Nash Equilibria for the Multiobjective Control of Linear Partial Differential Equations[1]

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Abstract. This article is concerned with the numerical solution of multiobjective control problems associated with linear partial differential equations. More precisely, for such problems, we look for the Nash equilibrium, which is the solution to a noncooperative game. First, we study the continuous case. Then, to compute the solution of the problem, we combine finite-difference methods for the time discretization, finite-element methods for the space discretization, and conjugate gradient algorithms for the iterative solution of the discrete control problems. Finally, we apply the above methodology to the solution of several tests problems.

Key Words. Linear partial differential equations, optimal control, Nash equilibria, adjoint systems, conjugate gradient methods, multiobjective optimization.

1 Introduction

In a classical *single-objective* control problem for a system modeled by a Differential Equation, there is an output control v, acting on the equation and trying to achieve a pre-determined goal, usually consisting of minimizing a functional $J(\cdot)$. When there is no constraint on the control space and functional J satisfies some suitable assumptions, there exists a unique solution u to the control problem, which is determined by the (optimality) condition $\nabla J(u) = 0$.

In a *multiobjective* control problem there are more than one goal and, possibly, more than one control acting on the equation. Now, in contrast with the single-objective case, there are several strategies in order to choose the controls, depending of the character of the problem. These strategies can be cooperative (when the controls cooperate between them in order to achieve the goals) and non-cooperative.

Nash equilibria define a *noncooperative multiple objective optimization strategy* first proposed by Nash (Ref. 1). Since it originated in *game theory* and *economics*, the notion of *player* is often used. For an optimization problem with G objectives (or functionals J_i to minimize), a *Nash strategy* consists in having G *players* (or controls v_i), each optimizing his own criterion. However, each player has to optimize his criterion given that all the other criteria are fixed by the rest of the *players*. When no *player* can further improve his criterion, it means that the system has reached a *Nash Equilibrium* state.

Of course there are other strategies for *multiobjective optimization*, such as the *Pareto* (cooperative) strategy (Ref. 2) and the *Stackelberg* (hierarchical-cooperative) strategy (Ref. 3), etc..

Some previous works about these strategies for the control of partial differential equations are the following:

In the articles by Lions (Refs. 4, 5), the author gives some results about the Pareto and Stackelberg strategies, respectively.

In the article by Díaz and Lions (Ref. 6), the authors prove an approximate controllability result for a system following a *Stackelberg-Nash strategy.* This result is based on the existence and uniqueness of a *Nash equilibrium*, which is proved by the authors for some particular cases satisfying some restrictions (in the present article we show the existence and uniqueness of a *Nash equilibrium* for more general situations).

In the article by Bristeau et al. (Ref. 7), the authors compare *Pareto and Nash strategies* by using *genetic algorithms* to compute numerically the solutions corresponding to these strategies.

2 Formulation of the Problem

Let us consider $T > 0$, $\Omega \subset I\!\!R^d$, $d = 1$ or 2, and two subsets Γ_1, $\Gamma_2 \subset \partial\Omega$, such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$. We define $Q = \Omega \times (0,T)$, $\Sigma_1 = \Gamma_1 \times (0,T)$ and $\Sigma_2 = \Gamma_2 \times (0,T)$. We define the control spaces $\mathcal{U}_1 = L^2(\omega_1 \times (0,T))$ and $\mathcal{U}_2 = L^2(\omega_2 \times (0,T))$, where $\omega_1, \omega_2 \subset \Omega$ and $\omega_1 \cap \omega_2 = \emptyset$. Finally, we consider the functionals J_1 and J_2 given by

$$\begin{aligned} J_i(v_1, v_2) &= \frac{\alpha_i}{2} \int_{\omega_i \times (0,T)} |v_i|^2 dxdt \\ &+ \frac{k_i}{2} \int_{\omega_{di} \times (0,T)} |y - y_{i,d}|^2 dxdt + \frac{l_i}{2} \int_{\omega_{Ti}} |y(T) - y_{i,T}|^2 dx, \end{aligned}$$

for every $(v_1, v_2) \in \mathcal{U}_1 \times \mathcal{U}_2$, where $\omega_{di}, \omega_{Ti} \subset \Omega$ $(i = 1, 2)$ and function y is defined as the solution of

$$\begin{array}{ll} \dfrac{\partial y}{\partial t} - \Delta y = f + v_1 \chi_{\omega_1} + v_2 \chi_{\omega_2} & \text{in } Q, \\ y(x, 0) = y_0(x) & \text{in } \Omega, \\ y = g_1 & \text{on } \Sigma_1, \\ \dfrac{\partial y}{\partial n} = g_2 & \text{on } \Sigma_2. \end{array}$$

with $f, g_i, y_0, y_{i,d}$ and $y_{i,T}$ being smooth enough functions, $\alpha_i > 0$, $k_i, l_i \geq 0$ and $k_i + l_i > 0$ $(i = 1, 2)$.

Remark 2.1 All the results to follow are also valid for more than two controls (and functionals) and for more general linear operators such as, for instance,

$$\mathcal{A}\varphi = \frac{\partial \varphi}{\partial t} - \nabla \cdot (A(x) \nabla \varphi) + V \cdot \nabla \varphi + c(x)\varphi.$$

The results are also valid for different type of controls such as, for instance, boundary or initial controls and for different type of functionals, such as, for instance,

$$\begin{aligned} J_i(v_1, v_2) &= \frac{\alpha_i}{2} \int_{\omega_i \times (0,T)} |v_i|^2 dxdt \\ &+ \frac{k_i}{2} \int_{\Omega \times (0,T)} \rho_i(x) |y - y_{i,d}|^2 dxdt + \frac{l_i}{2} \int_{\Omega} \eta_i(x) |y(T) - y_{i,T}|^2 dx, \end{aligned}$$

with $\rho_i, \eta_i \in L^\infty(\Omega)$ and $\rho_i(x), \eta_i(x) \geq 0$ $(i = 1, 2)$ (this kind of functionals is treated, for instance, in Ref. 6).

Now, for every $w_2 \in \mathcal{U}_2$ we consider the optimal control problem $(\mathcal{CP}_1(w_2))$: Find $u_1(w_2) \in \mathcal{U}_1$, such that

$$J_1(u_1(w_2), w_2) \leq J_1(v_1, w_2), \ \forall v_1 \in \mathcal{U}_1;$$

similarly for every $w_1 \in \mathcal{U}_1$ we consider the optimal control problem $(\mathcal{CP}_2(w_1))$: Find $u_2(w_1) \in \mathcal{U}_2$, such that

$$J_2(w_1, u_2(w_1)) \leq J_2(w_1, v_2), \ \forall v_2 \in \mathcal{U}_2.$$

The (unique) solution $u_1(w_2)$ (respectively $u_2(w_1)$) of problem $(\mathcal{CP}_1(w_2))$ (respectively $(\mathcal{CP}_2(w_1))$) is characterized by $\frac{\partial J_1}{\partial v_1}(u_1(w_2), w_2) = 0$ (respectively $\frac{\partial J_2}{\partial v_2}(w_1, u_2(w_1)) = 0$).
A Nash equilibrium is a pair $(u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that $u_1 = u_1(u_2)$ and $u_2 = u_2(u_1)$, i.e. (u_1, u_2) is a solution of the *coupled (optimality) system*:

$$\frac{\partial J_1}{\partial v_1}(u_1, u_2) = 0 \qquad (1.a)$$

$$\frac{\partial J_2}{\partial v_2}(u_1, u_2) = 0. \qquad (1.b)$$

We show that system (1) has a unique solution. Furthermore, we give a numerical method for the solution of this problem and present the results obtained with this method on some examples.

Remark 2.2 A special case is when $\omega_{T1} \cap \omega_{T2} \neq \emptyset$ and/or $\omega_{d1} \cap \omega_{d2} \neq \emptyset$. This case is a *competition-wise* problem, with each control (or *player*) trying to reach (possibly) different goals over a common *domain*. In some sense this is the case where the behavior of the solution y associated to the equilibrium (u_1, u_2) is most difficult to forecast.

3 Equivalent Formulation of (Optimality) System (1)

Let us consider $v_1 \in \mathcal{U}$ and a small perturbation $\delta_1 v_1$ of v_1. Then we have, with obvious notation,

$$\delta_1 J_1(v_1, w_2) = \int_{\omega_1\times(0,T)} \frac{\partial J_1}{\partial v_1}(v_1, w_2)\delta_1 v_1 dxdt = \alpha_1 \int_{\omega_1\times(0,T)} v_1 \delta_1 v_1 dxdt$$

$$+k_1 \int_{\omega_{d1}\times(0,T)} (y(v_1, w_2) - y_{1,d})\delta_1 y dxdt + l_1 \int_{\omega_{T1}} (y(T; v_1, w_2) - y_{1,T})\delta_1 y(t)dx,$$

where $\delta_1 y$ is the solution of

$$\frac{\partial \delta_1 y}{\partial t} - \Delta\delta_1 y = \delta_1 v_1 \chi_{\omega_1} \qquad \text{in } Q, \tag{2a}$$

$$\delta_1 y(x, 0) = 0 \qquad \text{in } \Omega, \tag{2b}$$

$$\delta_1 y = 0 \qquad \text{on } \Sigma_1, \tag{2c}$$

$$\frac{\partial \delta_1 y}{\partial n} = 0 \qquad \text{on } \Sigma_2. \tag{2d}$$

Let us introduce now a reasonably smooth function p_1 defined over $\overline{Q}$. Then, multiplying in (2) by p and integrating by parts, we obtain

$$\int_\Omega p_1(T)\delta_1 y(T)dx + \int_Q (-\frac{\partial p_1}{\partial t} - \Delta p_1)\delta_1 y dxdt - \int_{\Sigma_1} p_1 \frac{\partial}{\partial n}\delta_1 y d\Gamma dt$$

$$+ \int_{\Sigma_2} \frac{\partial p_1}{\partial n}\delta_1 y d\Gamma dt = \int_{\omega_1\times(0,T)} p_1 \delta_1 v_1 dxdt.$$

Now, in order to simplify the expression of $\frac{\partial J_1}{\partial v_1}(v_1, w_2)$, we choose p_1 as the solution of the following backward adjoint system:

$$\begin{aligned}
&-\frac{\partial p_1}{\partial t} - \Delta p_1 = k_1(y(v_1, w_2) - y_{1,d})\chi_{\omega_{d1}} && \text{in } Q,\\
&p_1(x, T) = l_1(y(T; v_1, w_2) - y_{1,T})\chi_{\omega_{T1}} && \text{in } \Omega,\\
&p_1 = 0 && \text{on } \Sigma_1,\\
&\frac{\partial p_1}{\partial n} = 0 && \text{on } \Sigma_2.
\end{aligned}$$

Therefore, we have that

$$\int_{\omega_1\times(0,T)} \frac{\partial J_1}{\partial v_1}(v_1, w_2)\delta_1 v_1 dxdt = \int_{\omega_1\times(0,T)} (\alpha_1 v_1 + p_1\chi_{\omega_1})\delta_1 v_1 dxdt.$$

Since $\delta_1 v_1$ is arbitrary, we have proved that

$$\frac{\partial J_1}{\partial v_1}(v_1, w_2) = \alpha_1 v_1 + p_1 \chi_{\omega_1}.$$

Thus, $\dfrac{\partial J_1}{\partial v_1}(u_1, w_2) = 0$ is equivalent to the following (optimality) system:

$$u_1 = -\frac{1}{\alpha_1} p_1 \chi_{\omega_1};$$

$$\begin{array}{ll} \dfrac{\partial y}{\partial t} - \Delta y = f + u_1 \chi_{\omega_1} + w_2 \chi_{\omega_2} & \text{in } Q, \\ y(x,0) = y_0(x) & \text{in } \Omega, \\ y = g_1 & \text{on } \Sigma_1, \\ \dfrac{\partial y}{\partial n} = g_2 & \text{on } \Sigma_2, \end{array}$$

and

$$\begin{array}{ll} -\dfrac{\partial p_1}{\partial t} - \Delta p_1 = k_1(y(u_1, w_2) - y_{1,d})\chi_{\omega_{d1}} & \text{in } Q, \\ p_1(x,T) = l_1(y(T; u_1, w_2) - y_{1,T})\chi_{\omega_{T1}} & \text{in } \Omega, \\ p_1 = 0 & \text{on } \Sigma_1, \\ \dfrac{\partial p_1}{\partial n} = 0 & \text{on } \Sigma_2. \end{array}$$

Similarly, $\dfrac{\partial J_2}{\partial v_2}(w_1, u_2) = 0$ is equivalent to the following (optimality) system:

$$u_2 = -\frac{1}{\alpha_2} p_2 \chi_{\omega_2};$$

$$\begin{array}{ll} \dfrac{\partial y}{\partial t} - \Delta y = f + w_1 \chi_{\omega_1} + u_2 \chi_{\omega_2} & \text{in } Q, \\ y(x,0) = y_0(x) & \text{in } \Omega, \\ y = g_1 & \text{on } \Sigma_1, \\ \dfrac{\partial y}{\partial n} = g_2 & \text{on } \Sigma_2, \end{array}$$

and

$$\begin{array}{ll} -\dfrac{\partial p_2}{\partial t} - \Delta p_2 = k_2(y(w_1, u_2) - y_{2,d})\chi_{\omega_{d2}} & \text{in } Q, \\ p_2(x,T) = l_2(y(T; w_1, u_2) - y_{2,T})\chi_{\omega_{T2}} & \text{in } \Omega, \\ p_2 = 0 & \text{on } \Sigma_1, \\ \dfrac{\partial p_2}{\partial n} = 0 & \text{on } \Sigma_2. \end{array}$$

Therefore, system (1) is equivalent to

$$
\begin{aligned}
&u_1 = -\frac{1}{\alpha_1} p_1 \chi_{\omega_1}, \\
&u_2 = -\frac{1}{\alpha_2} p_2 \chi_{\omega_2}, \\
\\
&\frac{\partial y}{\partial t} - \Delta y = f + u_1 \chi_{\omega_1} + u_2 \chi_{\omega_2} && \text{in } Q, \\
&y(x,0) = y_0(x) && \text{in } \Omega, \\
&y = g_1 && \text{on } \Sigma_1, \\
&\frac{\partial y}{\partial n} = g_2 && \text{on } \Sigma_2;
\end{aligned}
\tag{1}
$$

and for $i = 1, 2$,

$$
\begin{aligned}
&-\frac{\partial p_i}{\partial t} - \Delta p_i = k_i (y - y_{i,d}) \chi_{\omega_{di}} && \text{in } Q, \\
&p_i(x,T) = l_i (y(T) - y_{i,T}) \chi_{\omega_{Ti}} && \text{in } \Omega, \\
&p_i = 0 && \text{on } \Sigma_1, \\
&\frac{\partial p_i}{\partial n} = 0 && \text{on } \Sigma_2.
\end{aligned}
$$

4 Conjugate Gradient Solution of System (1)

4.1 Generalities

If we define V as the Hilbert space $V = \mathcal{U}_1 \times \mathcal{U}_2$ equipped with the scalar product

$$((u_1, u_2)\ ,\ (v_1, v_2))_V = \int_{\omega_1 \times (0,T)} u_1 v_1 dxdt + \int_{\omega_2 \times (0,T)} u_2 v_2 dxdt,$$

then we claim that the linear system (1) is a particular case of finding $u \in V$, such that

$$a(u, v) = L(v),\ \forall\ v \in V, \tag{3}$$

where $a : V \times V \to I\!R$ is bilinear continuous, symmetric and V-elliptic, and $L : V \to I\!R$ is linear and continuous. Therefore (see, e.g., Glowinski Ref.8), problem (1) has a unique solution and this solution can be computed by the following *conjugate gradient* algorithm:
Step 1. $u^0 \in V$ is given.
Step 2. Find $g^0 \in V$ such that

$$(g^0, v) = a(u^0, v) - L(v), \quad \forall v \in V.$$

Step 3. Set $w^0 = g^0$.

Then for $k \geq 0$, assuming that u^k, g^k and w^k are known, compute u^{k+1}, g^{k+1} and (if necessary) w^{k+1} as follows:
Step 4. Compute $\rho^k = \| g^k \|^2 / a(w^k, w^k)$.
Step 5. Update u^k via $u^{k+1} = u^k - \rho^k w^k$.
Step 6. Update g^k via the solution $g^{k+1} \in V$ of

$$(g^{k+1}, v) = (g^k, v) - \rho_k a(w^k, v), \quad \forall v \in V.$$

If $\dfrac{\| g^{k+1} \|^2}{\| g^0 \|^2} \leq \varepsilon$ take $u = u^{k+1}$; else:
Step 7. Compute $\gamma^k = \| g^{k+1} \|^2 / \| g^k \|^2$.
Step 8. Update w^k via $w^{k+1} = g^{k+1} + \gamma^k w^k$.
Step 9. Do $k = k + 1$ and return to Step 4.

Remark 4.1 In the algorithm, ε is a number that we choose for the stopping test. If the problem is finite dimensional, a typical value is $\varepsilon = 10^{-8}$ if one works in double precision. Concerning the convergence of the above algorithm we have, with $\varepsilon = 0$, that

$$\lim_{k \to +\infty} \| u^k - u \| = 0,$$

where u is the solution of problem (3). In fact, it can be shown (see, e.g., Ref. 9) that

$$\| u^k - u \| \leq c \| u^0 - u \| \left(\frac{\sqrt{\nu_a} - 1}{\sqrt{\nu_a} + 1} \right)^k,$$

where c is a constant, and where the *condition number* ν_a is defined by $\nu_a = \| A \| \| A^{-1} \|$, where A is the (unique) linear operator in $\mathcal{L}(V, V)$ defined by

$$a(v, w) = (Av, w), \quad \forall v, w \in V.$$

4.2 Construction of $a(\cdot, \cdot)$ and $L(\cdot)$

It is obvious that the mapping

$$\left(\frac{\partial J_1}{\partial v_1}, \frac{\partial J_2}{\partial v_2}\right) : (v_1, v_2) \in \mathcal{U}_1 \times \mathcal{U}_2 \longrightarrow \left(\frac{\partial J_1}{\partial v_1}(v_1, v_2), \frac{\partial J_2}{\partial v_2}(v_1, v_2)\right) \in \mathcal{U}_1 \times \mathcal{U}_2 \tag{4}$$

is an affine mapping of $\mathcal{U}_1 \times \mathcal{U}_2$. Therefore, there exist a linear continuous mapping $\mathcal{A} \in \mathcal{L}(V, V)$ and a vector $b \in V$ such that

$$\left(\frac{\partial J_1}{\partial v_1}(v_1, v_2), \frac{\partial J_2}{\partial v_2}(v_1, v_2)\right) = \mathcal{A}(v_1, v_2) - b.$$

Let us identify mapping $\mathcal{A}$: For every $(v_1, v_2) \in V$, the linear part of the affine mapping in relation (4) is defined by

$$\mathcal{A}(v_1, v_2) = (\alpha_1 v_1 + p_1 \chi_{\omega_1}, \alpha_2 v_2 + p_2 \chi_{\omega_2}),$$

where p_i, $i = 1, 2$, is the solution of

$$\begin{array}{ll} -\dfrac{\partial p_i}{\partial t} - \Delta p_i = k_i y \chi_{\omega_{di}} & \text{in } Q, \\ p_i(x,T) = l_i y(T) \chi_{\omega_{Ti}} & \text{in } \Omega, \\ p_i = 0 & \text{on } \Sigma_1, \\ \dfrac{\partial p_i}{\partial n} = 0 & \text{on } \Sigma_2, \end{array}$$

and y is the solution of

$$\begin{array}{ll} \dfrac{\partial y}{\partial t} - \Delta y = v_1 \chi_{\omega_1} + v_2 \chi_{\omega_2} & \text{in } Q, \\ y(x,0) = 0 & \text{in } \Omega, \\ y = 0 & \text{on } \Sigma_1, \\ \dfrac{\partial y}{\partial n} = 0 & \text{on } \Sigma_2. \end{array}$$

Proposition 4.1 Mapping $\mathcal{A}$ is linear, continuous, symmetric and strongly positive.

