowinski and J.L. Lions, J.L., "Exact and approximate controllability for distributed parameter systems, Part II", *Acta Numerica*, 159–333 (1995).

[11] A.M. Ramos, R. Glowinski and J. Periaux, "Pointwise Control of the Burgers Equation and Related Nash Equilibrium Problems: A Computational Approach". To appear.

[12] D.C. Liu and J. Nocedal, "On the limited memory BFGS method for large scale optimization", *Mathematical Programming*, **45**, 503–528 (1989).

[13] R. Glowinski and A.M. Ramos, "Control and Stabilization of the Cahn-Hilliard Equation. A Numerical Approach". In *Actas del XVI C.E.D.Y.A./ VI Congreso de Matemática Aplicada*, Vol. III (Anexo), Servicio de Publicaciones de la Universidad de Las Palmas de Gran Canaria. Eds.: R. Montenegro, G. Montero y G. Winter, (1999).

[14] J.I. Díaz and J.L. Lions, "On the Approximate Controllability for some Explosive Parabolic Problems". In *International Series of Numerical Mathematics*, Vol. 133, 115–132 (1999).