Proof. It is obvious that $\mathcal{A}$ is a linear mapping and it is easy to show (see Ref. 10) that is a continuous mapping. Let us prove that $\mathcal{A}$ is symmetric and strongly positive. Let us consider $(v_1, v_2),\ (w_1, w_2) \in V$. We have then

$$\mathcal{A}(v_1, v_2) \cdot (w_1, w_2) = (\alpha_1 v_1 + p\chi_{\omega_1}, \alpha_2 v_2 + p\chi_{\omega_2}) \cdot (w_1, w_2)$$

$$= \int_{\omega_1 \times (0,T)} (\alpha_1 v_1 + p_1 \chi_{\omega_1}) w_1 dxdt + \int_{\omega_2 \times (0,T)} (\alpha_2 v_2 + p_2 \chi_{\omega_2}) w_2 dxdt.$$

Let us focus on the term $\int_{\omega_1 \times (0,T)} p_1 \chi_{\omega_1} w_1 dxdt$. We have

$$\int_{\omega_1 \times (0,T)} p_1 w_1 dxdt = \int_{\Omega \times (0,T)} p_1 \chi_{\omega_1} (\frac{\partial}{\partial t} y(w_1, w_2) - \Delta y(w_1, w_2) - w_2 \chi_{\omega_2}) dxdt$$

$$= \int_{\Omega \times (0,T)} (-\frac{\partial p_1}{\partial t} - \Delta p_1) y(w_1, w_2) dxdt + l_1 \int_{\omega_{T1}} y(T; v_1, v_2) y(T; w_1, w_2) dx$$

$$= k_1 \int_{\omega_{d1} \times (0,T)} y(v_1, v_2) y(w_1, w_2) dxdt + l_1 \int_{\omega_{T1}} y(T; v_1, v_2) y(T; w_1, w_2) dx.$$

Then,

$$\mathcal{A}(v_1, v_2) \cdot (w_1, w_2) = \alpha_1 \int_{\omega_1 \times (0,T)} v_1 w_1 dxdt + \alpha_2 \int_{\omega_2 \times (0,T)} v_2 w_2 dxdt$$

$$+ k_1 \int_{\omega_{d1} \times (0,T)} y(v_1, v_2) y(w_1, w_2) dxdt + l_1 \int_{\omega_{T1}} y(T; v_1, v_2) y(T; w_1, w_2) dx$$

$$+k_2 \int_{\omega_{d2}\times(0,T)} y(v_1,v_2)y(w_1,w_2)dxdt + l_2 \int_{\omega_{T2}} y(T;v_1,v_2)y(T;w_1,w_2)dx.$$

This proves that $\mathcal{A}$ is a symmetric mapping. Furthermore

$$\mathcal{A}(v_1,v_2)\cdot(v_1,v_2) \geq \inf(\alpha_1,\alpha_2)(\| v_1 \|^2_{\mathcal{U}_1} + \| v_2 \|^2_{\mathcal{U}_2}) = \inf(\alpha_1,\alpha_2) \| (v_1,v_2) \|^2_V,$$

which proves that $\mathcal{A}$ is strongly positive and completes the proof. □

Let us identify b: The constant part of the affine mapping (4) is the function $b \in V$ defined by $b = (p_1\chi_{\omega_1}, p_2\chi_{\omega_2})$, where p_i, $i = 1,2$, is the solution of

$$\begin{array}{ll} -\dfrac{\partial p_i}{\partial t} - \Delta p_i = k_i(Y - y_{i,d})\chi_{\omega_{di}} & \text{in } Q, \\ p_i(x,T) = l_i(Y(T) - y_{i,T})\chi_{\omega_{Ti}} & \text{in } \Omega, \\ p_i = 0 & \text{on } \Sigma_1, \\ \dfrac{\partial p_i}{\partial n} = 0 & \text{on } \Sigma_2, \end{array}$$

and Y is the solution of

$$\begin{array}{ll} \dfrac{\partial Y}{\partial t} - \Delta Y = f & \text{in } Q, \\ Y(x,0) = y_0(x) & \text{in } \Omega, \\ Y = g_1 & \text{on } \Sigma_1, \\ \dfrac{\partial Y}{\partial n} = g_2 & \text{on } \Sigma_2. \end{array}$$

Now, if we define $a(\cdot,\cdot) : V \times V \to I\!R$ by

$$a(v,w) = (\mathcal{A}(v), w)_V \quad \forall\, v,w \in V,$$

and $L : V \to I\!R$ by

$$L(v) = (b,v)_V, \quad \forall\, v \in V,$$

Proposition 4.1 proves that mapping $a(\cdot,\cdot)$ is bilinear continuous, symmetric and V-elliptic; mapping L is (obviously) linear and continuous. Thus, as mentioned in Section 4.1, system (1) has a unique solution, which can be computed by the *conjugate gradient* algorithm described there. In Section 4.3 we shall adapt this algorithm to the problem under consideration.

4.3 Conjugate Gradient Algorithm for System (1)

Step 1. (u_1^0, u_2^0) is given in V.
Step 2.a.

$$\begin{array}{ll} \dfrac{\partial y^0}{\partial t} - \Delta y^0 = f + u_1^0\chi_{\omega_1} + u_2^0\chi_{\omega_2} & \text{in } Q, \\ y^0(x,0) = y_0(x) & \text{in } \Omega, \\ y^0 = g_1 & \text{on } \Sigma_1, \\ \dfrac{\partial y^0}{\partial n} = g_2 & \text{on } \Sigma_2. \end{array}$$

Step 2.b. For $i = 1, 2$,

$$\begin{array}{ll} -\dfrac{\partial p_i^0}{\partial t} - \Delta p_i^0 = k_i(y^0 - y_{i,d})\chi_{\omega_{di}} & \text{in } Q, \\ p_i^0(x,T) = l_i((y^0(T) - y_{i,T})\chi_{\omega_{Ti}} & \text{in } \Omega, \\ p_i^0 = 0 & \text{on } \Sigma_1, \\ \dfrac{\partial p_i^0}{\partial n} = 0 & \text{on } \Sigma_2. \end{array}$$

Step 2.c. $(g_1^0, g_2^0) = (\alpha_1 u_1^0 + p_1^0 \chi_{\omega_1}, \alpha_2 u_2^0 + p_2^0 \chi_{\omega_2})$.

Step 3. $(w_1^0, w_2^0) = (g_1^0, g_2^0)$.

Then, for $k \geq 0$, assuming that (u_1^k, u_2^k), (g_1^k, g_2^k), (w_1^k, w_2^k) are known, all in V, we compute (u_1^{k+1}, u_2^{k+1}), (g_1^{k+1}, g_2^{k+1}) and (if necessary) (w_1^{k+1}, w_2^{k+1}) as follows:

Step 4.a.

$$\begin{array}{ll} \dfrac{\partial \overline{y}^k}{\partial t} - \Delta \overline{y}^k = w_1^k \chi_{\omega_1} + w_2^k \chi_{\omega_2} & \text{in } Q, \\ \overline{y}^k(x,0) = 0 & \text{in } \Omega, \\ \overline{y}^k = 0 & \text{on } \Sigma_1, \\ \dfrac{\partial \overline{y}^k}{\partial n} = 0 & \text{on } \Sigma_2. \end{array}$$

Step 4.b. For $i = 1, 2$,

$$\begin{array}{ll} -\dfrac{\partial \overline{p}_i^k}{\partial t} - \Delta \overline{p}_i^k = k_i \overline{y}^k \chi_{\omega_{di}} & \text{in } Q, \\ \overline{p}_i^k(x,T) = l_i \overline{y}^k(T)\chi_{\omega_{Ti}} & \text{in } \Omega, \\ \overline{p}_i^k = 0 & \text{on } \Sigma_1, \\ \dfrac{\partial \overline{p}_i^k}{\partial n} = 0 & \text{on } \Sigma_2. \end{array}$$

Step 4.c. $(\overline{g}_1^k, \overline{g}_2^k) = (\alpha_1 w_1^k + \overline{p}_i^k \chi_{\omega_1}, \alpha_2 w_2^k + \overline{p}_i^k \chi_{\omega_2})$.

Step 4.d. $\rho_k = \dfrac{\| (g_1^k, g_2^k) \|_V^2}{\int_{\omega_1 \times (0,T)} \overline{g}_1^k w_1^k dxdt + \int_{\omega_2 \times (0,T)} \overline{g}_2^k w_2^k dxdt}$.

Step 5. $(u_1^{k+1}, u_2^{k+1}) = (u_1^k, u_2^k) - \rho_k (w_1^k, w_2^k)$.

Step 6. $(g_1^{k+1}, g_2^{k+1}) = (g_1^k, g_2^k) - \rho_k (\overline{g}_1^k, \overline{g}_2^k)$.

If $\dfrac{\| (g_1^{k+1}, g_2^{k+1}) \|_V^2}{\| (g_1^0, g_2^0) \|_V^2} \leq \varepsilon$, then take $(u_1, u_2) = (u_1^{k+1}, u_2^{k+1})$; else:

Step 7. $\gamma_k = \dfrac{\| (g_1^{k+1}, g_2^{k+1}) \|_V^2}{\| (g_1^k, g_2^k) \|_V^2}$.

Step 8. $(w_1^{k+1}, w_2^{k+1}) = (g_1^{k+1}, g_2^{k+1}) + \gamma_k (w_1^k, w_2^k)$.

Step 9. Do $k = k + 1$, and go to Step 4.a.

5 Time Discretization

5.1 Formulation of the Semidiscrete Problem

For simplicity, we consider from now on the special *competition-wise* control problem (see Remark 2.2) given by the case where $k_1 = k_2 = k$, $l_1 = l_2 = l$, $\omega_{d1} = \omega_{d2} = \omega_d$ and $\omega_{T1} = \omega_{T2} = \omega_T$. For this special case, we point out that the mapping $\mathcal{A}$ defined in Section 4.2 is $\mathcal{A}(v_1, v_2) = (\alpha_1 v_1 + p\chi_{\omega_1}, \alpha_2 v_2 + p\chi_{\omega_2})$, where $p = p_1 = p_2$ (since p_1 and p_2, defined in Section 4.2, are solution of the same equation). Then, the functions $\overline{p}_1^k$ and $\overline{p}_2^k$ defined in the Step 4.2 of the *Conjugate Gradient* algorithm described in Section 4.3 are solution of the same equation and therefore $\overline{p}_1^k = \overline{p}_2^k = \overline{p}^k$.

We consider the time discretization step Δt, defined by $\Delta t = T/N$, where N is a positive integer. Then, if we denote $n\Delta t$ by t^n, we have $0 < t^1 < t^2 < \cdots < t^N = T$. For simplicity, we assume that $f, g_1, g_2, y_{1,d}$ and $y_{2,d}$ are continuous functions, at least with respect to the time variable (if not we can always use continuous approximations of these functions). Now, we approximate $\mathcal{U}_1$ by $\mathcal{U}_1^{\Delta t} = (L^2(\omega_1))^N$ and $\mathcal{U}_2$ by $\mathcal{U}_2^{\Delta t} = (L^2(\omega_2))^N$. Then, for every $w_2 \in \mathcal{U}_2^{\Delta t}$ we approximate problem $(\mathcal{CP}_1(w_2))$ by the following minimization problem $(\mathcal{CP}_1(w_2))^{\Delta t}$: Find $u_1^{\Delta t}(w_2) \in \mathcal{U}_1^{\Delta t}$, such that

$$J_1^{\Delta t}(u_1^{\Delta t}(w_2), w_2) \leq J_1^{\Delta t}(v_1, w_2), \ \forall v_1 \in \mathcal{U}_1^{\Delta t},$$

with

$$\begin{aligned} J_1^{\Delta t}(v_1, v_2) &= \alpha_1 \frac{\Delta t}{2} \sum_{n=1}^{N} \int_{\omega_1} |v_1^n|^2 dx \\ &+ k \frac{\Delta t}{2} \sum_{n=1}^{N} \int_{\omega_d} |y^n - y_{1,d}(t^n)|^2 dx + \frac{l}{2} \int_{\omega_T} |y^N - y_{1,T}|^2 dx, \end{aligned}$$

where $\{y^n\}_{n=1}^N$ is defined by the solution of the following semi-discrete parabolic problem:

$$y^0 = y_0, \tag{5}$$

and for $n = 1, ..., N$,

$$\frac{y^n - y^{n-1}}{\Delta t} - \Delta y^n = f(t^n) + v_1^n \chi_{\omega_1} + v_2^n \chi_{\omega_2} \quad \text{in } \Omega, \tag{6a}$$

$$y^n = g_1(t^n) \quad \text{in } \Gamma_1, \tag{6b}$$

$$\frac{\partial y^n}{\partial n} = g_2(t^n) \quad \text{in } \Gamma_2. \tag{6c}$$

Similarly, for every $w_1 \in \mathcal{U}_1^{\Delta t}$, we approximate then problem $(\mathcal{CP}_2(w_1))$ by the following minimization problem $(\mathcal{CP}_2(w_1))^{\Delta t}$: Find $u_2^{\Delta t}(w_1) \in \mathcal{U}_2^{\Delta t}$, such that

$$J_2^{\Delta t}(w_1, u_2^{\Delta t}(w_1)) \leq J_2^{\Delta t}(w_1, v_2), \ \forall v_2 \in \mathcal{U}_2^{\Delta t},$$

with

$$
\begin{aligned}
J_2^{\Delta t}(v_1, v_2) &= \alpha_2 \frac{\Delta t}{2} \sum_{n=1}^{N} \int_{\omega_2} |v_2^n|^2 dx \\
&+ k \frac{\Delta t}{2} \sum_{n=1}^{N} \int_{\omega_d} |y^n - y_{2,d}(t^n)|^2 dx + \frac{l}{2} \int_{\omega_T} |y^N - y_{2,T}|^2 dx,
\end{aligned}
$$

where $\{y^n\}_{n=1}^N$ is again defined by the solution of problem (5)–(6).

5.2 Optimality Conditions for the Semidiscretized Optimal Control Problems $(\mathcal{CP}_1(w_2))^{\Delta t}$ and $(\mathcal{CP}_2(w_1))^{\Delta t}$

We suppose that the discrete control space $\mathcal{U}_i^{\Delta t}$, $i = 1, 2$, is equipped with the scalar product $(\cdot, \cdot)_{i,\Delta t}$ defined by

$$
(v_i, w_i)_{i,\Delta t} = \Delta t \sum_{n=1}^{N} \int_{\omega_i} v_i^n w_i^n dx.
$$

The (unique) vector $u_1^{\Delta t}(w_2)$ solution of problem $(\mathcal{CP}_1(w_2))^{\Delta t}$ is characterized by the optimality condition $\frac{\partial}{\partial v_1} J_1^{\Delta t}(u_1^{\Delta t}(w_2), w_2) = 0$. Similarly, the (unique) vector $u_2^{\Delta t}(w_1)$ solution of problem $(\mathcal{CP}_2(w_2))^{\Delta t}$ is characterized by the optimality condition $\frac{\partial}{\partial v_2} J_2^{\Delta t}(w_1, u_2^{\Delta t}(w_1)) = 0$.

Let us concentrate on problem $(\mathcal{CP}_1(w_2))^{\Delta t}$. We consider $v_1 \in \mathcal{U}_1^{\Delta t}$ and a small perturbation $\delta_1 v_1$ of v_1. Then we have, with obvious notation,

$$
\delta_1 J_1^{\Delta t}(v_1, w_2) = (\frac{\partial}{\partial v_1} J_1^{\Delta t}(v_1, w_2), \delta_1 v_1)_{1,\Delta t} = \Delta t \alpha_1 \sum_{n=1}^{N} \int_{\omega_1} v_1^n \delta_1 v_1^n dx
$$

$$
+ k \Delta t \sum_{n=1}^{N} \int_{\omega_d} (y^n - y_{1,d}(t^n)) \delta_1 y^n dx + l \int_{\omega_T} (y^N - y_{1,T}) \delta_1 y^N dx,
$$

where

$$
\delta_1 y^0 = 0, \tag{7}
$$

and for $n = 1, ..., N$,

$$
\begin{aligned}
&\frac{\delta_1 y^n - \delta_1 y^{n-1}}{\Delta t} - \Delta \delta_1 y^n = \delta_1 v_1^n \chi_{\omega_1} && \text{in } \Omega, && \text{(8a)} \\
&\delta_1 y^n = 0 && \text{on } \Gamma_1, && \text{(8b)} \\
&\frac{\partial}{\partial n} \delta_1 y^n = 0 && \text{on } \Gamma_2. && \text{(8c)}
\end{aligned}
$$

Let us introduce now $\{p_1^n\}_{n=1}^{N+1}$, where each p_1^n is a smooth function defined on $\overline{\Omega}$. Then, multiplying in (8) by p_1^n, integrating continuously over Ω and discretely over $(0,T)$, we obtain

$$\begin{aligned} 0 &= \Delta t \sum_{n=1}^{N} \int_\Omega \left(\frac{p_1^n - p_1^{n+1}}{\Delta t} - \Delta p_1^n\right)\delta_1 y^n dx + \int_\Omega p_1^{N+1}\delta y^N dx \\ &+ \Delta t \sum_{n=1}^{N} \left(\int_{\Gamma_2} \frac{\partial p_1^n}{\partial n}\delta_1 y^n d\Gamma - \int_{\Gamma_1} p_1^n \frac{\partial}{\partial n}\delta_1 y^n d\Gamma \right) - \Delta t \sum_{n=1}^{N} \int_{\omega_1} \delta_1 v_1^n p_1^n dx. \end{aligned}$$

Now, in order to simplify the expression of $\frac{\partial}{\partial v_1} J_1^{\Delta t}(v_1, w_2)$, we choose $\{p_1^n\}_{n=1}^{N+1}$ satisfying the following conditions:

$$p_1^{N+1} = l(y^N(v_1, w_2) - y_{1,T})\chi_{\omega_T},$$

and for $n = N, ..., 1$,

$$\begin{array}{ll} \dfrac{p_1^n - p_1^{n+1}}{\Delta t} - \Delta p_1^n = k(y^n(v_1, w_2) - y_{1,d}(t^n))\chi_{\omega_d} & \text{in } \Omega, \\ p_1^n = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial p_1^n}{\partial n} = 0 & \text{on } \Gamma_2. \end{array}$$

Taking the above relations into account, we obtain that

$$\delta_1 J_1^{\Delta t}(v_1, w_2) = \Delta t \sum_{n=1}^{N} \int_{\omega_1} (\alpha_1 v_1^n + p_1^n \chi_{\omega_1})\, \delta_1 v_1^n dx.$$

Since $\delta_1 v_1$ is arbitrary, we have proved that

$$\frac{\partial}{\partial v_1} J_1^{\Delta t}(v_1, w_2) = \{\alpha_1 v_1^n + p_1^n \chi_{\omega_1}\}_{n=1}^{N}. \tag{9}$$

Therefore, $\frac{\partial}{\partial v_1} J_1^{\Delta t}(u_1, w_2) = 0$ is equivalent to the following (optimality) system:

$$u_1 = -\frac{1}{\alpha_1}\{p_1^n \chi_{\omega_1}\}_{n=1}^{N};$$

$$y^0 = y_0,$$

and for $n = 1, ..., N$,

$$\begin{array}{ll} \dfrac{y^n - y^{n-1}}{\Delta t} - \Delta y^n = f(t^n) + u_1^n \chi_{\omega_1} + w_2^n \chi_{\omega_2} & \text{in } \Omega, \\ y^n = g_1(t^n) & \text{in } \Gamma_1, \\ \dfrac{\partial y^n}{\partial n} = g_2(t^n) & \text{in } \Gamma_2; \end{array}$$

$$p_1^{N+1} = l(y^N(u_1, w_2) - y_{1,T})\chi_{\omega_T},$$

and for $n = N, ..., 1,$

$$\begin{aligned}
&\frac{p_1^n - p_1^{n+1}}{\Delta t} - \Delta p_1^n = k(y^n(u_1, w_2) - y_{1,d}(t^n))\chi_{\omega_d} && \text{in } \Omega,\\
&p_1^n = 0 && \text{on } \Gamma_1,\\
&\frac{\partial p_1^n}{\partial n} = 0 && \text{on } \Gamma_2.
\end{aligned}$$

Similarly, we can find the (optimality) system equivalent to $\frac{\partial}{\partial v_2} J_2^{\Delta t}(w_1, u_2) = 0$. Therefore, the problem of finding $(u_1, u_2) \in \mathcal{U}_1^{\Delta t} \times \mathcal{U}_2^{\Delta t}$, such that

$$\frac{\partial}{\partial v_1} J_1^{\Delta t}(u_1, u_2) = 0, \tag{10a}$$

$$\frac{\partial}{\partial v_2} J_2^{\Delta t}(u_1, u_2) = 0, \tag{10b}$$

is equivalent to

$$u_1 = -\frac{1}{\alpha_1}\{p_1^n \chi_{\omega_1}\}_{n=1}^N;$$

$$u_2 = -\frac{1}{\alpha_2}\{p_2^n \chi_{\omega_2}\}_{n=1}^N;$$

$$y^0 = y_0,$$

and for $n = 1, ..., N,$

$$\begin{aligned}
&\frac{y^n - y^{n-1}}{\Delta t} - \Delta y^n = f(t^n) + u_1^n \chi_{\omega_1} + u_2^n \chi_{\omega_2} && \text{in } \Omega,\\
&y^n = g_1(t^n) && \text{in } \Gamma_1,\\
&\frac{\partial y^n}{\partial n} = g_2(t^n) && \text{in } \Gamma_2;
\end{aligned}$$

$$p_1^{N+1} = l(y^N(u_1, u_2) - y_{1,T})\chi_{\omega_T},$$

and for $n = N, ..., 1,$ and $i = 1, 2,$

$$\begin{aligned}
&\frac{p_i^n - p_i^{n+1}}{\Delta t} - \Delta p_i^n = k(y^n(u_1, u_2) - y_{i,d}(t^n))\chi_{\omega_d} && \text{in } \Omega,\\
&p_i^n = 0 && \text{on } \Gamma_1,\\
&\frac{\partial p_i^n}{\partial n} = 0 && \text{on } \Gamma_2.
\end{aligned}$$

6 Conjugate Gradient Solution of Problem (10)

6.1 Generalities

If we define $V^{\Delta t}$ as the Hilbert space $V^{\Delta t} = \mathcal{U}_1^{\Delta t} \times \mathcal{U}_2^{\Delta t}$ equipped with the scalar product

$$((u_1, u_2)\ ,\ (v_1, v_2))_V^{\Delta t} = \Delta t \left(\sum_{n=1}^{N} \int_{\omega_1} u_1^n v_1^n dx + \sum_{n=1}^{N} \int_{\omega_2} u_2^n v_2^n dx \right),$$

then we claim that the linear system (10) is a particular case of finding $u \in V^{\Delta t}$, such that

$$a^{\Delta t}(u, v) = L^{\Delta t}(v),\ \forall\ v \in V^{\Delta t},$$

where $a^{\Delta t} : V^{\Delta t} \times V^{\Delta t} \to I\!R$ is bilinear continuous, symmetric and $V^{\Delta t}$-elliptic; and $L^{\Delta t} : V^{\Delta t} \to I\!R$ is linear and continuous. Therefore, as explained in Section 4.1, problem (10) has a unique solution and this solution can be computed by a *conjugate gradient* algorithm.

6.2 Construction of $a^{\Delta t}(\cdot, \cdot)$ and $L^{\Delta t}(\cdot)$

It is obvious that the mapping

$$(\frac{\partial J_1^{\Delta t}}{\partial v_1}, \frac{\partial J_2^{\Delta t}}{\partial v_2}) : (v_1, v_2) \in \mathcal{U}_1^{\Delta t} \times \mathcal{U}_2^{\Delta t} \longrightarrow (\frac{\partial J_1^{\Delta t}}{\partial v_1}(v_1, v_2), \frac{\partial J_2^{\Delta t}}{\partial v_2}(v_1, v_2)) \in \mathcal{U}_1^{\Delta t} \times \mathcal{U}_2^{\Delta t} \tag{11}$$

is an affine mapping of $\mathcal{U}_1^{\Delta t} \times \mathcal{U}_2^{\Delta t}$. Therefore, there exist a linear continuous mapping $\mathcal{A}^{\Delta t} \in \mathcal{L}(V^{\Delta t}, V^{\Delta t})$ and a vector $b^{\Delta t} \in V^{\Delta t}$ such that

$$(\frac{\partial J_1^{\Delta t}}{\partial v_1}(v_1, v_2), \frac{\partial J_2^{\Delta t}}{\partial v_2}(v_1, v_2)) = \mathcal{A}^{\Delta t}(v_1, v_2) - b^{\Delta t}.$$

Let us identify mapping $\mathcal{A}^{\Delta t}$: For every $(v_1, v_2) \in V^{\Delta t}$, the linear part of the affine mapping in relation (11) is defined by

$$\mathcal{A}^{\Delta t}(v_1, v_2) = (\alpha_1 v_1 + p\chi_{\omega_1}, \alpha_2 v_2 + p\chi_{\omega_2}),$$

where $p = \{p^n\}_{n=1}^{N+1}$ is the solution of

$$p^{N+1} = l y^N \chi_{\omega_T},$$

and for $n = N, ..., 1,$

$$\begin{aligned} &\frac{p^n - p^{n+1}}{\Delta t} - \Delta p^n = k y^n \chi_{\omega_d} && \text{in } \Omega, \\ &p^n = 0 && \text{on } \Gamma_1, \\ &\frac{\partial p^n}{\partial n} = 0 && \text{on } \Gamma_2, \end{aligned}$$

and $y = \{y^n\}_{n=0}^N$ is the solution of

$$y^0 = 0,$$

and for $n = 1, ..., N$,

$$\begin{array}{ll} \dfrac{y^n - y^{n-1}}{\Delta t} - \Delta y^n = v_1^n \chi_{\omega_1} + v_2^n \chi_{\omega_2} & \text{in } \Omega, \\ y^n = 0 & \text{in } \Gamma_1, \\ \dfrac{\partial y^n}{\partial n} = 0 & \text{in } \Gamma_2. \end{array}$$

Proposition 6.1 Mapping $\mathcal{A}^{\Delta t}$ is linear, continuous, symmetric and strongly positive.

Let us identify b: The constant part of the affine mapping (11) is the function $b^{\Delta t} \in V^{\Delta t}$ defined by $b^{\Delta t} = (p_1 \chi_{\omega_1}, p_2 \chi_{\omega_2})$, where, for $i = 1, 2$, $p_i = \{p_i^n\}_{n=1}^{N+1}$ is the solution of

$$p_i^{N+1} = l(Y^N - y_{i,T}) \chi_{\omega_T},$$

and for $n = N, ..., 1$,

$$\begin{array}{ll} \dfrac{p_i^n - p_i^{n+1}}{\Delta t} - \Delta p_i^n = k(Y^n - y_{i,d}(t^n)) \chi_{\omega_d} & \text{in } \Omega, \\ p_i^n = 0 & \text{on } \Gamma_1, \\ \dfrac{\partial p_i^n}{\partial n} = 0 & \text{on } \Gamma_2, \end{array}$$

and $Y = \{Y^n\}_{n=0}^N$ is the solution of

$$Y^0 = y_0,$$

and for $n = 1, ..., N$,

$$\begin{array}{ll} \dfrac{Y^n - Y^{n-1}}{\Delta t} - \Delta Y^n = f(t^n) & \text{in } \Omega, \\ Y^n = g_1(t^n) & \text{in } \Gamma_1, \\ \dfrac{\partial Y^n}{\partial n} = g_2(t^n) & \text{in } \Gamma_2. \end{array}$$

Now, if we define $a^{\Delta t}(\cdot, \cdot) : V^{\Delta t} \times V^{\Delta t} \to I\!R$ by

$$a^{\Delta t}(v, w) = (\mathcal{A}^{\Delta t}(v), w)_{V^{\Delta t}} \quad \forall \; v, w \in V^{\Delta t},$$

and $L^{\Delta t} : V^{\Delta t} \to I\!R$ by

$$L^{\Delta t}(v) = (b, v)_{V^{\Delta t}} \quad \forall \; v \in V^{\Delta t},$$

Proposition 6.1 proves that mapping $a(\cdot, \cdot)$ is bilinear continuous, symmetric and $V^{\Delta t}$-elliptic; and mapping $L^{\Delta t}$ is (obviously) linear and continuous. Thus, as explained in Section 4.1 problem (10) has a unique solution, which can be computed by the *conjugate gradient* algorithm described in Section 6.3.

6.3 Conjugate Gradient Algorithm for the Solution of Problem (10)

Step 1. (u_1^0, u_2^0) is given in $V^{\Delta t}$.
Step 2.a. $y^{0,0} = y_0$, and for $n = 1, ..., N$,

$$\begin{aligned}
&\frac{y^{0,n} - y^{0,n-1}}{\Delta t} - \Delta y^{0,n} = f(t^n) + u_1^{0,n}\chi_{\omega_1} + u_2^{0,n}\chi_{\omega_2} && \text{in } \Omega,\\
&y^{0,n} = g_1(t^n) && \text{on } \Gamma_1,\\
&\frac{\partial y^{0,n}}{\partial n} = g_2(t^n), && \text{on } \Gamma_2.
\end{aligned}$$

Step 2.b. For $i = 1, 2$, $p_i^{0,N+1} = l(y^{0,N} - y_{i,T})\chi_{\omega_T}$, and for $n = N, ..., 1$,

$$\begin{aligned}
&\frac{p_i^{0,n} - p_i^{0,n+1}}{\Delta t} - \Delta p_i^{0,n} = k(y^{0,n} - y_{i,d}(t^n))\chi_{\omega_d} && \text{in } \Omega,\\
&p_i^{0,n} = 0 && \text{on } \Gamma_1,\\
&\frac{\partial p_i^{0,n}}{\partial n} = 0 && \text{on } \Gamma_2.
\end{aligned}$$

Step 2.c. $(g_1^0, g_2^0) = (\alpha_1 u_1^0 + p_1^0\chi_{\omega_1}, \alpha_2 u_2^0 + p_2^0\chi_{\omega_2})$.
Step 3. $(w_1^0, w_2^0) = (g_1^0, g_2^0)$.

Then, for $k \geq 0$, assuming that (u_1^k, u_2^k), (g_1^k, g_2^k), (w_1^k, w_2^k) are known, all in $V^{\Delta t}$, we compute (u_1^{k+1}, u_2^{k+1}), (g_1^{k+1}, g_2^{k+1}) and (if necessary) (w_1^{k+1}, w_2^{k+1}) as follows:

Step 4.a. $\overline{y}^{k,0} = 0$, and for $n = 1, ..., N$,

$$\begin{aligned}
&\frac{\overline{y}^{k,n} - \overline{y}^{k,n-1}}{\Delta t} - \Delta \overline{y}^{k,n} = w_1^{k,n}\chi_{\omega_1} + w_2^{k,n}\chi_{\omega_2} && \text{in } \Omega,\\
&\overline{y}^{k,n} = 0 && \text{on } \Gamma_1,\\
&\frac{\partial \overline{y}^{k,n}}{\partial n} = 0 && \text{on } \Gamma_2.
\end{aligned}$$

Step 4.b. $\overline{p}^{k,N+1} = l\overline{y}^{k,N}\chi_{\omega_T}$, and for $n = N, ..., 1$,

$$\begin{aligned}
&\frac{\overline{p}^{k,n} - \overline{p}^{k,n+1}}{\Delta t} - \Delta \overline{p}^{k,n} = k\overline{y}^{k,n}\chi_{\omega_d} && \text{in } \Omega,\\
&\overline{p}^{k,n} = 0 && \text{on } \Gamma_1,\\
&\frac{\partial \overline{p}^{k,n}}{\partial n} = 0 && \text{on } \Gamma_2.
\end{aligned}$$

Step 4.c. $(\overline{g}_1^k, \overline{g}_2^k) = (\alpha_1 w_1^k + \overline{p}^k\chi_{\omega_1}, \alpha_2 w_2^k + \overline{p}^k\chi_{\omega_2})$.
Step 4.d. $\rho_k = \dfrac{\| (g_1^k, g_2^k) \|_{V^{\Delta t}}^2}{\Delta t \sum_{n=1}^N \left(\int_{\omega_1} \overline{g}_1^{k,n} w_1^{k,n} dx + \int_{\omega_2} \overline{g}_2^{k,n} w_2^{k,n} dx \right)}$.
Step 5. $(u_1^{k+1}, u_2^{k+1}) = (u_1^k, u_2^k) - \rho_k(w_1^k, w_2^k)$.

Step 6. $(g_1^{k+1}, g_2^{k+1}) = (g_1^k, g_2^k) - \rho_k(\overline{g}_1^k, \overline{g}_2^k)$.

If $\dfrac{\| (g_1^{k+1}, g_2^{k+1}) \|^2_{V^{\Delta t}}}{\| (g_1^0, g_2^0) \|^2_{V^{\Delta t}}} \leq \varepsilon$, then take $(u_1, u_2) = (u_1^{k+1}, u_2^{k+1})$; else:

Step 7. $\gamma_k = \dfrac{\| (g_1^{k+1}, g_2^{k+1}) \|^2_{V^{\Delta t}}}{\| (g_1^k, g_2^k) \|^2_{V^{\Delta t}}}$.

Step 8. $(w_1^{k+1}, w_2^{k+1}) = (g_1^{k+1}, g_2^{k+1}) + \gamma_k(w_1^k, w_2^k)$.

Step 9. Do $k = k + 1$, and go to Step 7.

7 Full Discretization of Problem (1)

7.1 Generalities

We suppose from now on that Ω is a polygonal domain of $I\!R^2$. We also consider ω_d, ω_T, ω_1 and ω_2 subdomains of Ω. We introduce a finite-element triangulation $\mathcal{T}_h$ of Ω, with h the largest length of the edges of the triangles of $\mathcal{T}_h$, and $\partial\omega_T$, $\partial\omega_d$, $\partial\omega_1$ and $\partial\omega_2$ being part of the triangulation. Next, we approximate $L^2(\Omega\times(0,T))$ and $L^2(0,T : H^1(\Omega))$ by $W_h^{\Delta t}$ defined by $W_h^{\Delta t} = (W_h)^N$, with

$$W_h = \{z \in \mathcal{C}^0(\overline{\Omega}), z_{|T} \in P_1, \ \forall T \in \mathcal{T}_h\},$$

where P_1 is the space of those polynomials of degree ≤ 1 ($dim(P_1) = 3$ and $dim(W_h) = N_h$, where N_h is the number of vertex in $\mathcal{T}_h$). Now, for $i \in \{1, 2\}$, we approximate $\mathcal{U}_i$ by $\mathcal{U}_{i,h}^{\Delta t}$ defined by $\mathcal{U}_{i,h}^{\Delta t} = (\mathcal{U}_{i,h})^N$, where

$$\mathcal{U}_{i,h} = \{z : z \in \mathcal{C}^0(\overline{\omega_i}), \ z_{|T} \in P_1, \ \forall T \in \omega_i\}.$$

Finally, we approximate the space $\{z \in L^2(0,T : H^1(\Omega)) : z(t)_{|\Gamma_1} = 0\}$ by $W_{h,0}^{\Delta t}$ defined by $W_{h,0}^{\Delta t} = (W_{h,0})^N$, where

$$W_{h,0} = \{z \in W_h : z_{|\Gamma_1} = 0\}.$$

7.2 Formulation of the Fully Discrete Control Problem

Let us suppose (for simplicity) that $g_1 = g_2 = 0$. For every $w_2 \in \mathcal{U}_{2,h}^{\Delta t}$ we approximate then problem $(\mathcal{CP}_1(w_2))$ by the following finite dimensional minimization problem $(\mathcal{CP}_1(w_2))_h^{\Delta t}$: Find $u_{1,h}^{\Delta t}(w_2) \in \mathcal{U}_{1,h}^{\Delta t}$, such that

$$J_{1,h}^{\Delta t}(u_{1,h}^{\Delta t}(w_2), w_2) \leq J_{1,h}^{\Delta t}(v_1, w_2), \ \forall v_1 \in \mathcal{U}_{1,h}^{\Delta t},$$

with

$$\begin{aligned} J_{1,h}^{\Delta t}(v_1, v_2) &= \alpha_1 \frac{\Delta t}{2} \sum_{n=1}^{N} \int_{\omega_1} |v_1^n|^2 dxdt \\ &+ k\frac{\Delta t}{2} \sum_{n=1}^{N} \int_{\omega_d} |y_h^n - y_{1,d}(t^n)|^2 dx + \frac{l}{2} \int_{\omega_T} |y_h^N - y_{1,T}|^2 dx, \end{aligned}$$

where $y_h = \{y_h^n\}_{n=1}^N \in W_{h,0}^{\Delta t}$ is defined by the solution of the following discrete parabolic problem (written in variational form):

$$
\begin{aligned}
& y_{0,h} \in W_h, \quad \forall\ h, && \text{(12a)}\\
& \lim_{h\to 0} y_{0,h} = y_0 \ \text{ in } L^2(\Omega), && \text{(12b)}\\
& y_h^0 = y_{0,h}, && \text{(13)}
\end{aligned}
$$

and for $n = 1, ..., N$,

$$\int_\Omega \left(\frac{y_h^n - y_h^{n-1}}{\Delta t} z + \nabla y_h^n \cdot \nabla z - f(t^n) - v_1^n \chi_{\omega_1} - v_2^n \chi_{\omega_2}\right) z dx, \quad \forall\ z \in W_{h,0}. \qquad (14)$$

Similarly, for every $w_1 \in \mathcal{U}_{1,h}^{\Delta t}$ we approximate problem $(\mathcal{CP}_2(w_1))$ by the following finite dimensional minimization problem $(\mathcal{CP}_2(w_1))_h^{\Delta t}$: Find $u_{2,h}^{\Delta t}(w_1) \in \mathcal{U}_{2,h}^{\Delta t}$, such that

$$J_{2,h}^{\Delta t}(w_1, u_{2,h}^{\Delta t}(w_1)) \leq J_{2,h}^{\Delta t}(w_1, v_2),\ \forall v_2 \in \mathcal{U}_{2,h}^{\Delta t},$$

with

$$
\begin{aligned}
J_{2,h}^{\Delta t}(v_1, v_2) \quad &= \quad \alpha_2 \frac{\Delta t}{2} \sum_{n=1}^{N} \int_{\omega_2} |v_2^n|^2 dx dt \\
&\quad + k \frac{\Delta t}{2} \sum_{n=1}^{N} \int_{\omega_d} |y_h^n - y_{2,d}(t^n)|^2 dx + \frac{l}{2} \int_{\omega_T} |y_h^N - y_{2,T}|^2 dx,
\end{aligned}
$$

where $y_h = \{y_h^n\}_{n=1}^N \in W_{h,0}^{\Delta t}$ is again defined by the solution of (12)–(14).

7.3 Optimality Conditions for the Fully-Discrete Control Problems $(\mathcal{CP}_1(w_2))_h^{\Delta t}$ and $(\mathcal{CP}_2(w_1))_h^{\Delta t}$

We suppose that the discrete control space $\mathcal{U}_{i,h}^{\Delta t}$, $i = 1, 2$, is equipped with the scalar product $(\cdot, \cdot)_{i,h,\Delta t}$, defined by

$$(v_i, w_i)_{i,h,\Delta t} = \Delta t \sum_{n=1}^{N} \int_{\omega_i} v_i^n w_i^n dx.$$

The (unique) vector $u_{1,h}^{\Delta t}(w_2)$ solution of problem $(\mathcal{CP}_1(w_2))_h^{\Delta t}$ is characterized by the optimality condition $\dfrac{\partial}{\partial v_1} J_{1,h}^{\Delta t}(u_{1,h}^{\Delta t}(w_2), w_2) = 0$. Similarly, the (unique) vector $u_{2,h}^{\Delta t}(w_1)$ solution of problem $(\mathcal{CP}_2(w_2))_h^{\Delta t}$ is characterized by the optimality condition $\dfrac{\partial}{\partial v_2} J_{2,h}^{\Delta t}(w_1, u_{2,h}^{\Delta t}(w_1)) = 0$.

Now, in a way similar to that followed for problem $(\mathcal{CP}_1(w_2))^{\Delta t}$, it is easy to deduce that, for every $(v_1, v_2) \in \mathcal{U}_{1,h}^{\Delta t} \times \mathcal{U}_{2,h}^{\Delta t}$ and $i = 1, 2$,

$$\frac{\partial}{\partial v_i} J_{i,h}^{\Delta t}(v_1, v_2) = \{\alpha_i v_i^n + p_i^n \chi_{\omega_i}\}_{n=1}^N \in \mathcal{U}_{i,h}^{\Delta t},$$

where $\{p_i^n\}_{n=1}^{N+1} \in W_{h,0}^{\Delta t}$ is defined by the solution of

$$p_i^{N+1} = l(y_h^N(v_1, v_2) - y_{i,T})\chi_{\omega_T},$$

and for $n = N, ..., 1$,

$$\int_\Omega (\frac{p_i^n - p_i^{n+1}}{\Delta t} z + \nabla p_i^n \cdot \nabla z) dx = k \int_{\omega_d} (y_h^n(v_1, v_2) - y_{i,d}(t^n)) z dx, \quad \forall\ z \in W_{h,0},$$

where $\{y_h^n(v_1, v_2)\}_{n=1}^N$ is defined by the solution of (12)–(14).

Therefore, the problem of finding $(u_1, u_2) \in \mathcal{U}_{1,h}^{\Delta t} \times \mathcal{U}_{2,h}^{\Delta t}$, such that

$$\frac{\partial}{\partial v_1} J_{1,h}^{\Delta t}(u_1, u_2) = 0, \tag{15a}$$

$$\frac{\partial}{\partial v_2} J_{2,h}^{\Delta t}(u_1, u_2) = 0, \tag{15b}$$

is equivalent to

$$u_1 = -\frac{1}{\alpha_1}\{p_1^n \chi_{\omega_1}\}_{n=1}^N;$$

$$u_2 = -\frac{1}{\alpha_2}\{p_2^n \chi_{\omega_2}\}_{n=1}^N;$$

$$y_h^0 = y_{0,h},$$

and for $n = 1, ..., N$, $y_h^n \in W_{h,0}$ is the solution of

$$\int_\Omega (\frac{y_h^n - y_h^{n-1}}{\Delta t} z + \nabla y_h^n \cdot \nabla z) dx = \int_\Omega (f(t^n) + u_1^n \chi_{\omega_1} + u_2^n \chi_{\omega_2}) z dx, \quad \forall\ z \in W_{h,0};$$

$$p_i^{N+1} = l(y_h^N - y_{i,T})\chi_{\omega_T},\ i = 1, 2,$$

and for $n = N, ..., 1$, $p_i^n \in W_{h,0}$, $i = 1, 2$, is the solution of

$$\int_\Omega (\frac{p_i^n - p_i^{n+1}}{\Delta t} z + \nabla p_i^n \cdot \nabla z) dx = k \int_{\omega_d} (y_h^n - y_{i,d}(t^n)) z dx, \quad \forall\ z \in W_{h,0}.$$

8 Conjugate Gradient Solution of Problem (15)

8.1 Generalities

As done for the the semi-discrete problem (10), we can solve problem (15) by the following *conjugate gradient* algorithm:
Step 1. (u_1^0, u_2^0) is given in $\mathcal{U}_{1,h}^{\Delta t} \times \mathcal{U}_{2,h}^{\Delta t}$.
Step 2.a. $y_h^{0,0} = y_{0,h}$, and for $n = 1, ..., N$, $y_h^{0,n} \in W_{h,0}$ is the solution of

$$\int_\Omega (\frac{y_h^{0,n} - y_h^{0,n-1}}{\Delta t} z + \nabla y_h^{0,n} \cdot \nabla z) dx = \int_\Omega (f(t^n) + u_1^{0,n} \chi_{\omega_1} + u_2^{0,n} \chi_{\omega_2}) z dx, \ \forall \ z \in W_{h,0}.$$

Step 2.b. For $i = 1, 2$, $p_i^{0,N+1} = l(y_h^{0,N} - y_{i,T}) \chi_{\omega_T}$, and for $n = N, ..., 1$, $p_i^{0,n} \in W_{h,0}$ is the solution of

$$\int_\Omega (\frac{p_i^{0,n} - p_i^{0,n+1}}{\Delta t} z + \nabla p_i^{0,n} \cdot \nabla z) dx = k \int_{\omega_d} (y_h^{0,n} - y_{i,d}(t^n)) z dx, \ \forall \ z \in W_{h,0}.$$

Step 2.c. $(g_1^0, g_2^0) = (\alpha_1 u_1^0 + p_1^0 \chi_{\omega_1}, \alpha_2 u_2^0 + p_2^0 \chi_{\omega_2})$.

Step 3. $(w_1^0, w_2^0) = (g_1^0, g_2^0)$.

Then, for $k \geq 0$, assuming that (u_1^k, u_2^k), (g_1^k, g_2^k), (w_1^k, w_2^k) are known, all in $\mathcal{U}_{1,h}^{\Delta t} \times \mathcal{U}_{2,h}^{\Delta t}$, we compute (u_1^{k+1}, u_2^{k+1}), (g_1^{k+1}, g_2^{k+1}) and (if necessary) (w_1^{k+1}, w_2^{k+1}) as follows:

Step 4.a. $\overline{y}^{k,0} = 0$, and for $n = 1, ..., N$, $\overline{y}_h^{k,n} \in W_{h,0}$ is the solution of

$$\int_\Omega (\frac{\overline{y}_h^{k,n} - \overline{y}_h^{k,n-1}}{\Delta t} z + \nabla \overline{y}_h^{k,n} \cdot \nabla z) dx = \int_\Omega (w_1^{k,n} \chi_{\omega_1} + w_2^{k,n} \chi_{\omega_2}) z dx, \quad \forall \ z \in W_{h,0}.$$

Step 4.b. $\overline{p}^{k,N+1} = l \overline{y}_h^{k,N} \chi_{\omega_T}$, and for $n = N, ..., 1$, $\overline{p}^{k,n} \in W_{h,0}$ is the solution of

$$\int_\Omega (\frac{\overline{p}^{k,n} - \overline{p}^{k,n+1}}{\Delta t} z + \nabla \overline{p}^{k,n} \cdot \nabla z) dx = k \int_{\omega_d} \overline{y}_h^{k,n} z dx, \quad \forall \ z \in W_{h,0}.$$

Step 4.c. $(\overline{g}_1^k, \overline{g}_2^k) = (\alpha_1 w_1^k + \overline{p}^k \chi_{\omega_1}, \alpha_2 w_2^k + \overline{p}^k \chi_{\omega_2})$.

Step 4.d. $\rho_k = \dfrac{\| (g_1^k, g_2^k) \|^2_{\mathcal{U}_{1,h}^{\Delta t} \times \mathcal{U}_{2,h}^{\Delta t}}}{\Delta t \sum_{n=1}^N \left(\int_{\omega_1} \overline{g}_1^{k,n} w_1^{k,n} dx + \int_{\omega_2} \overline{g}_2^{k,n} w_2^{k,n} dx \right)}$.

Step 5. $(u_1^{k+1}, u_2^{k+1}) = (u_1^k, u_2^k) - \rho_k (w_1^k, w_2^k)$.
Step 6. $(g_1^{k+1}, g_2^{k+1}) = (g_1^k, g_2^k) - \rho_k (\overline{g}_1^k, \overline{g}_2^k)$.

If $\dfrac{\| (g_1^{k+1}, g_2^{k+1}) \|^2_{\mathcal{U}_{1,h}^{\Delta t} \times \mathcal{U}_{2,h}^{\Delta t}}}{\| (g_1^0, g_2^0) \|^2_{\mathcal{U}_{1,h}^{\Delta t} \times \mathcal{U}_{2,h}^{\Delta t}}} \leq \varepsilon$, then take $(u_1, u_2) = (u_1^{k+1}, u_2^{k+1})$; else:

Step 7. $\gamma_k = \dfrac{\| (g_1^{k+1}, g_2^{k+1}) \|^2_{\mathcal{U}_{1,h}^{\Delta t} \times \mathcal{U}_{2,h}^{\Delta t}}}{\| (g_1^k, g_2^k) \|^2_{\mathcal{U}_{1,h}^{\Delta t} \times \mathcal{U}_{2,h}^{\Delta t}}}$.

Step 8. $(w_1^{k+1}, w_2^{k+1}) = (g_1^{k+1}, g_2^{k+1}) + \gamma_k (w_1^k, w_2^k)$.

Step 9. Do $k = k + 1$, and go to Step 7.

Remark 8.1 The practical application of the above algorithm requires the calculation of various integrals over the triangles of $\mathcal{T}_h$. Integrals such as $\int_T y_h^{k,n} z dx$, $\int_T w_i^{k,n} z dx$ $(i = 1, 2)$ can be computed exactly without difficulty for the one-dimensional case and by using the *two-dimensional Simpson's rule* for the two-dimensional case (see, e.g., page 96 of Ref. 8). Numerical integration is necessary in order to compute integrals such as $\int_T f(t^n) z dx$, $\int_T y_{i,d}(t^n) z dx$, $\int_T y_{i,T} z dx$ $(i = 1, 2)$. Some quadrature schemes for computing approximately these integrals can be found in, e.g., pp. 178–185 of Ref. 11.

9 Numerical Experiments.

9.1 Generalities.

We consider the domain Ω defined by $\Omega = (0, 1) \times (0, 1)$ and the space discretization step h defined by $h = 1/(I - 1)$, where I is a positive integer (I^2 =number of vertices). Then, for every $i, j \in \{1, \cdots I\}$, we take the triangulation $\mathcal{T}_h$ with vertex $x_{i,j} = ((i-1)h, (j-1)h)$ and the triangles as in the typical case showed in Figure 1 (we point out that, due to the special domain consider in this example, we have denote by h a parameter different to the one defined in Section 7.1).

We consider the case $\Gamma = \Gamma_1$ (i.e. $\Gamma_2 = \emptyset$), $\omega_1 = (0, 0.25) \times (0, 0.25)$, $\omega_2 = (0.75, 1) \times (0, 0.25)$, $\omega_T = \Omega$ and $\omega_d = (0.25, 0.75) \times (0.25, 0.75)$ (see Figure 2). For the data of the problem we take $f \equiv 1$, $y_0 \equiv 0$, $g_1 = 0$ and $g_2 = 0$. In the conjugate gradient algorithm we take the initial guess $(u_1^0, u_2^0) = (0, 0)$ and the stopping criterion $\varepsilon = 10^{-8}$.

9.2 Stabilization Type Test Problems ($k > 0$ and $l = 0$).

We consider the finite horizon time $T = 1.5$ and the time step discretization $\Delta t = 1.5/45$. For the space discretization we consider the step $h = 1/36$ ($I = 37$). In order to see how the non-controlled solution behaves, we have visualized in Figure 3 the computed solution of the non-controlled equation at time $t = 1.5$.

Throughout this section we use $k = 1$ and $l = 0$.

Same Goal: $y_{1,d} = y_{2,d} = 1$.

In Figures 4–5 we have visualized the graph of the computed solution of the controlled equation with $\alpha_1 = \alpha_2 = 10^{-6}$.

In Figures 6–7 we have visualized the graph of the computed solution of the controlled equation with $\alpha_1 = 10^{-8}$ and $\alpha_2 = 10^{-2}$.

In Figure 8 we have visualized the graph of $\| y(t) - 1 \|^2_{L^2(\omega_d)}$ for different cases. In Table 1 we give some further results about our solutions.

Different Goals: $y_{1,d} = 1$ **and** $y_{2,d} = -1$**.**

In Figures 9–10 we have visualized the graph of the computed solution of the controlled equation with $\alpha_1 = \alpha_2 = 10^{-6}$.

In Figures 11–12 we have visualized the graph of the computed solution of the controlled equation with $\alpha_1 = 10^{-8}$ and $\alpha_2 = 10^{-2}$.

In Figures 13–14 we have visualized the graph of $\| y(t) - 1 \|^2_{L^2(\omega_d)}$ and $\| y(t) - (-1) \|^2_{L^2(\omega_d)}$ for different cases. In Table 2 we give some further results about our solutions.

Remark 9.1 We point out (see Figures 13 and 14) that, when the goals are different, the controlled solution can be worse (in terms of $L^2(\omega_d \times (0,T))$ with respect to both goals than the uncontrolled solution.

9.3 Controllability Type Test Problems ($k = 0$ and $l > 0$).

Throughout this section we consider $k = 0$ and $l = 1$. We also consider the finite horizon time $T = 3$ and the time step discretization $\Delta t = 3/90$. For the space discretization we consider the step $h = 1/67$ ($I = 68$). The solution without controls behaves again as showed in Figure 3.

Same Goal: $y_{1,T} = y_{2,T} = 1$**.**

In Figures 15–16 we have visualized the graph of the computed solution of the controlled equation with $\alpha_1 = \alpha_2 = 10^{-6}$.

In Figures 17–18 we have visualized the graph of the computed solution of the controlled equation with $\alpha_1 = 10^{-6}$ and $\alpha_2 = 10^{-3}$.

In Figure 19 we have visualized the graph of $\| y(t) - 1 \|^2_{L^2(\omega_T)}$ for different cases, for $t \in [2.5, 3]$ (for $t < 2.5$ the solutions are almost identical). In Table 3 we give some further results about our solutions.

Different Goals: $y_{1,T} = 1$ **and** $y_{2,T} = -1$**.**

In Figures 20–21 we have visualized the graph of the computed solution of the controlled equation with $\alpha_1 = \alpha_2 = 10^{-6}$.

In Figures 22–23 we have visualized the graph of the computed solution of the controlled equation with $\alpha_1 = 10^{-6}$ and $\alpha_2 = 10^{-3}$.

In Figures 24–25 we have visualized the graph of $\| y(t) - 1 \|^2_{L^2(\omega_T)}$ and $\| y(t) - (-1) \|^2_{L^2(\omega_T)}$ for different cases, for $t \in [2.5, 3]$ (for $t < 2.5$ the solutions are almost identical). In Table 4 we give some further results about our solutions.

10 Conclusions

The numerical results obtained in this article (see also Ref. 12) are consistent with what we can expect from a non-cooperative strategy as the Nash strategy;

more precisely:

(i) In a neighborhood of each control domain ω_i (i=1,2), the controlled solution tends to be close to the target function y_{di}, for the stabilization type problems, and close to the target function y_{Ti}, at time T, for the controllability type problems.

(ii) In the same way as two companies having *opposite* goals can go bankrupt if they do not cooperate, in our problems there are cases, when the target functions are not compatible (i.e. when being close to one of the target functions implies to be far from the other), where the solution without control is closer to some of the desired states (or both of them) than the solution obtained with the Nash strategy. This can be seen by comparing, for the different cases, the graphs of $\| y(u;x,t) - y_{di} \|^2_{L^2(\omega_d \times (0,T))}$ with the graphs of $\| y(0;x,t) - y_{di} \|^2_{L^2(\omega_d \times (0,T))}$, and the graphs of $\| y(u;x,T) - y_{Ti} \|^2_{L^2(\omega_T)}$ with the graphs of $\| y(0;x,T) - y_{Ti} \|^2_{L^2(\omega_T)}$, for $i = 1,2$.

The results obtained in this article, for the linear heat equation, call for an investigation of Nash equilibria for more complicated nonlinear models such as the Burgers equation (see Ref. 13) or the Navier-Stokes equations.

References

1. Nash, J.F., *Noncooperative Games*, Annals of Mathematics, Vol. 54, pp. 286–295, 1951.

2. Pareto, V., *Cours d'Économie Politique*, Rouge, Lausanne, Switzerland, 1896.

3. von Stackelberg, H., *Marktform und Gleichgewicht*, Springer, Berlin, Germany, 1934.

4. Lions, J.L., *Contrôle de Pareto de Systèmes Distribués: Le Cas d'Évolution*, Comptes Rendus de l'Académie des Sciences, Serie I, Vol. 302, pp. 413–417, 1986.

5. Lions, J.L. *Some Remarks on Stackelberg's Optimization*, Mathematical Models and Methods in Applied Sciences, Vol. 4, pp. 477–487, 1994.

6. Díaz, J.I. and Lions, J.L., *On the Approximate Controllability of Stackelberg-Nash Strategies.* Proceedings of the Diderot Mathematical Forum *Mathematics and Environment: Problems Related to Water* (to appear).

7. Bristeau, M.O., Glowinski, R., Mantel, B., Periaux, J., and Sefrioui, M., *Genetic Algorithms for Electromagnetic Backscattering Multiobjective Optimization*, Electromagnetic Optimization by Genetic Algorithms, Edited by Y. Rahmat-Samii and E. Michielssen, John Wiley, New York, NY, pp. 399–434, 1999.

8. Glowinski, R., *Numerical Methods for Nonlinear Variational Problems*, Springer, New York, NY, 1984.

9. Daniel, J., *The Approximate Minimization of Functionals*, Prentice-Hall, Englewood Cliffs, New Jersey, 1970.

10. Lions, J.L and Magenes, E., *Problèmes aux Limites Non-Homogènes et Applications*, Volumes I and II, Dunod, Paris, France, 1968.

11. Ciarlet, P.G., *The Finite Element Method for Elliptic Problems*, North-Holland, New York, NY, 1978.

12. Ramos, A.M., Glowinski, R. and Periaux, J., *Nash Equilibria for the Multiobjective Control of Linear Partial Differential Equations*, Universidad Complutense de Madrid, Matemática Aplicada Report No. MA-UCM 2001-5, 2001.

13. Ramos, A.M., Glowinski, R. and Periaux, J., *Pointwise Control of the Burgers Equation and related Nash Equilibria Problems: A Computational Approach.* To appear in Journal of Optimization Theory and Applications.

List of Tables.

	No control	$\alpha_1 = 10^{-4}$ $\alpha_2 = 10^{-4}$	$\alpha_1 = 10^{-8}$ $\alpha_2 = 10^{-2}$	$\alpha_1 = 10^{-6}$ $\alpha_2 = 10^{-6}$
$\parallel y(t) - 1 \parallel^2_{L^2(\omega_d \times (0,1.5))}$	0.330592	0.218454	0.075907	0.0556384

Table 1:

	No control	$\alpha_1 = 10^{-4}$ $\alpha_2 = 10^{-4}$	$\alpha_1 = 10^{-8}$ $\alpha_2 = 10^{-2}$	$\alpha_1 = 10^{-6}$ $\alpha_2 = 10^{-6}$
$\parallel y(t) - 1 \parallel^2_{L^2(\omega_d \times (0,1.5))}$	0.330592	0.343811	0.0763473	1.07423
$\parallel y(t) + 1 \parallel^2_{L^2(\omega_d \times (0,1.5))}$	0.422275	0.420292	1.20273	1.09692

Table 2:

	No control	$\alpha_1 = 10^{-4}$ $\alpha_2 = 10^{-4}$	$\alpha_1 = 10^{-6}$ $\alpha_2 = 10^{-3}$	$\alpha_1 = 10^{-6}$ $\alpha_2 = 10^{-6}$
$\parallel y(t) - 1 \parallel^2_{L^2(\omega_T)}$	0.931462	0.436426	0.306953	0.263431

Table 3:

	No control	$\alpha_1 = 10^{-4}$ $\alpha_2 = 10^{-4}$	$\alpha_1 = 10^{-6}$ $\alpha_2 = 10^{-3}$	$\alpha_1 = 10^{-6}$ $\alpha_2 = 10^{-6}$
$\parallel y(t) - 1 \parallel^2_{L^2(\omega_d)}$	0.931462	1.35798	0.605875	4.72735
$\parallel y(t) + 1 \parallel^2_{L^2(\omega_d)}$	1.07194	1.42389	2.7438	4.7366

Table 4:

List of Figures.

Fig. 21: Computed solution of the controlled equation with $\alpha_1 = \alpha_2 = 10^{-6}$ at time $t = 3$.

Fig. 22: Computed solution of the controlled equation with $\alpha_1 = 10^{-6}, \alpha_2 = 10^{-3}$ at time $t = 2.5$.

Fig. 23: Computed solution of the controlled equation with $\alpha_1 = 10^{-6}, \alpha_2 = 10^{-3}$ at time $t = 3$.

Fig. 24: $\| y(t) - 1 \|^2_{L^2(\omega_T)}$, y is the computed solution for the following cases: uncontrolled equation (–), $\alpha_1 = \alpha_2 = 10^{-4}$ (oo), $\alpha_1 = \alpha_2 = 10^{-6}$ (- -), $\alpha_1 = 10^{-6}$ and $\alpha_2 = 10^{-3}$ ($**$).

Fig. 25: $\| y(t) + 1 \|^2_{L^2(\omega_T)}$, y is the computed solution for the following cases: uncontrolled equation (–), $\alpha_1 = \alpha_2 = 10^{-4}$ (oo), $\alpha_1 = \alpha_2 = 10^{-6}$ (- -), $\alpha_1 = 10^{-6}$ and $\alpha_2 = 10^{-3}$ ($**$).

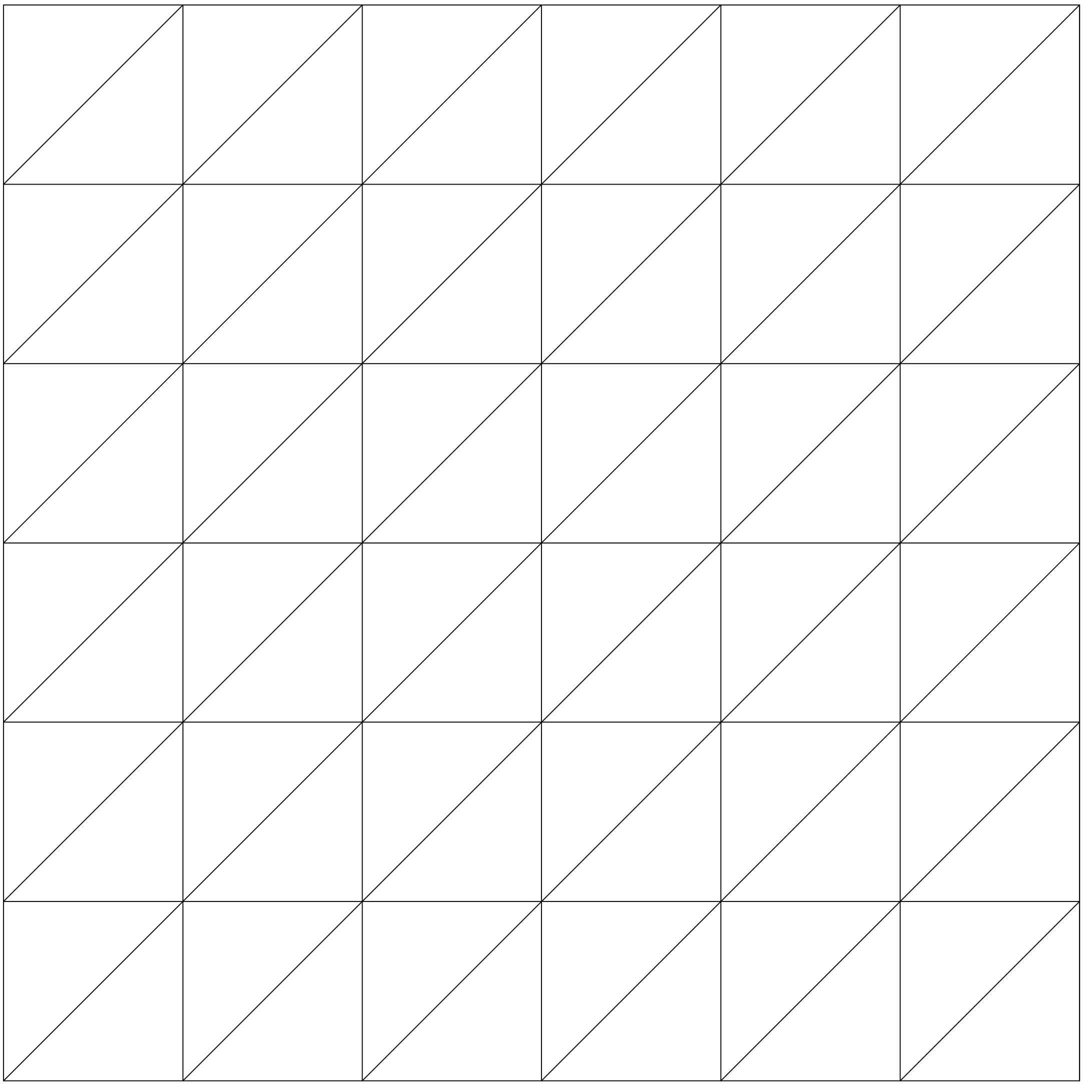


Fig. 1:

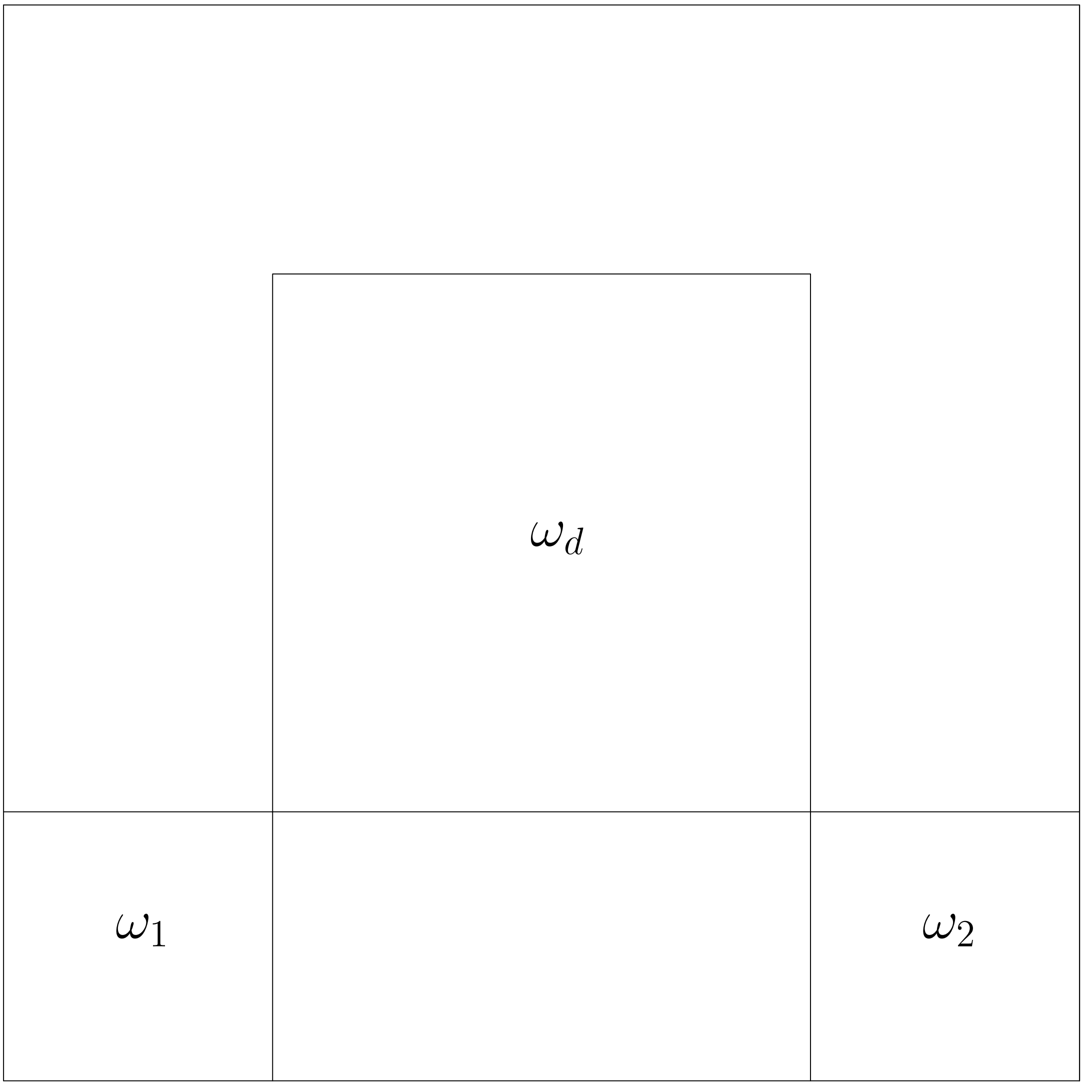


Fig. 2:

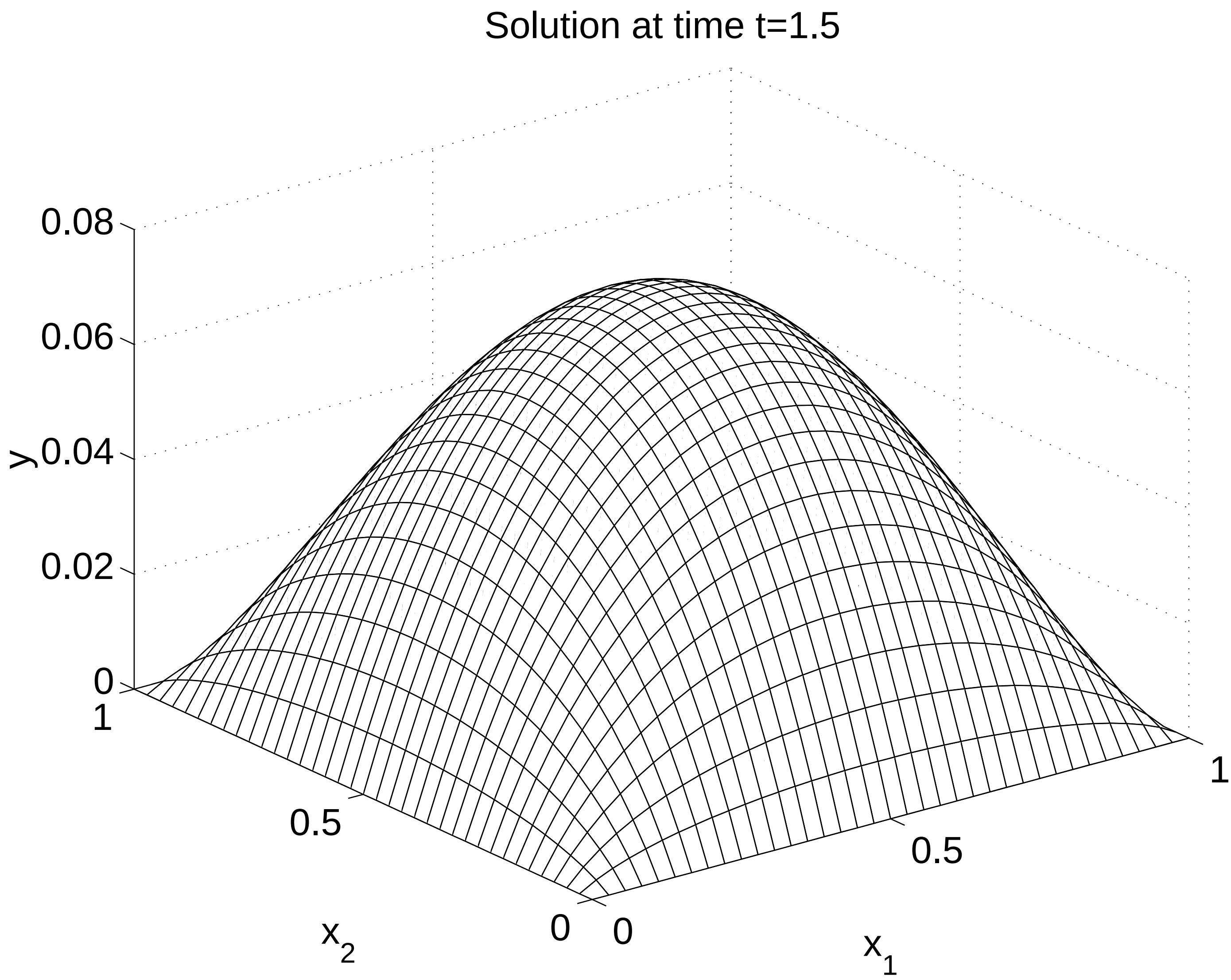
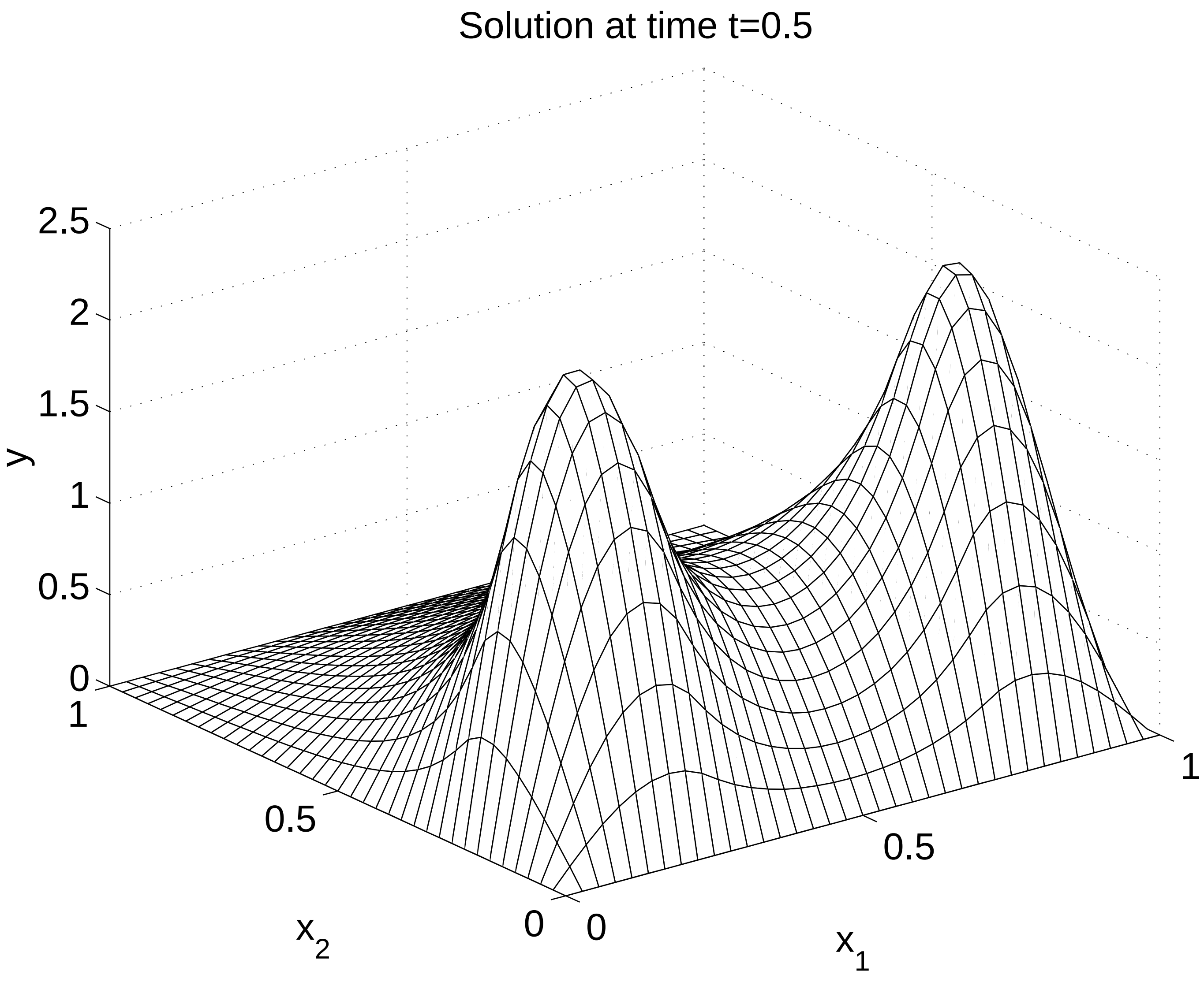


Fig. 3:



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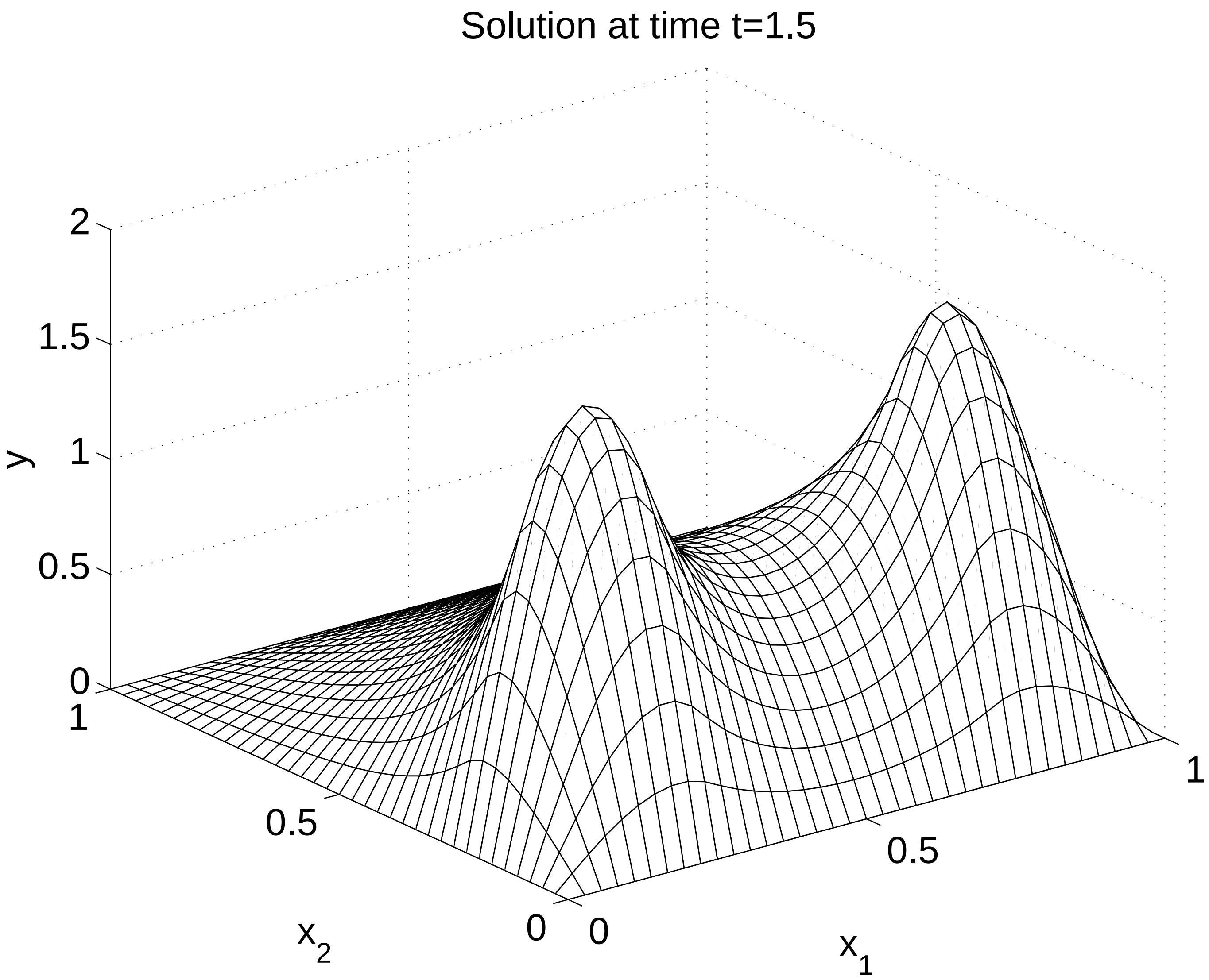

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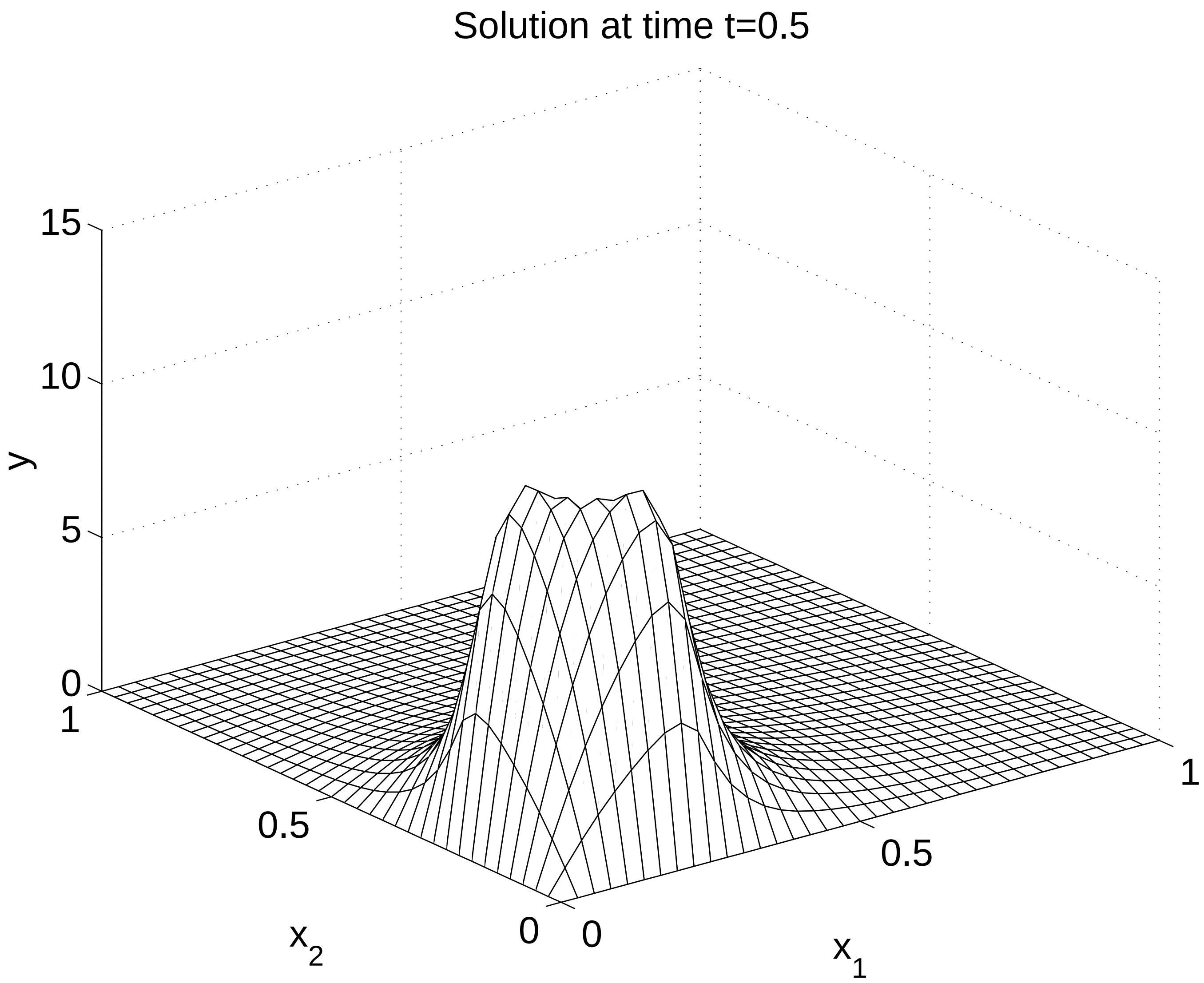

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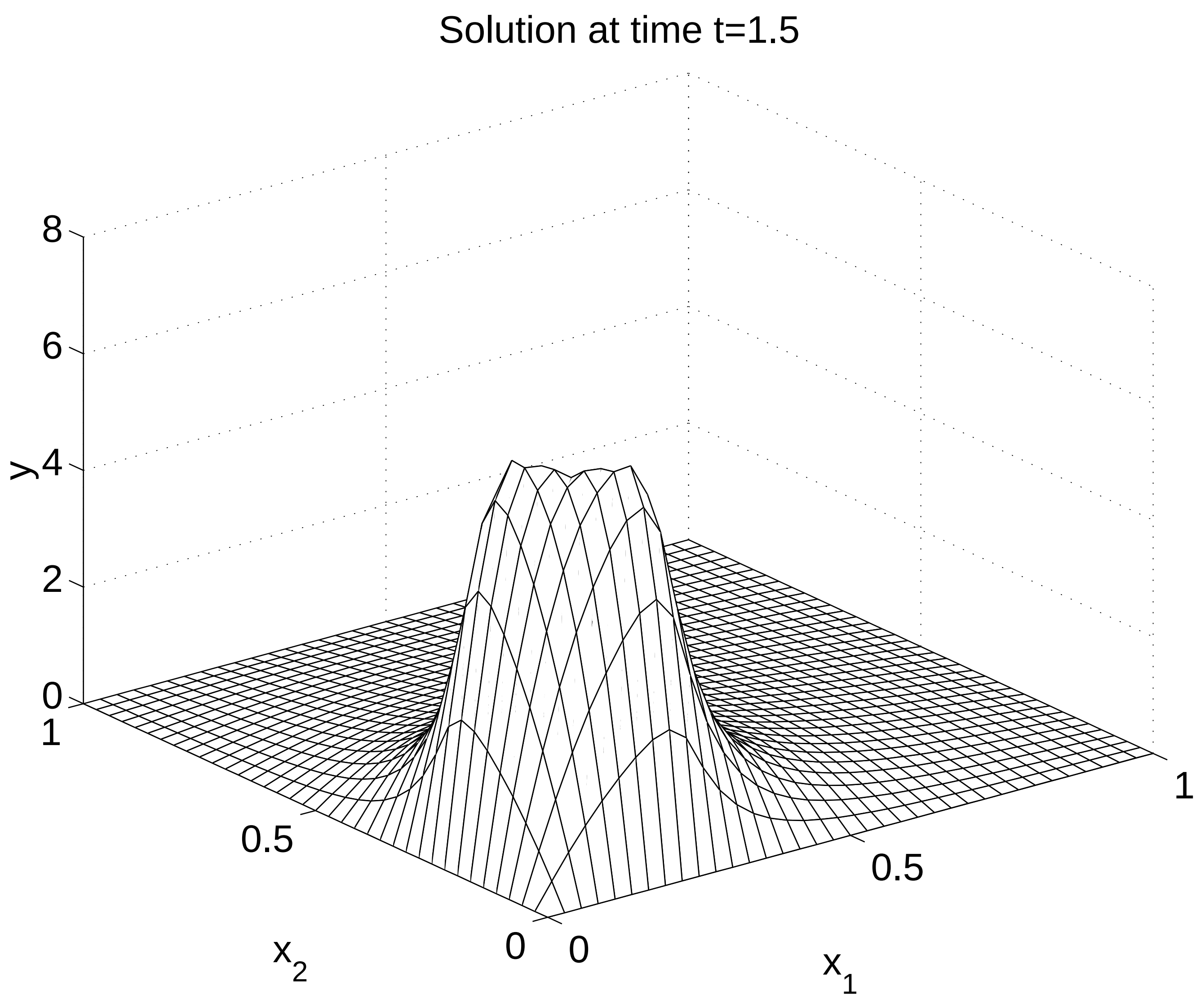

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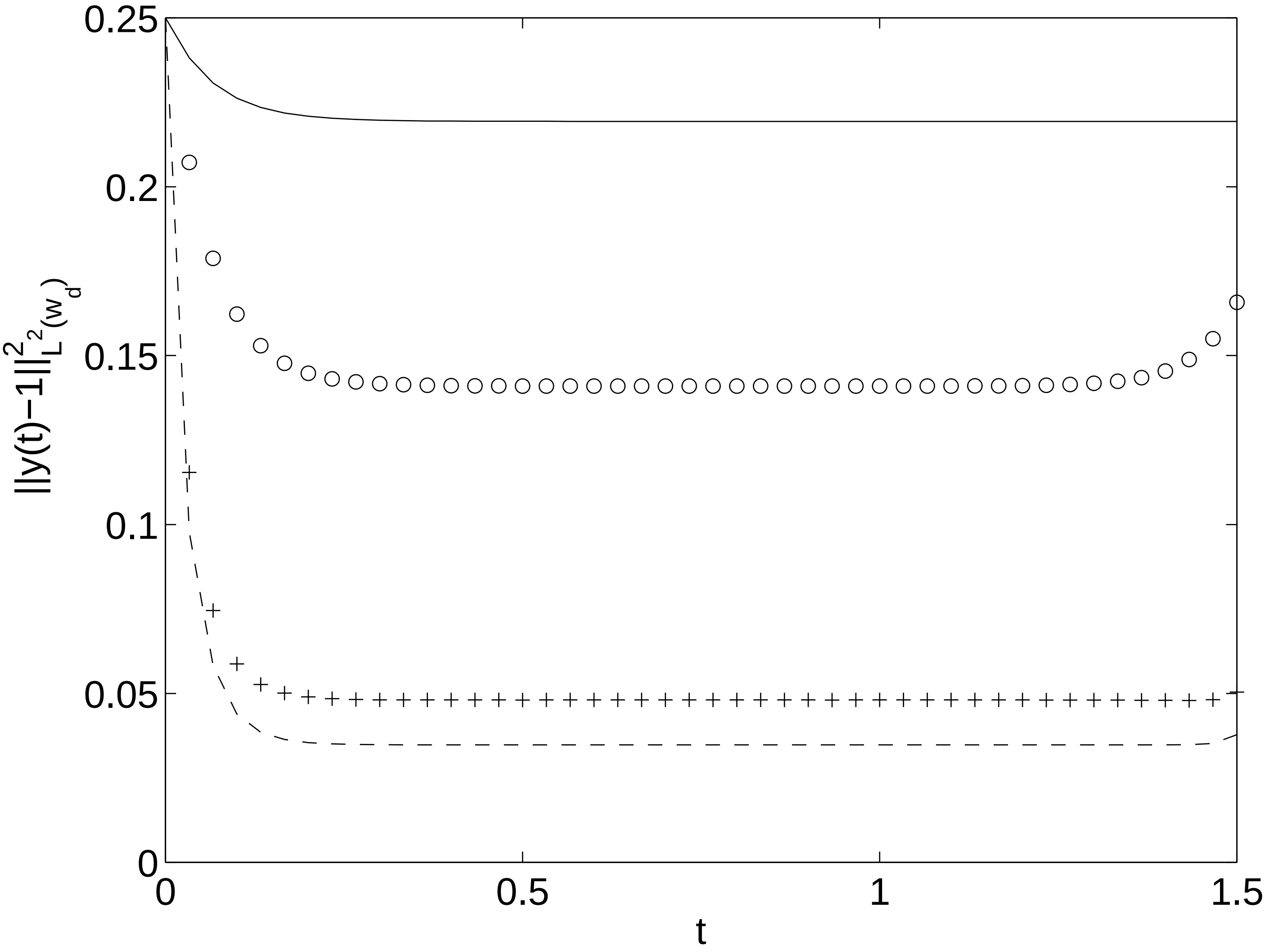

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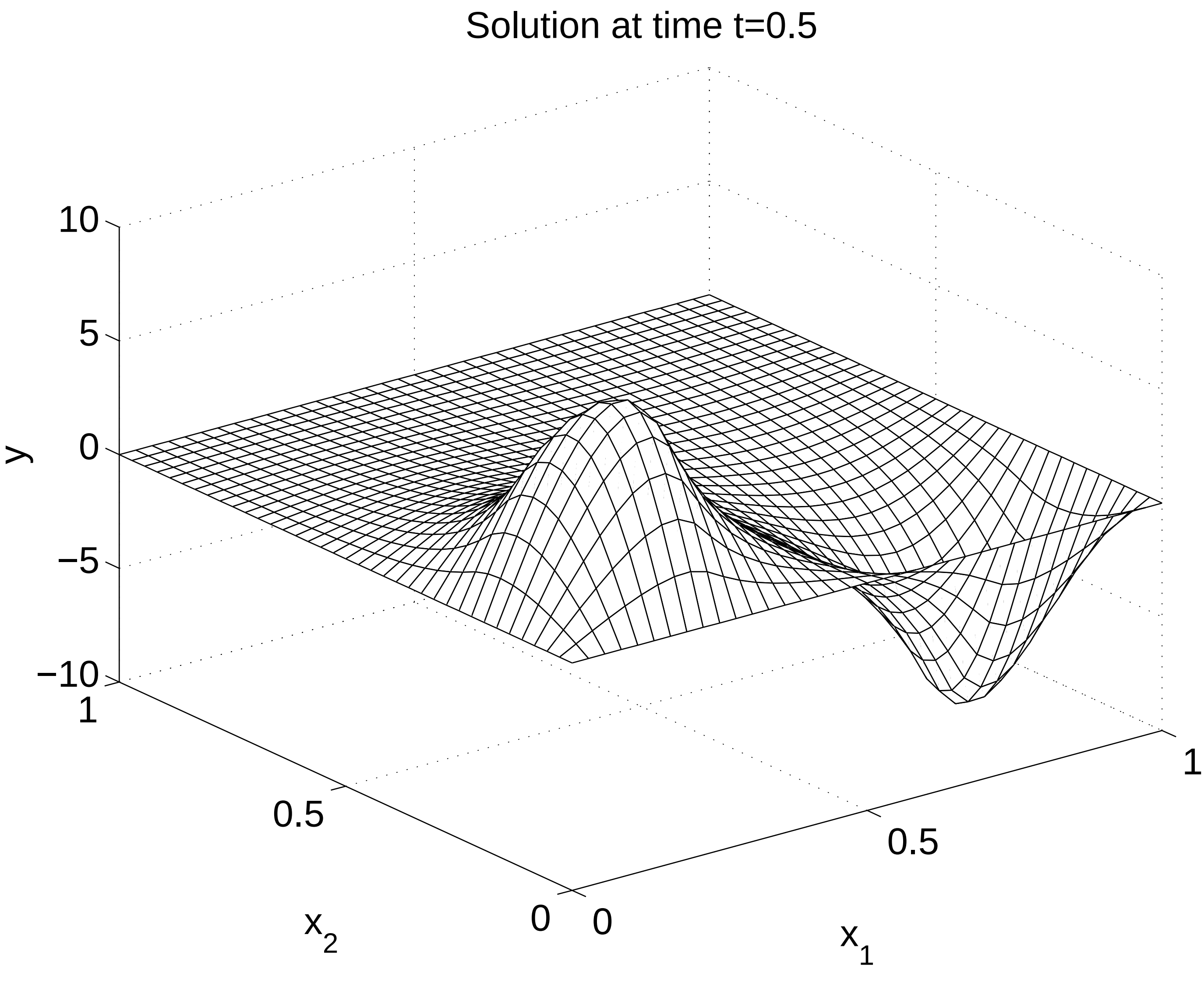

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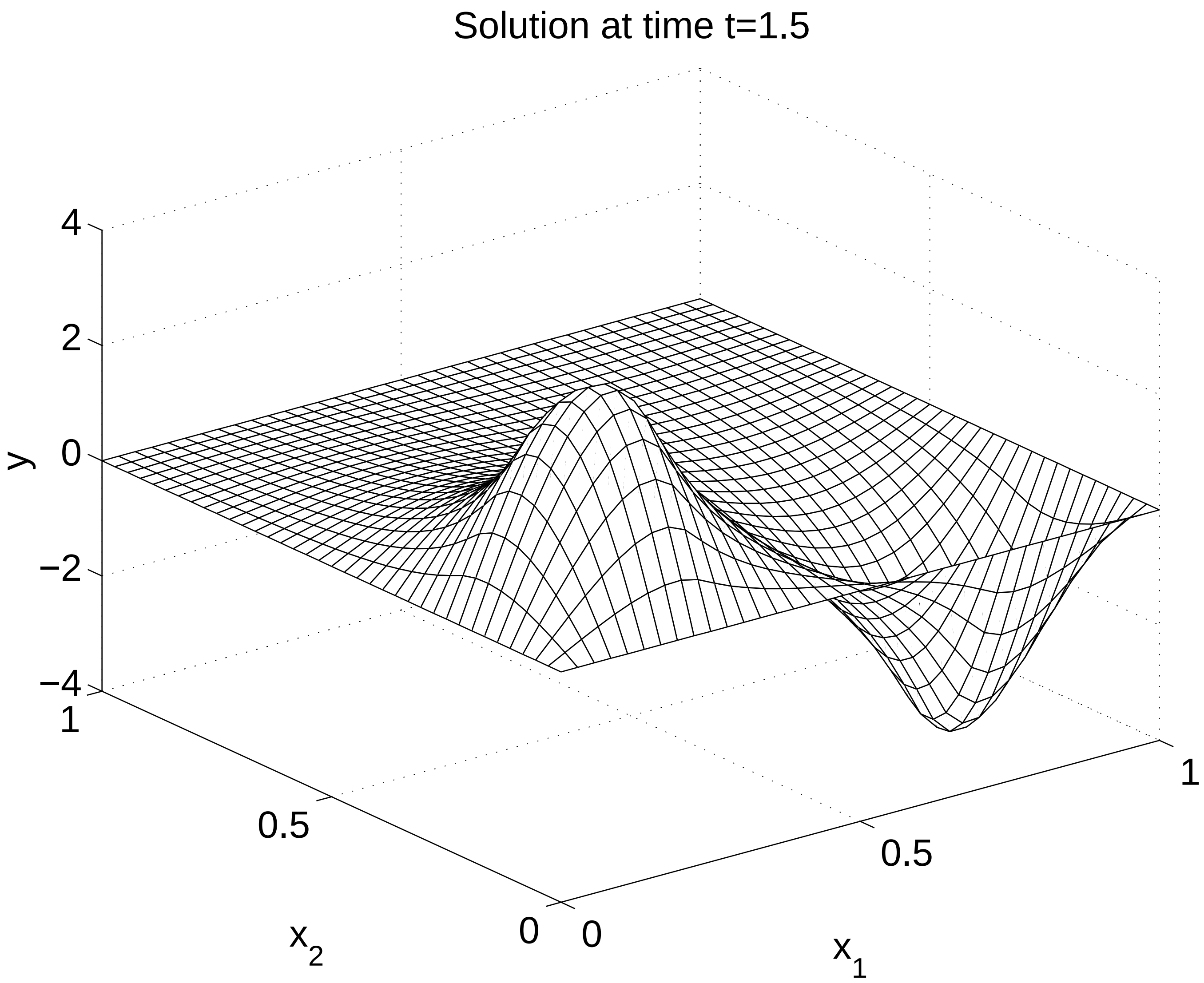

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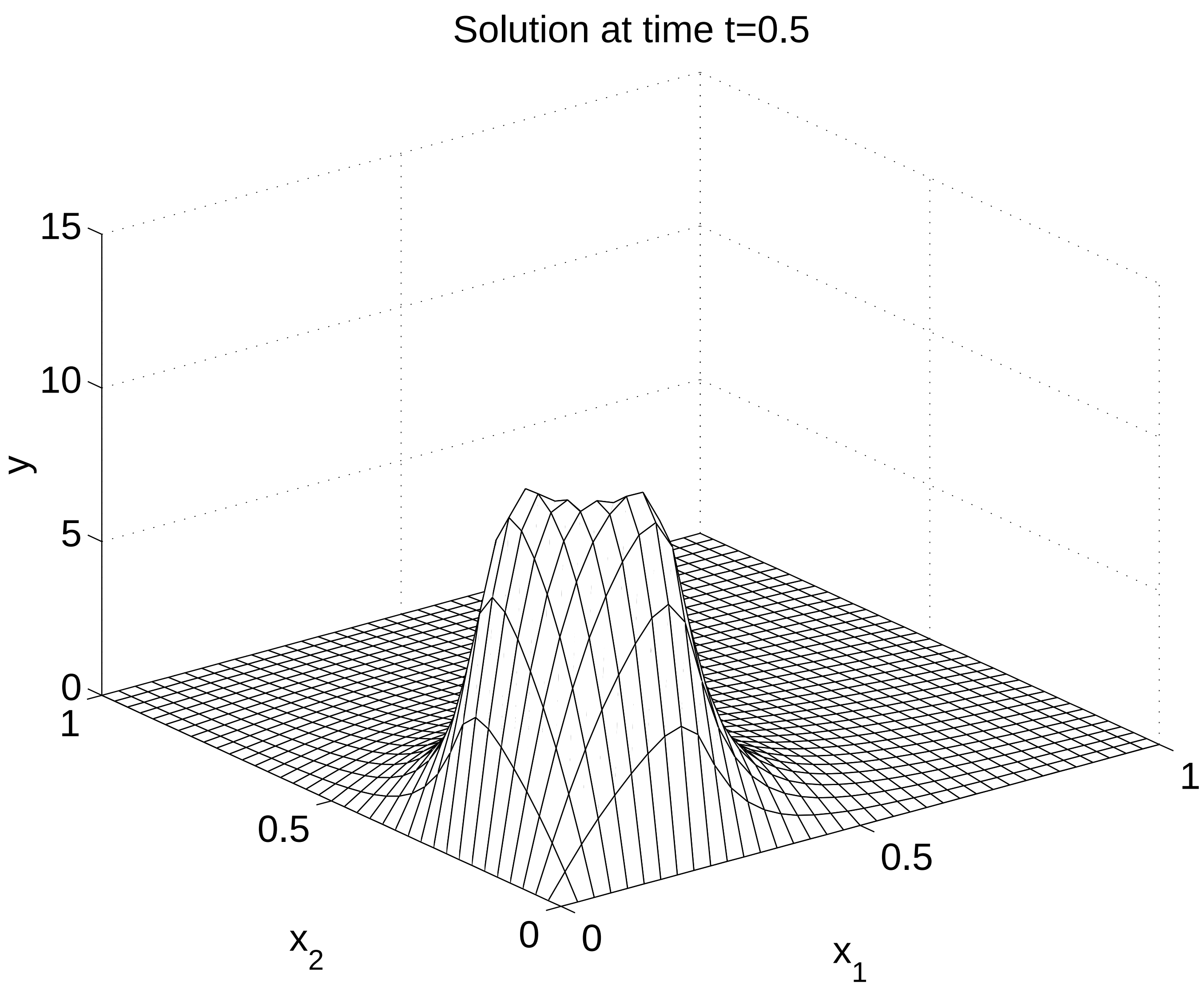

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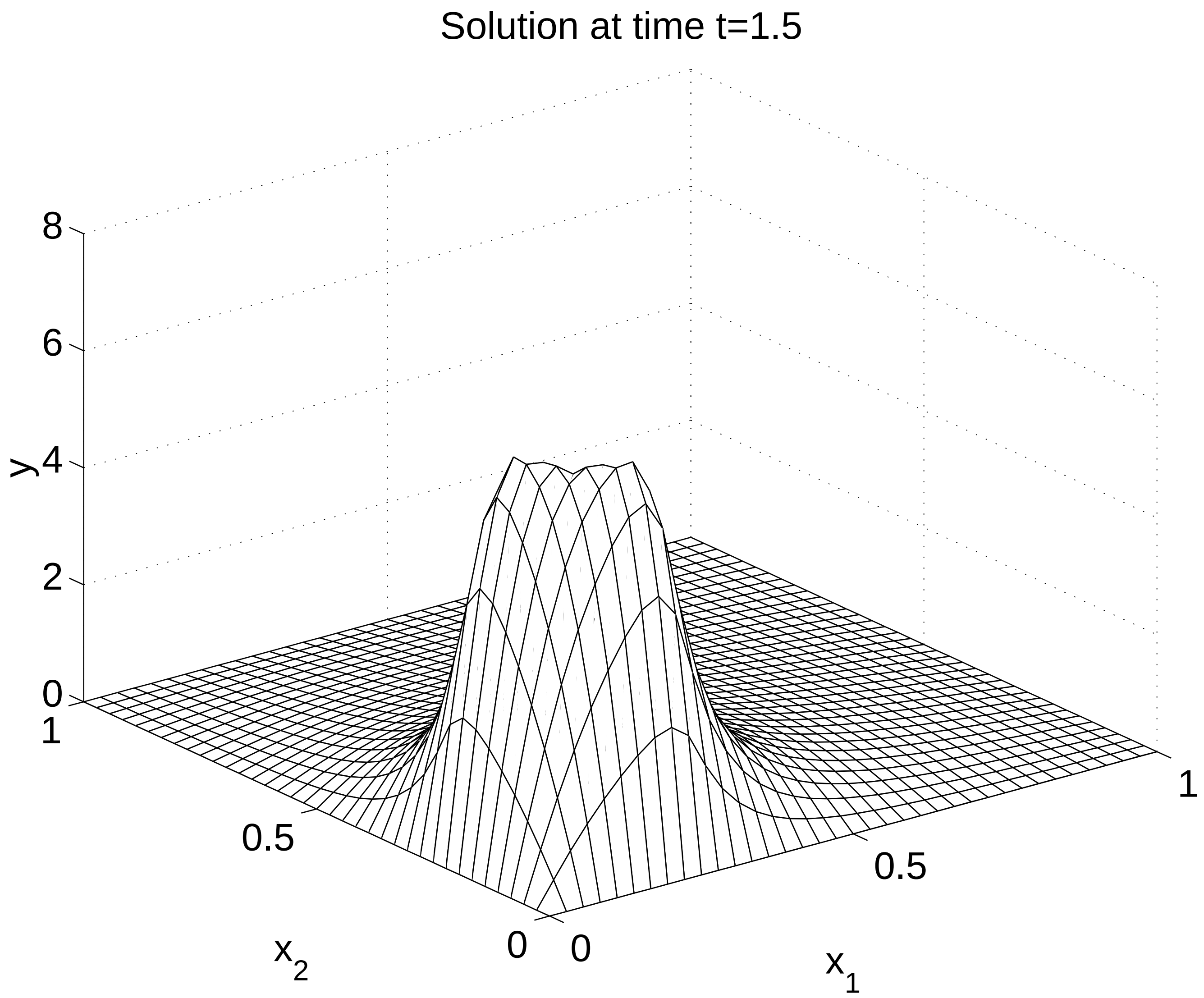

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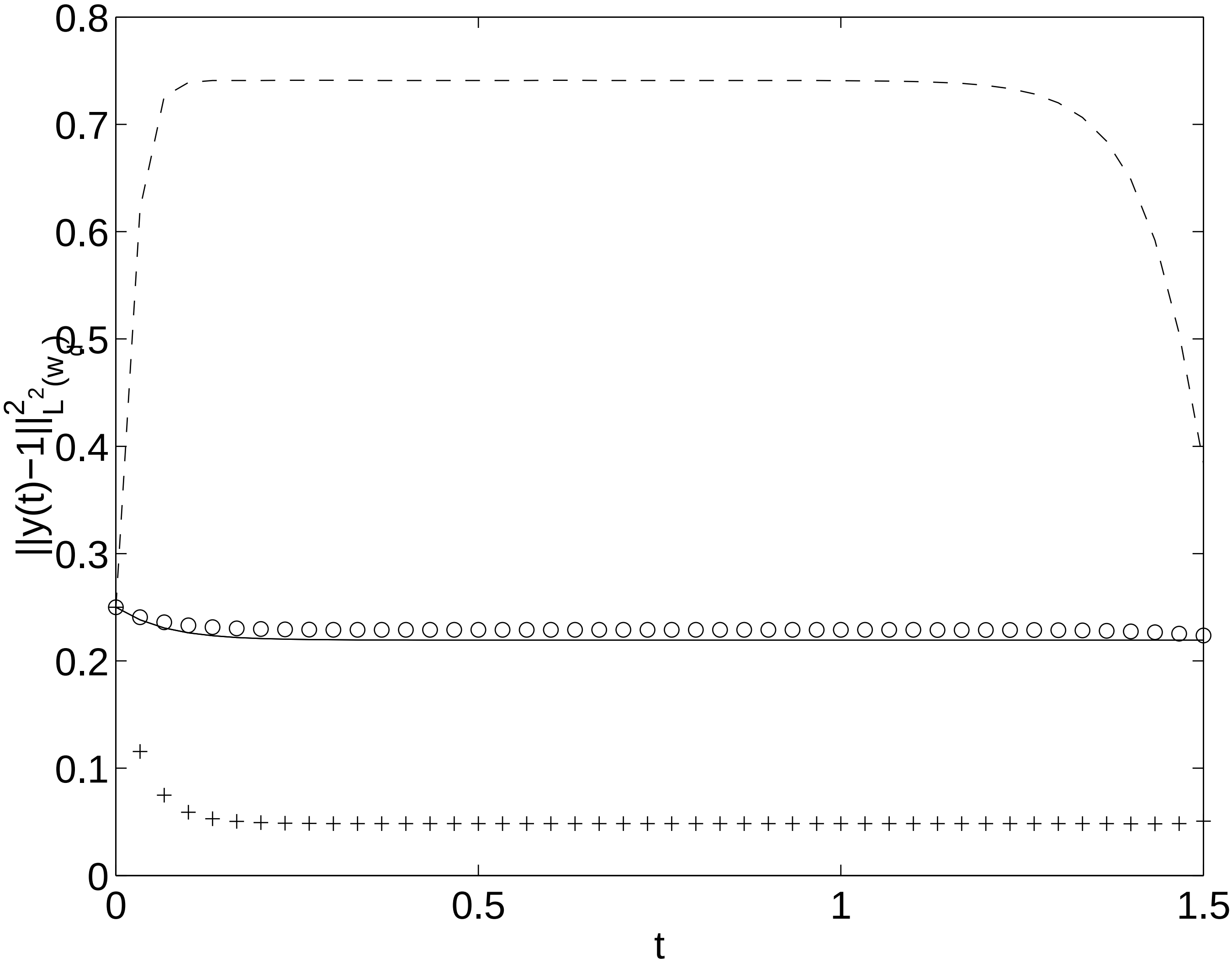

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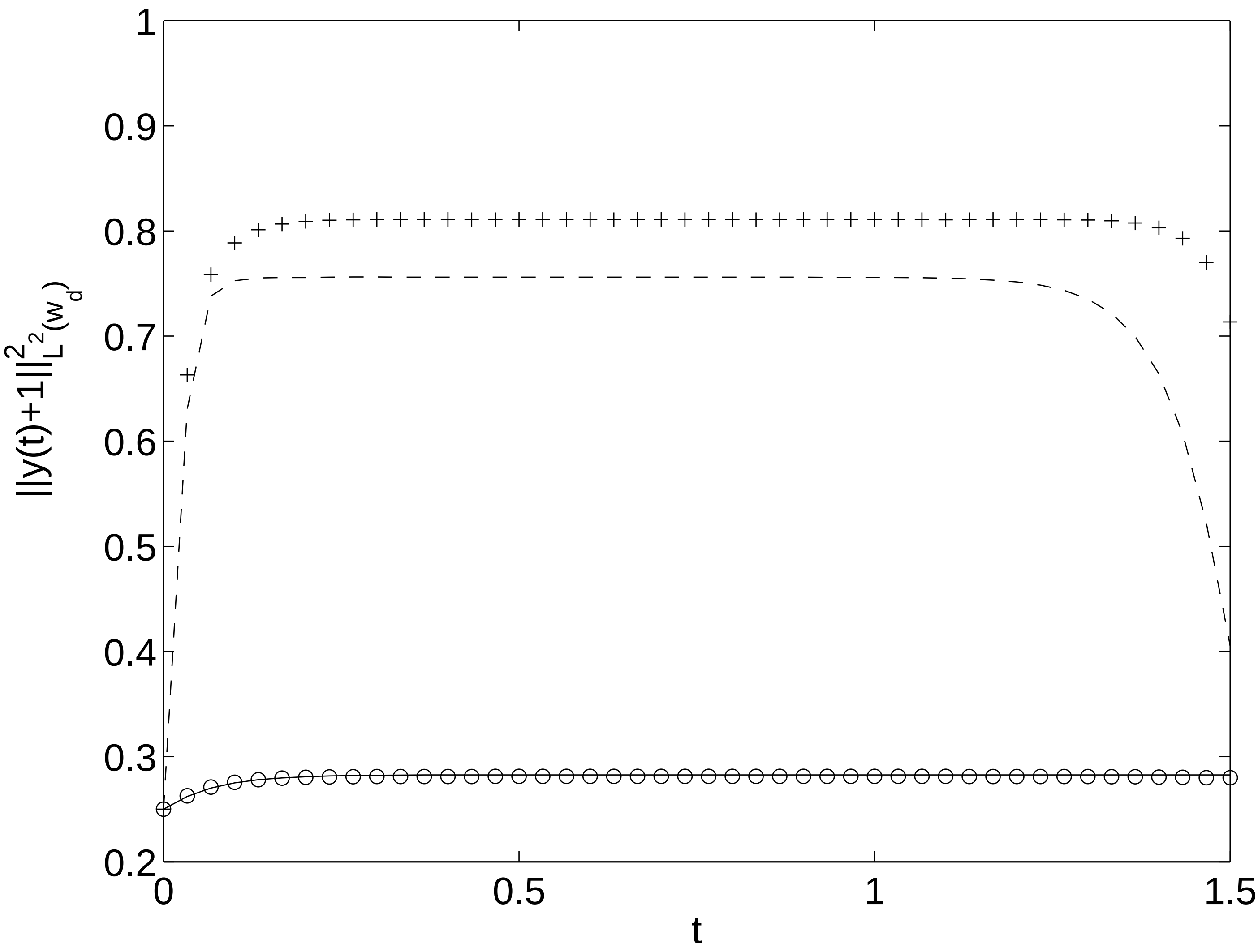

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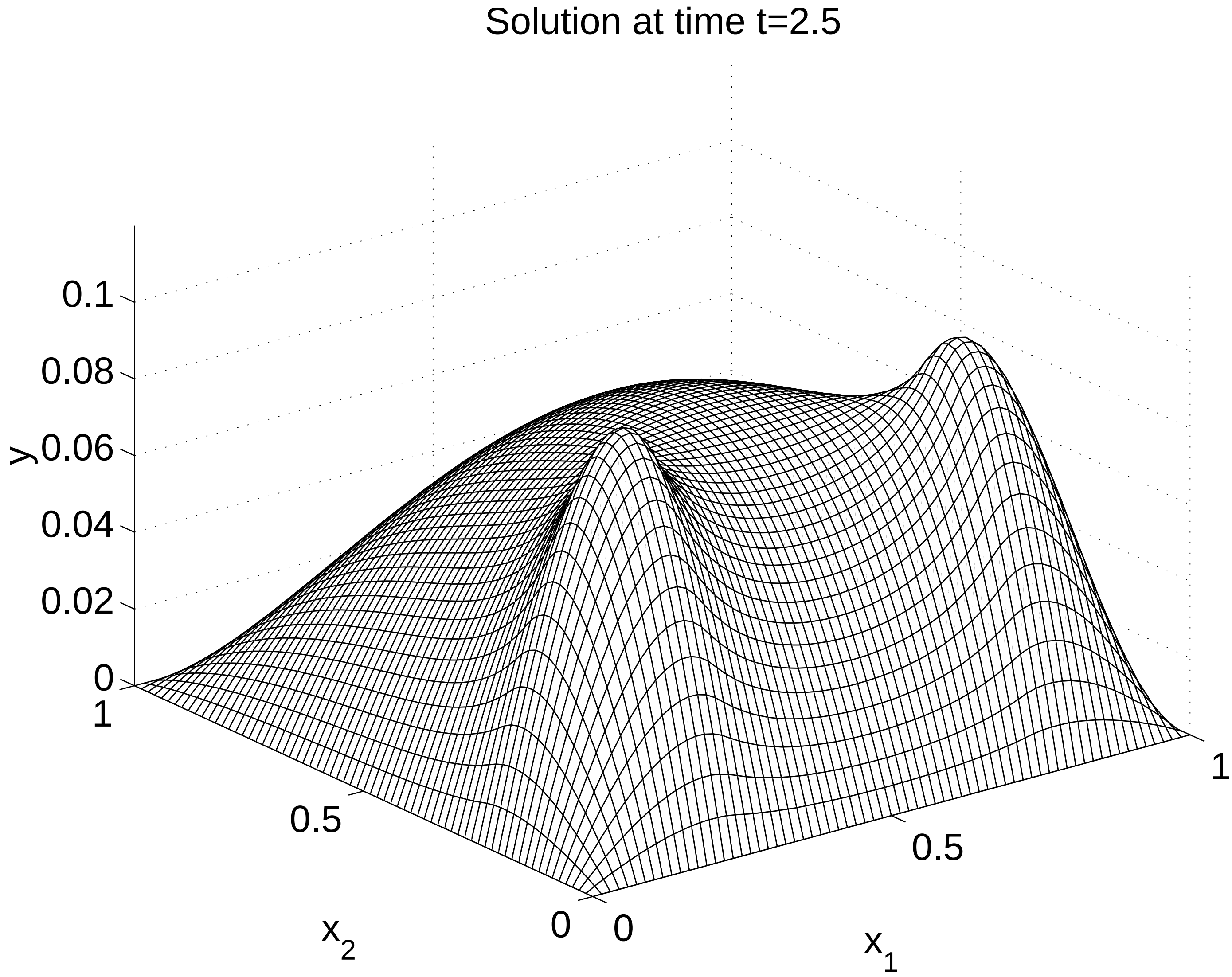

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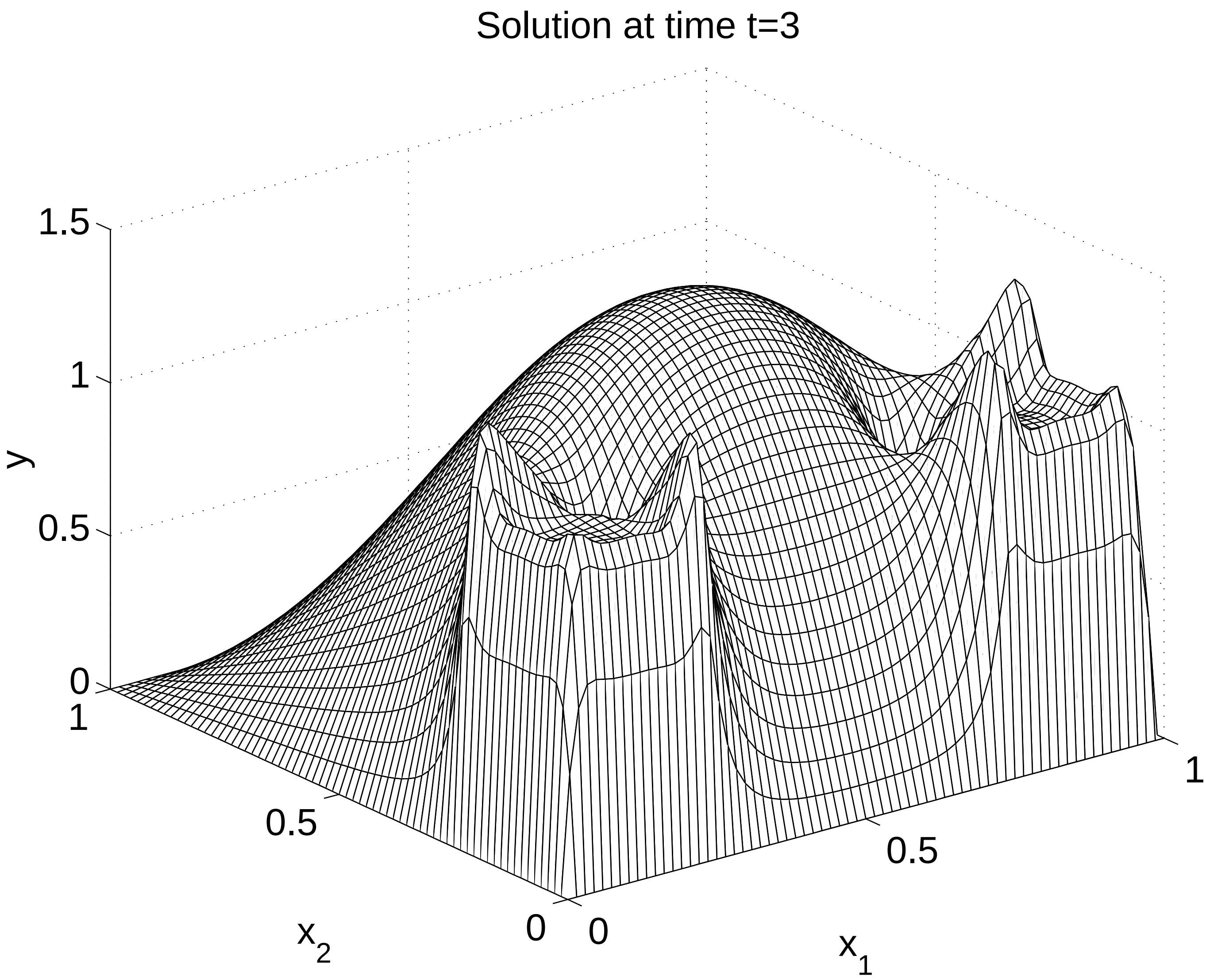

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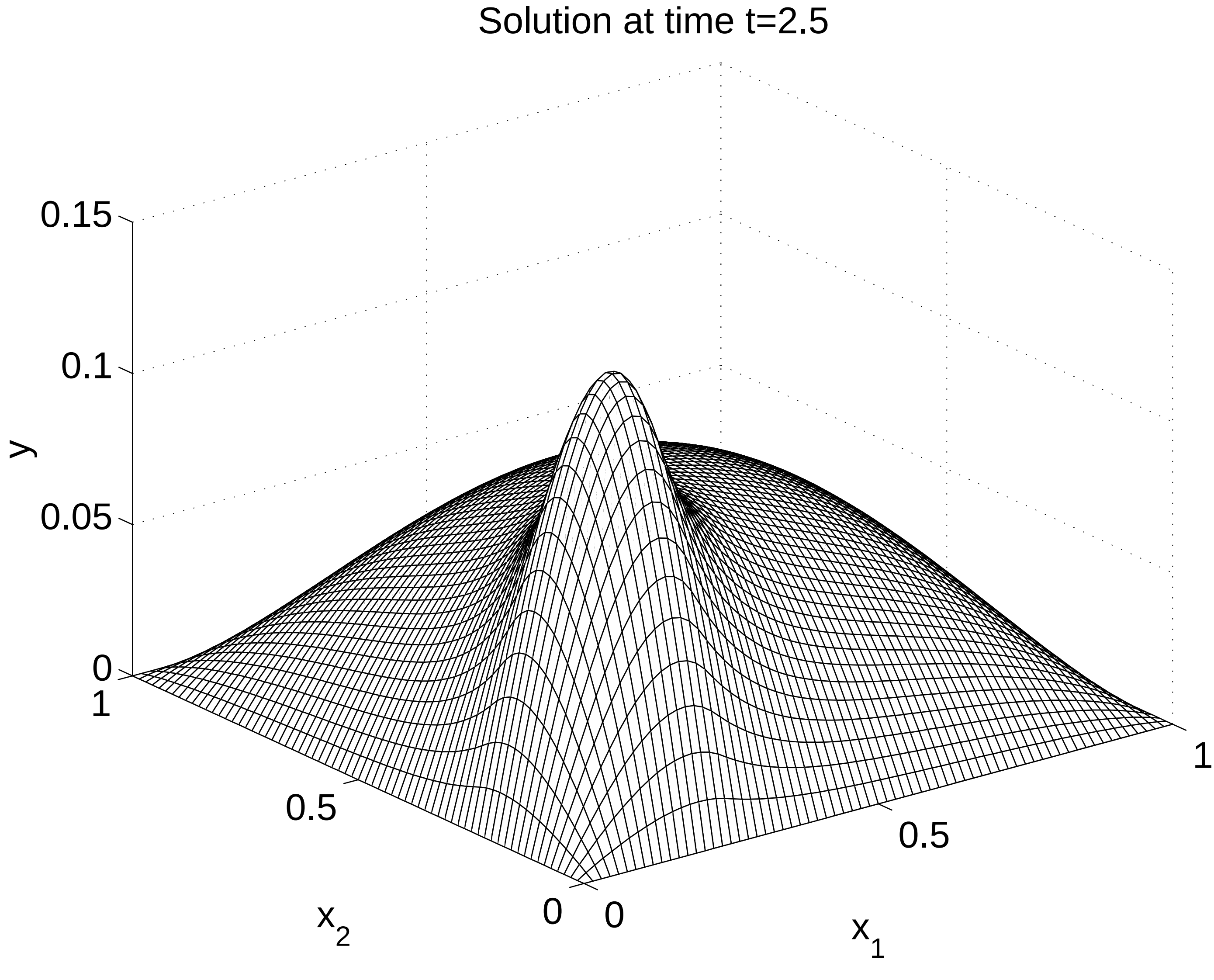

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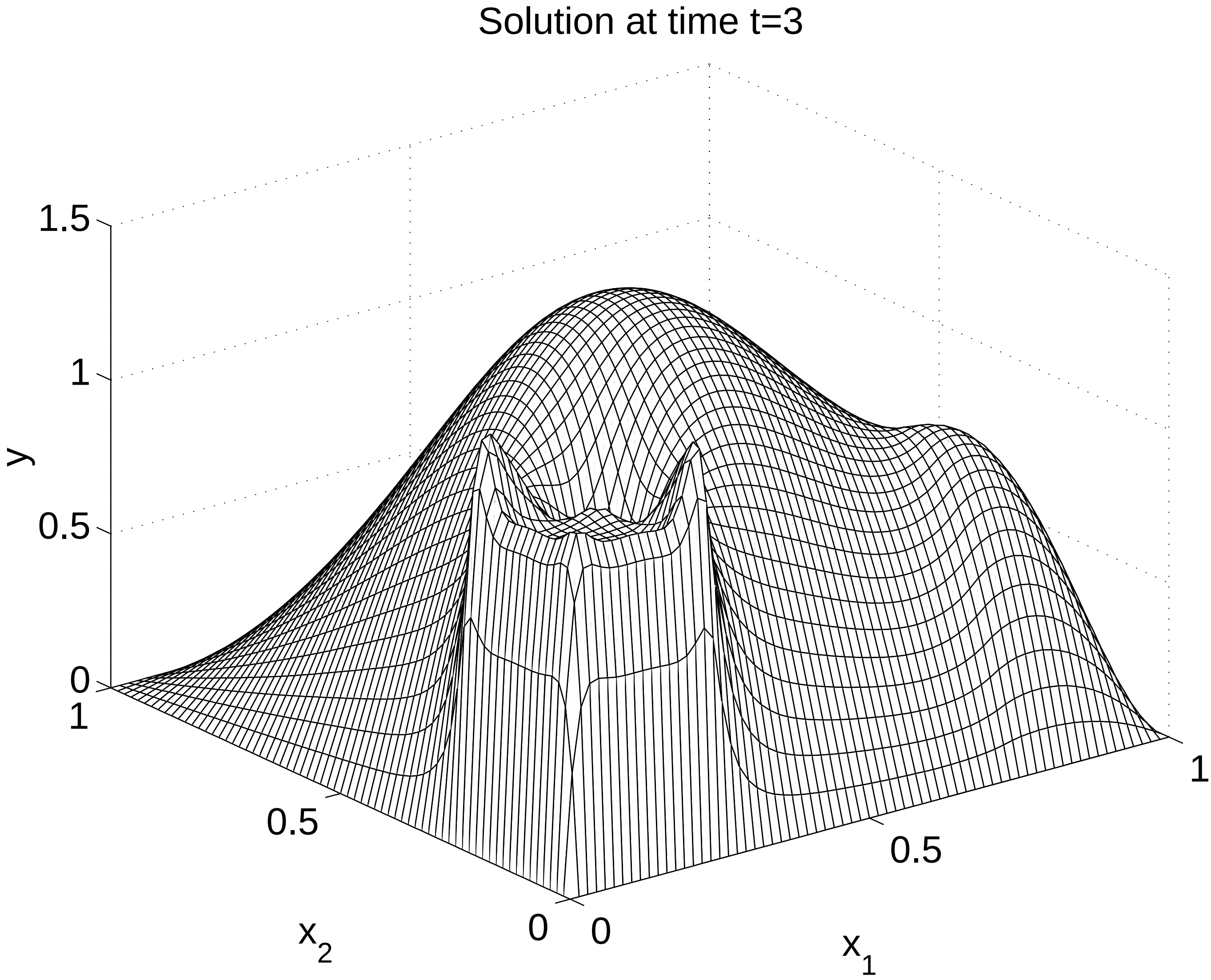

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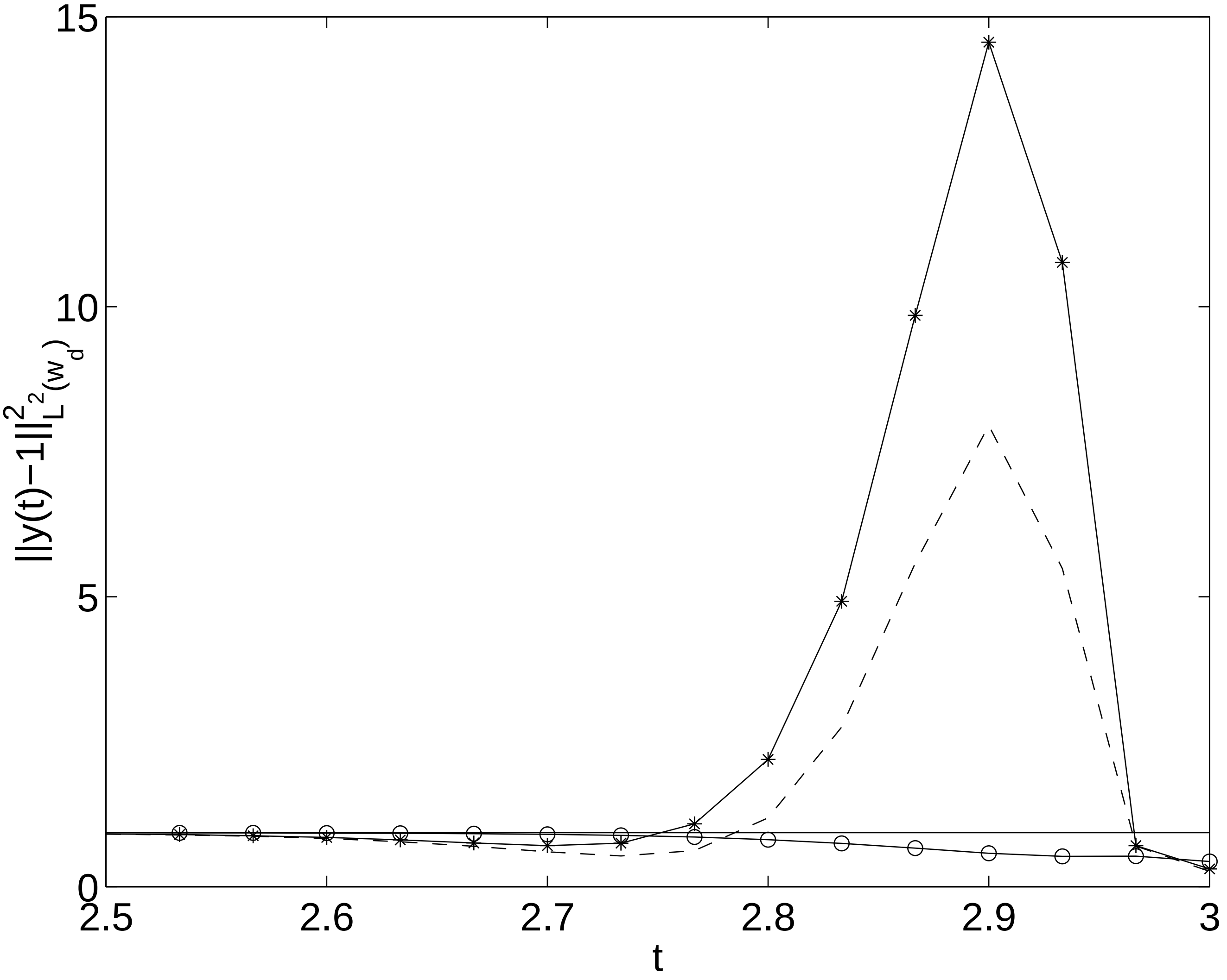

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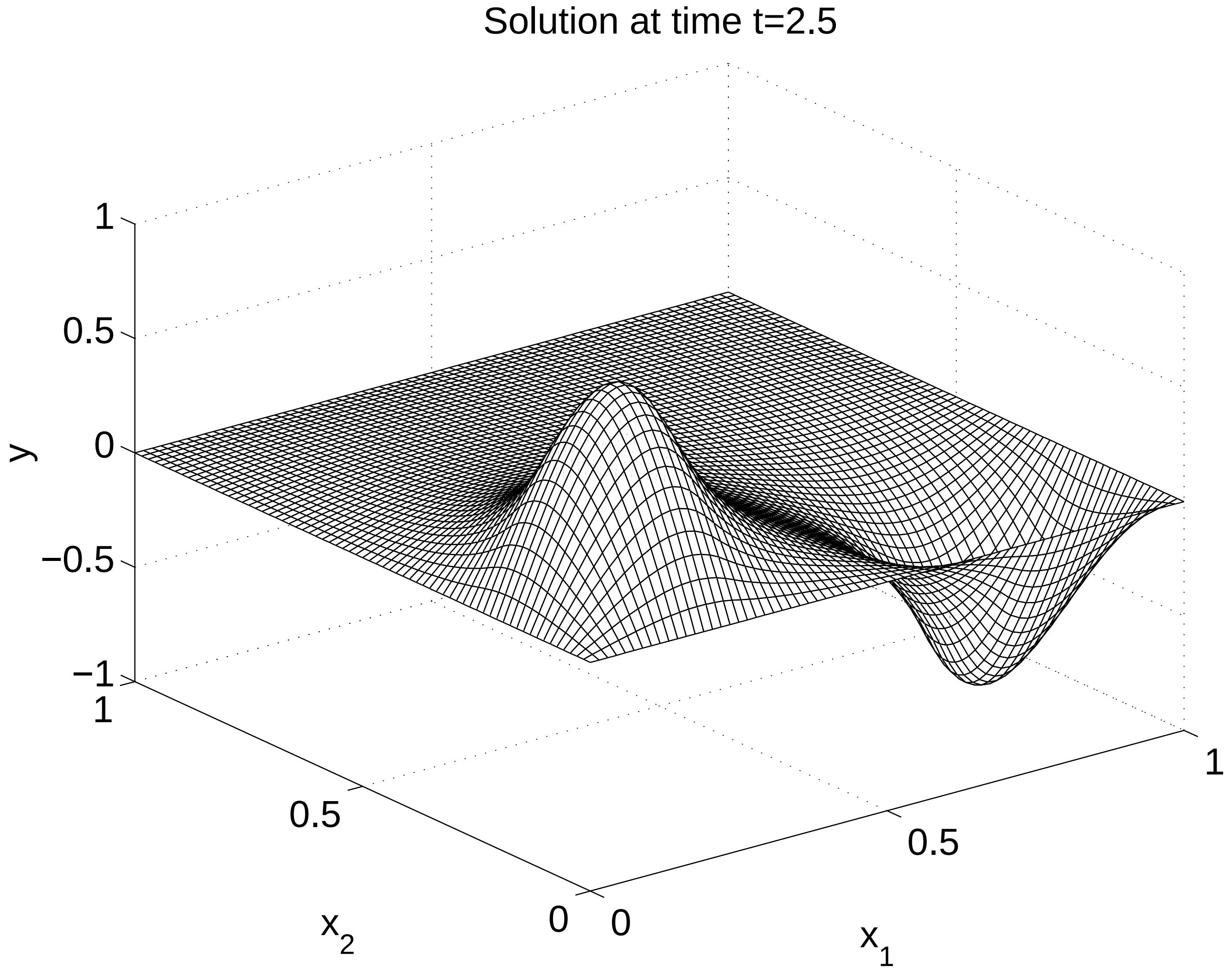

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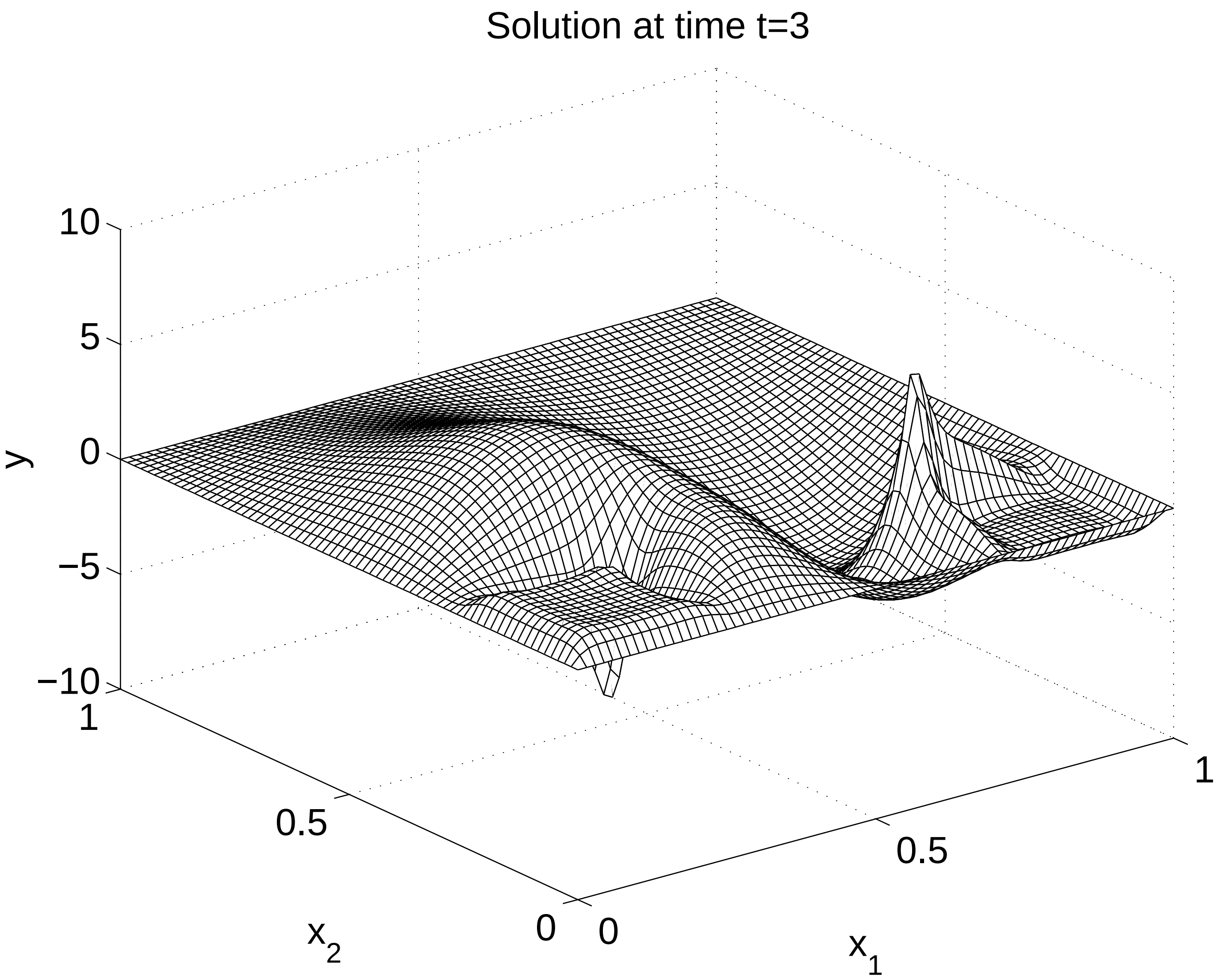

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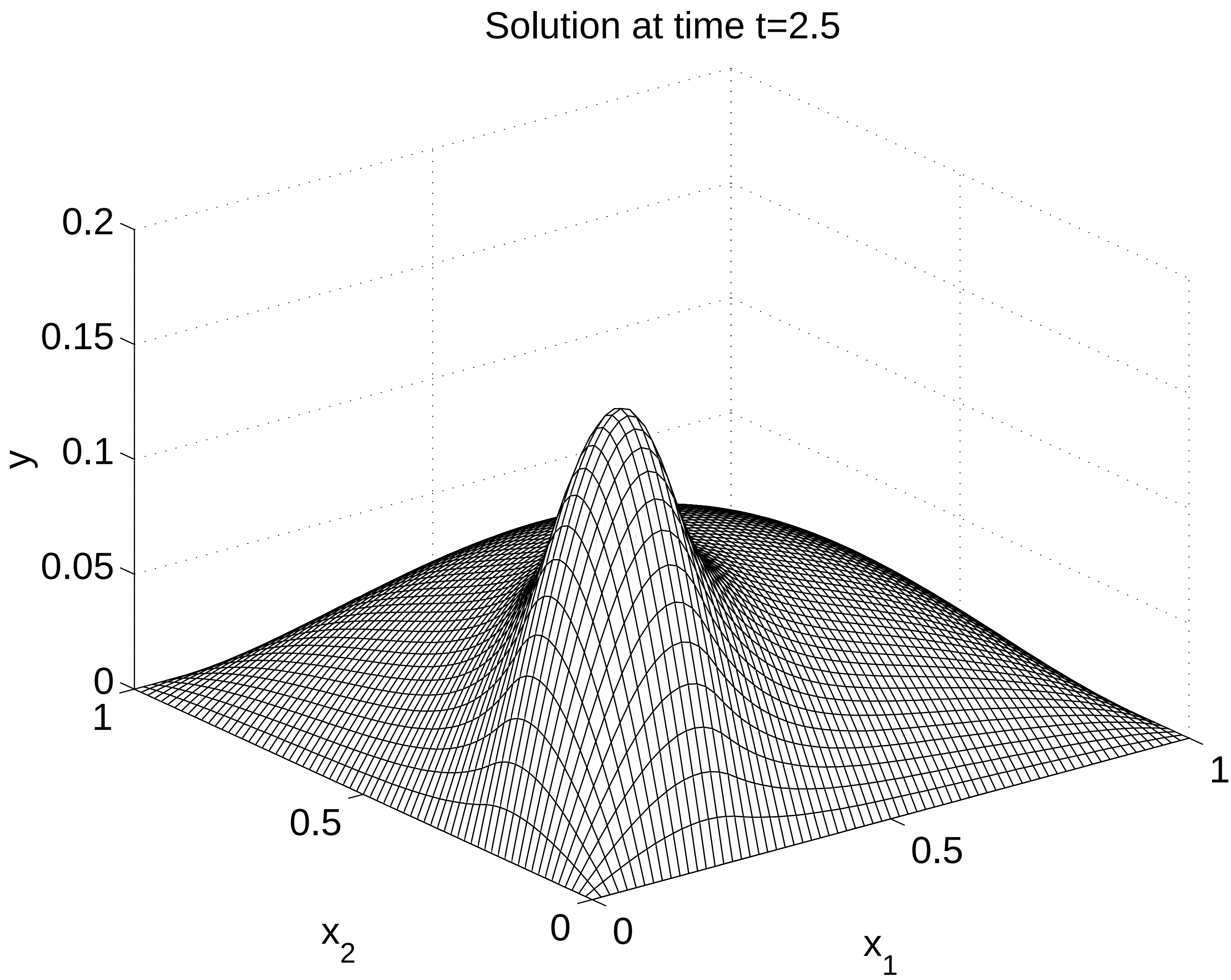

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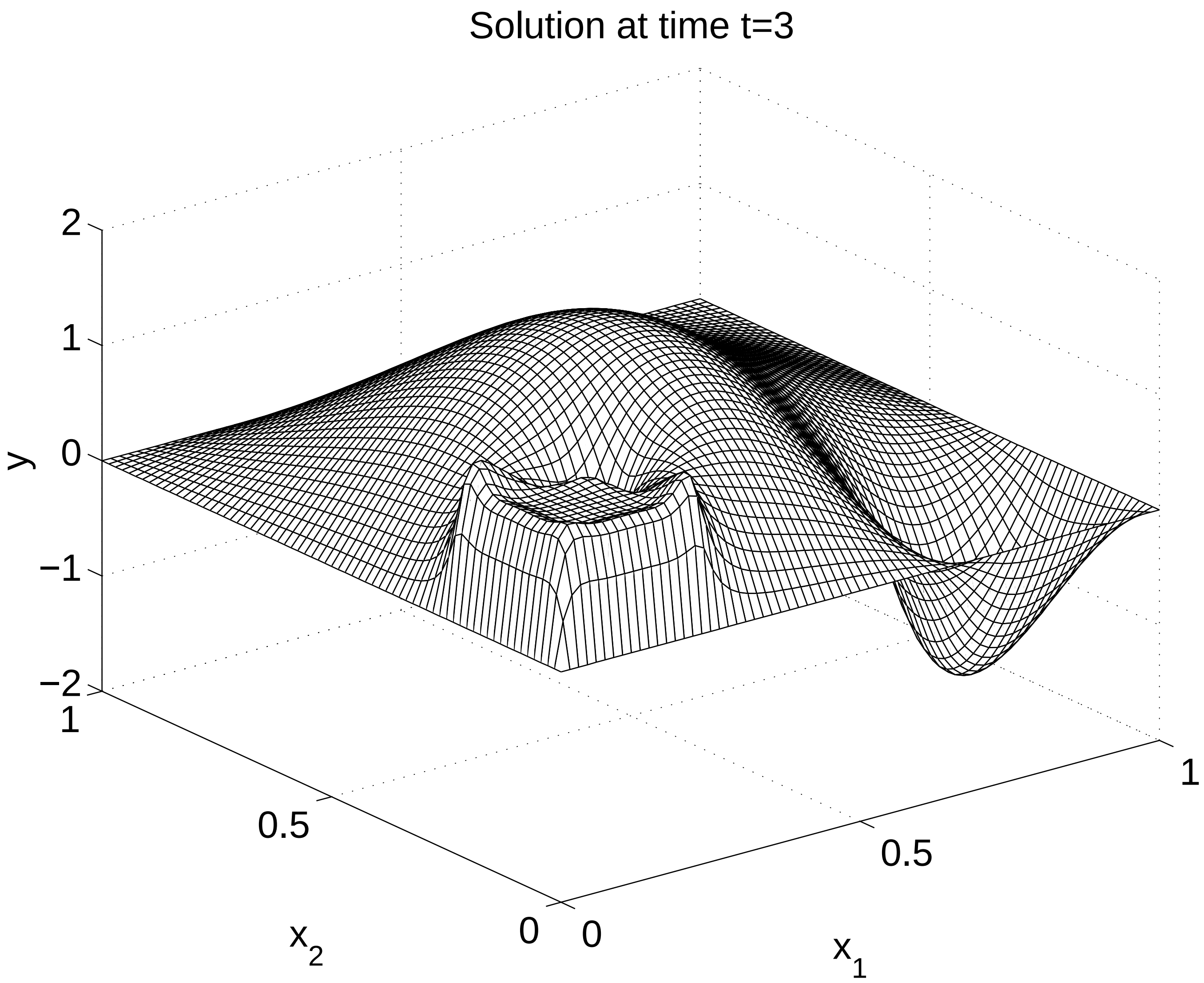

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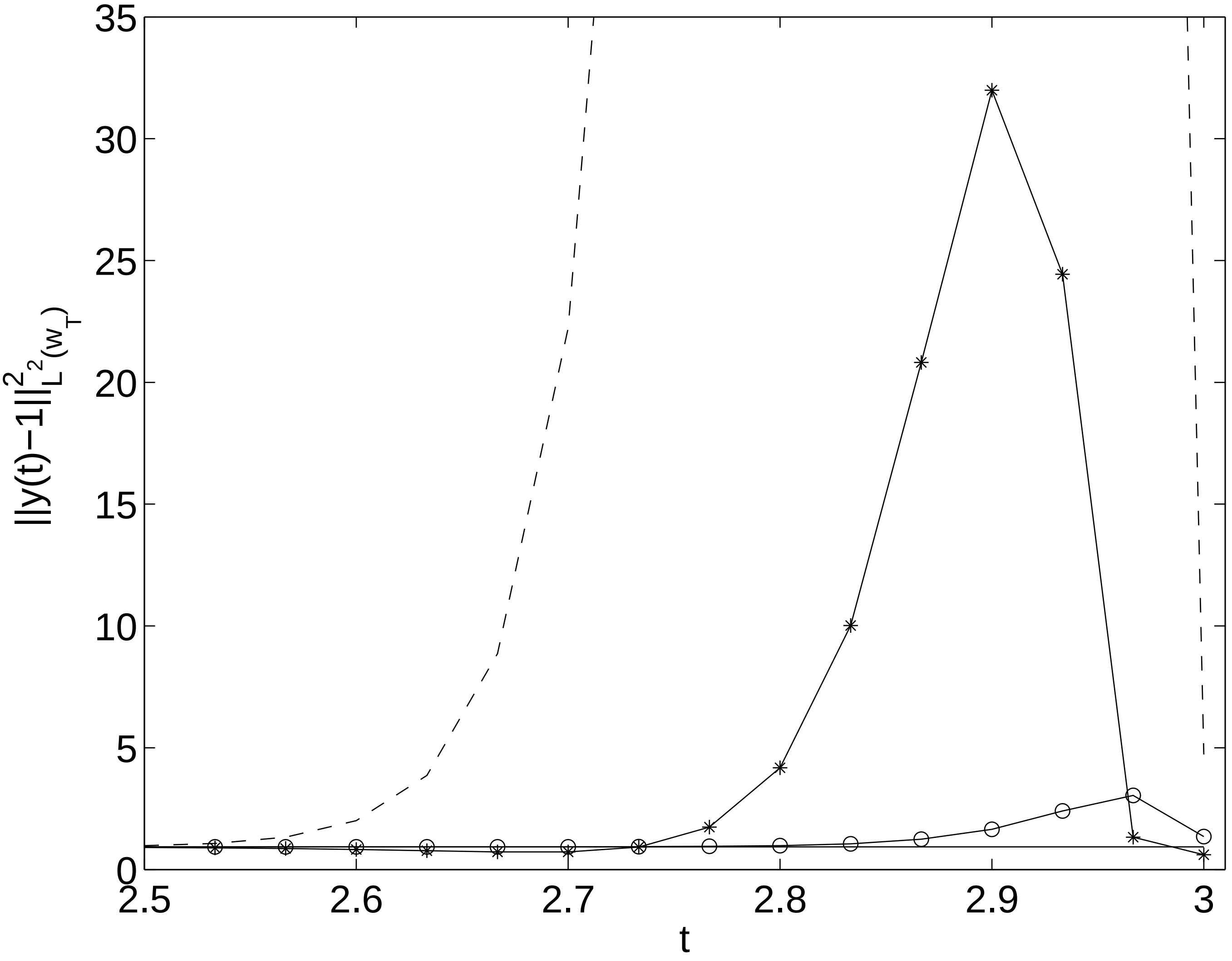

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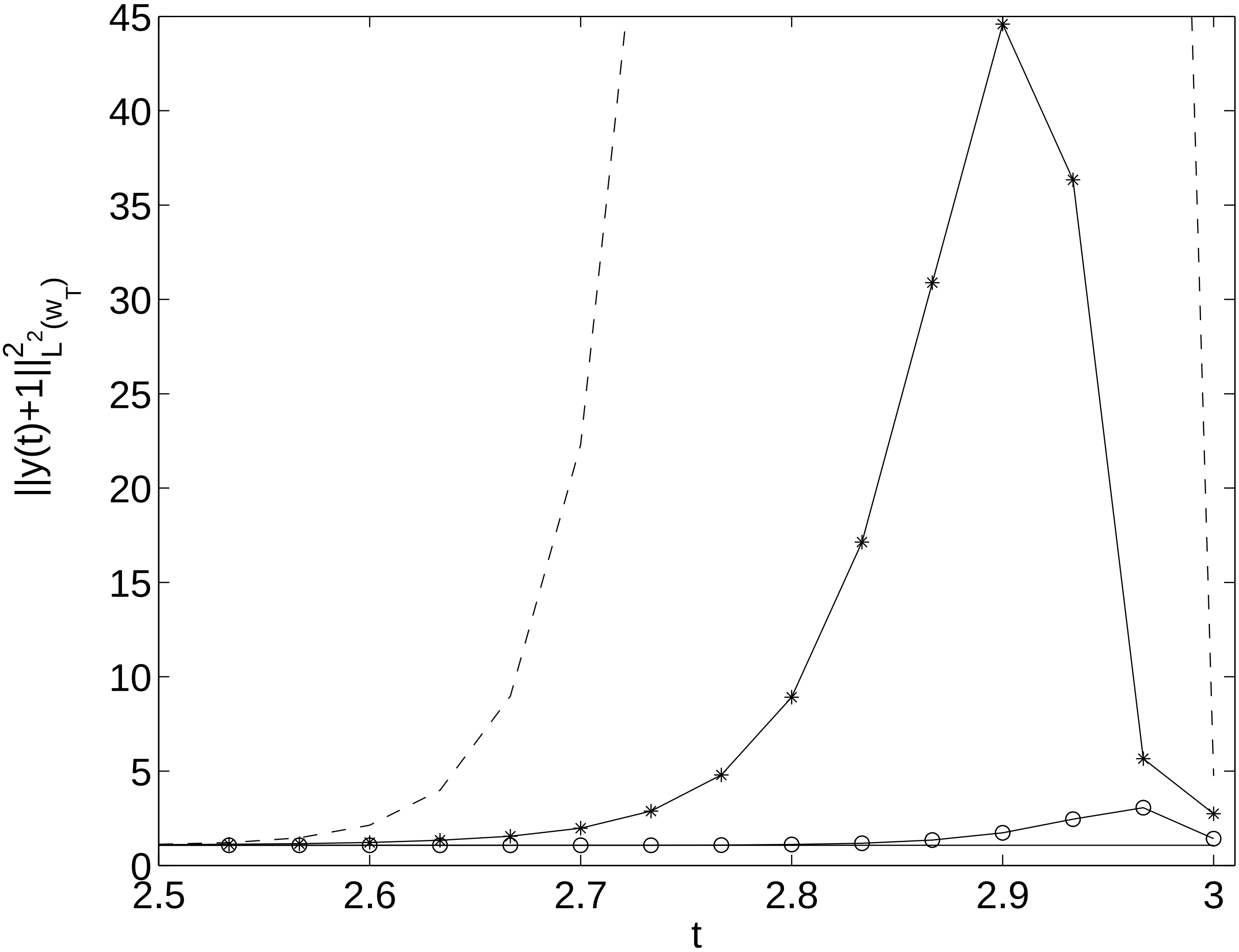

Fig. 25: