IDENTIFICATION OF A PRESSURE DEPENDENT HEAT TRANSFER COEFFICIENT

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ABSTRACT

This paper deals with an inverse problem concerning the identification of the heat exchange coefficient H (assumed to be dependent on the pressure) between a certain material and the external environment, when only experimental measurements of the temperature are supposed to be known. The main difficulty is that the experimental data are affected by error. We set two scenarios for the inverse problem. For each scenario, knowing the initial and ambient temperatures, we identify function H through different methods and we obtain estimates for the error. Finally, we perform numerical tests.

Keywords: Function identification, Inverse Problems, Heat transfer.

INTRODUCTION

In this work, we focus our attention on an inverse problem concerning the identification of the heat transfer coefficient H (assuming it depends on pressure) between a certain material with the external environment. Some practical applications in which this coefficient appears can be seen in [4], [5], [6] and [7]). The goal is to identify H to get a solution for the corresponding model, approximating some given temperature measurements.

The physical problem modeled in the references mentioned above is the evolution of the temperature in a homogeneous sample of a material placed in an equipment capable of compressing it (which will increase its temperature) and, that is also warming up (respectively, cooling down) due to heat exchange with an external environment that is warmer (respectively, cooler). To describe the temperature distribution within the sample complex models based on partial differential equations are often used (see, e.g., [6]). These equations involve functions and parameters that must be known before computing the solution. These functions and parameters are usually determined either by experimentation based protocols ([6]) or by solving inverse problems posed in an appropriate mathematical framework (see, e.g., [1], [2] and [3]).

In some contexts, and under certain conditions, it can be assumed that H has a known expression (e.g., H is constant or a function with a few real parameters to identify). In these cases, the least squares method may provide a good tool to solve inverse problems (see, e.g., [4]). However, when the goal is to identify a function, the problem becomes more complicated, especially if the experimental data are given with measurement errors, due to measurement equipment accuracy limitations. The challenge in this work is to identify function H when continuity and positivity are the only information available about H.

For simplicity, let us consider an homogeneous sample and let us assume that the temperature gradient inside it is negligible. The Newton Cooling Law and the relation describing the change in temperature due to the pressure variation, when isentropic changes of temperature are assumed (see [7]), provide a simple mathematical model for this phenomenon through the following initial value problem (*direct problem*):

$$\begin{cases} T' = H(P)(T^{\mathrm{e}} - T) + \alpha P'T, \; t \in [t_0, t_{\mathrm{f}}] \\ T(t_0) = T_0. \end{cases} \tag{1}$$

Here $T(t)$ (K) is the temperature of the sample at time t; $P(t)$ (Pa) is the pressure of the equipment at time t; T^{e} is the ambient temperature; T_0 is the temperature at the initial time t_0; $\alpha \geq 0$ is a parameter involving thermal expansion, density and specific heat capacity; and H is the pressure dependent heat exchange coefficient. In order to solve problem (1), constants $T_0,\ T^e \in \mathbb{R}$, pressure P and function $H : [P_{\min}, P_{\max}] \to \mathbb{R}$ are needed ($[P_{\min}, P_{\max}]$ is a suitable range of pressure).

The values of T_0 and T^e can be obtained by measuring devices (*thermocouples*), the coefficient α is assumed to be known and the pressure is provided by the equipment. However, function H cannot be obtained easily. We will design strategies to enable, from experimental measurements, the identification of function H (*inverse problem*); by doing so, we will be able to approximate the solution of model (1) for other values of T_0, T^e and P (provided it is kept in the initial ranges of pressure $[P_{\min}, P_{\max}]$) without requiring new measurements.

In order to define a suitable framework to carry out this identification, we suppose that:

- The ambient temperature T^e is constant.
- The initial temperature T_0 is higher than T^e.
- P is a known, non-decreasing, continuous and piecewise $\mathcal{C}^1$ function on the time interval $[t_0, t_{\rm f}]$.
- H is a positive and continuous function on the pressure range $[P_0, P_{\rm f}] = [P(t_0), P(t_{\rm f})]$.

The temperature measurements are assumed to have been taken during an experiment in which the entire range of pressures has been covered. In practice, a linear pressure can be used.

We note that H is not relevant when T is close to T^e. We set a *threshold* μ to separate it from $T^{\rm e}$ ($H(P)$ is not identified when T is too close to $T^{\rm e}$).

SCENARIOS OF THE INVERSE PROBLEM

Depending on the knowledge one has on the solution T in $[t_0, t_{\rm f}]$, we consider the inverse problem immersed in various scenarios:

- The first one arises when a function $\widetilde{T}$ that represents the approximate value of the temperature at any instant of time is assumed known.
- However, the usual situation is that only a discrete amount of values $\widehat{T}_k$ approximating the values of T at the corresponding instants is known.

For these scenarios, we will develop a "stable" method to approximate T' from the data and thereby obtain a discrete number of approximate values of H for points of the interval $[P_0, P_{\rm f}]$. To this end, we must ensure that the temperature values are sufficiently far from T^e, otherwise the coefficient H would have a negligible influence in the equation, and its identification could not be performed. We set a "threshold" as follows:

a) In the first scenario, assuming $\left\|T - \widetilde{T}\right\| < \delta$, we consider the threshold $\mu = \widetilde{m} - T^e$, where $\widetilde{m} = \min_{t\in[t_0,t_{\rm f}]} \widetilde{T}(t)$. If $\mu \leq \delta$, we would need to perform an experiment starting from a higher value of the initial temperature T_0, in order to obtain a higher approximate temperature.

b) In the second scenario, we assume a set of measurements $\widehat{T}_k$ such that $|T(\tau_k) - \widehat{T}_k| < \widehat{\delta}$, with $\widehat{\delta} > 0$, where $\{\tau_0 = t_0, \tau_1, \tau_2, \ldots, \tau_p = t_{\rm f}\}$ is a sequence of instants, is avalaible. We will denote by $\widetilde{T}$ a function that interpolates the values $\{\widehat{T}_0, \widehat{T}_1, \ldots, \widehat{T}_p\}$ at points $\{\tau_0, \tau_1, \ldots, \tau_p\}$ and consider $\delta > 0$, a bound of the norm of the difference between T and $\widetilde{T}$ in the interval $[t_0, t_{\rm f}]$, i.e., $\left\|T - \widetilde{T}\right\| < \delta$. The threshold μ is defined from $\widetilde{T}$ as in the previous scenario.

AD HOC EXPERIMENT

This section presents a method which identifies function H on the assumption that we can perform an experiment designed *ad hoc* as follows: we assume measurements of temperature in an even number of instants $\{t_k\}_{k=0}^n$, which form an equally spaced partition of $[t_0, t_{\rm f}]$ with step h are known. We choose the pressure applied by the equipment as a continuous function that increases linearly with the same slope in the intervals $[t_{2k-1}, t_{2k}]$ and remains constant in the rest of the intervals $[t_{2k}, t_{2k+1}]$, $k = 0, 1, \ldots, \frac{n-1}{2}$. Thus $P(t_{2k}) = P(t_{2k+1})$ and the values $\{P(t_{2k})\}_{k=0}^{\frac{n-1}{2}}$ form a partition of the range of pressures $[P_0, P_{\rm f}]$.

Denoting by $\{\widehat{T}_k\}_{k=0}^n$ the temperature measurements, we can find the approximations $\widetilde{H}_k \simeq H(P(t_{2k}))$ through the following methodology: for each $k \in \left\{0, 1, \ldots, \frac{n-1}{2}\right\}$ we consider the interval $[t_{2k}, t_{2k+1}]$. Here, since pressure is constant, the solution of problem (1) verifies

$$T'(t) = H(P(t_{2k}))(T^e - T(t)),\ t \in (t_{2k}, t_{2k+1}).$$

Then, $T(t) = T^e + (T_{2k} - T^e)e^{-H(P(t_{2k}))(t-t_{2k})}$, $t \in [t_{2k}, t_{2k+1}]$, where $T_k = T(t_k)$. For $t = t_{2k+1}$,

$$H(P(t_{2k})) = \frac{1}{h} \ln\left(\frac{T_{2k} - T^e}{T_{2k+1} - T^e}\right).$$

This suggests to take as an approximation of the value of H at $P(t_{2k})$ the value

$$\widetilde{H}_k = \frac{1}{h} \ln \left(\frac{\widehat{T}_{2k} - T^e}{\widehat{T}_{2k+1} - T^e} \right). \tag{2}$$

Proposition 1: *Denoting by* $\sigma_k = \frac{\widehat{T}_k - T_k}{T_k - T^e}$,

$$\widetilde{H}_k - H(P(t_{2k})) = \frac{1}{h} \ln \left(\frac{1 + \sigma_{2k}}{1 + \sigma_{2k+1}} \right).$$

Remark 2: a) If the temperature measurements are exact (and, consequently, all σ_k vanish) then this method provides the exact values of H.

b) Proposition 1 also shows that the error committed when approaching function H using this method only depends on the values of $1 + \sigma_k$, i.e., the relative errors of $\widehat{T}_k - T^e$ with respect to $T_k - T^e$. As these errors are a feature of the equipment, there is the unusual fact that the error in the approximation of function H by this methodology is independent of the function.

ITERATIVE ALGORITHM

It may happen that the equipment does not allow an experiment as described previously. Thus, another strategies are developed here, in order to identify coefficient H.

Identifying from a function that approximates the temperature

In this context, an approximation $\widetilde{T} \in \mathcal{C}([t_0, t_f])$ of T is assumed to be known. More precisely,

$$\left|\left| T - \widetilde{T} \right|\right| < \delta \tag{3}$$

for $0 < \delta < \mu = \widetilde{m} - T^e$. Since

$$H(P(t)) = \frac{T'(t) - \alpha P'(t) T(t)}{T^e - T(t)},$$

we define function

$$u(t) = \frac{T'(t) - \alpha P'(t) T(t)}{T^e - T(t)}, \ t_0 < t < t_f$$

and its approximation

$$\widetilde{u}_h(t) = \frac{R_h(\widetilde{T})(t) - \alpha P'(t) \widetilde{T}(t)}{T^e - \widetilde{T}(t)},$$

where $R_h : \mathcal{C}([t_0, t_f]) \to \mathcal{C}([t_0, t_f])$ is the approximate differentiation operator given by

$$R_h(v)(t) = \begin{cases} D_h(v)(t) + \Psi_h(v)(t_0), \\ \qquad t \in [t_0, t_0 + h] \\ \dfrac{v(t+h) - v(t-h)}{2h}, \\ \qquad t \in [t_0 + h, t_f - h] \\ D_{-h}(v)(t) + \Psi_h(v)(t_f - 3h), \\ \qquad t \in [t_f - h, t_f], \end{cases}$$

with

$$D_h(v)(t) = \frac{-3v(t) + 4v(t+h) - v(t+2h)}{2h},$$

$$\Psi_h(v)(t) = $$
$$= \frac{v(t+3h) - 3v(t+2h) + 3v(t+h) - v(t)}{2h}.$$

Remark 3: This approximate derivation operator provides an error bound slightly worse than that provided by the standard operator of order 2 (without $\Psi_h(v)$). However, with this definition, $R_h\left(\mathcal{C}([t_0, t_f])\right) \subset \mathcal{C}([t_0, t_f])$.

Proposition 4: *Let* $T \in \mathcal{C}^3([t_0, t_f])$ *and* $\widetilde{T} \in \mathcal{C}([t_0, t_f])$ *verifying* (3) *with* $0 < \delta < \mu$. *Then*

$$\begin{aligned} ||u - \widetilde{u}_h|| \leq & \frac{29 M_3}{6(\mu - \delta)} h^2 \\ & + \frac{4\delta\left(\widetilde{M} - \widetilde{m} + 2\mu\right)}{\mu(\mu - \delta)} \frac{1}{h} + \frac{\alpha P'_M T^e \delta}{\mu(\mu - \delta)}, \end{aligned} \tag{4}$$

where $M_3 = ||T'''||$, $\widetilde{M} = \left|\left|\widetilde{T}\right|\right|$ *and* $P'_M = ||P'||$.

In (4) the step h appears (squared) multiplying one term and dividing another one. Hence, the optimal estimate is obtained when choosing a value of h that balances both terms to get the minimum value. The next result (its proof is straightforward by using Proposition 4) indicates how to choose such a value of h and its corresponding estimate:

Proposition 5: *Under the assumptions in Proposition 4, the smallest value of the bound in* (4) *is reached when taken as a time step*

$$h^* = \left(\frac{12(\widetilde{M} - \widetilde{m} + 2\mu)}{29 \mu M_3} \delta \right)^{\frac{1}{3}}. \tag{5}$$

For this optimal value of time step, we have the following error bound

$$||u - \widetilde{u}_{h^*}|| \leq \left(522 M_3 \frac{(\widetilde{M} - \widetilde{m} + 2\mu)^2}{\mu^2(\mu - \delta)^3} \right)^{\frac{1}{3}} \delta^{\frac{2}{3}} + \frac{\alpha P'_M T^e}{\mu(\mu - \delta)}\, \delta.$$

Let h^* be as in (5). Let us denote by n the integer part of $\frac{t_{\mathrm{f}} - t_0}{h^*}$, $t_k = t_0 + k h^*$ and $\widetilde{T}_k = \widetilde{T}(t_k)$. If we approximate $H(P_k)$ by $\widetilde{H}_k = \widetilde{u}_{h^*}(t_k)$, i.e.,

$$\widetilde{H}_k = \frac{R_{h^*}(\widetilde{T})(t_k) - \alpha P'(t_k)\widetilde{T}_k}{T^e - \widetilde{T}_k}, \qquad (6)$$

for $k = 0, 1, \ldots, n$, Proposition 5 leads to:

Theorem 6 : *Under the assumptions in Proposition 4 one has*

$$\max_{k=0,1,\ldots,n} \left| H(P_k) - \widetilde{H}_k \right|$$

$$\leq \left(522 M_3 \frac{(\widetilde{M} - \widetilde{m} + 2\mu)^2}{\mu^2(\mu - \delta)^3} \right)^{\frac{1}{3}} \delta^{\frac{2}{3}} + \frac{\alpha P'_M T^e}{\mu(\mu - \delta)}\, \delta,$$

where $\widetilde{H}_k,\ k = 0, 1, \ldots, n$, *are given in* (6).

Remark 7 : Theorem 6 provides an estimate of the error committed when taking the optimum value h^* as the time step. The difficulty is that this value is unknown, since it depends on M_3. Next, we introduce an iterative algorithm in order to compute the values given in (6), from measurements of temperature, and successive approximations of h^*.

Identifying from a finite number of approximated values of the temperature

We start from measurements of temperature $\{\widehat{T}_0, \ldots, \widehat{T}_p\}$ corresponding to $\{\tau_0 = t_0, \ldots, \tau_p = t_{\mathrm{f}}\}$ and we assume that the error is of order $\widehat{\delta}$. We consider a function $\widetilde{T}$ that interpolates the previous values and assume that the interpolation method used is such that the error δ in T is of the order of the measurement error $\widehat{\delta}$, i.e., $\delta = C\widehat{\delta}$ (increasing the number of measurements, if necessary).

Once function $\widetilde{T}$ is defined in this way, we are in the same situation as in the previous section; hence it suffices to consider the threshold $\mu = \widetilde{m} - T^e$, take the time step h as in (5), n as the integer part of $\frac{t_{\mathrm{f}} - t_0}{h}$ and $\widetilde{T}_k = \widetilde{T}(t_k)$, where $t_k = t_0 + kh$ for $k = 0, 1, \ldots, n$. Thus, the values $\widetilde{H}_k$ in (6) provide an approximation of H as in Theorem 6.

Next, we describe an algorithm to approximate the values of H at points $P_k \in [P_0, P_{\mathrm{f}}]$, for instants t_k in equally spaced partitions of $[t_0, t_{\mathrm{f}}]$. The time step of these partitions should be defined, in an iterative way, in order to approximate h^*.

The input data are: $\{\widehat{T}_k\}_{k=0}^p$ and $\widehat{\delta} > 0$. First of all, we construct a function $\widetilde{T}(t)$ interpolating $\{\widehat{T}_k\}_{k=0}^p$. Next, we estimate the error $\delta > 0$ due to the interpolation. Then, the admissible threshold $\mu = \widetilde{m} - T^e$, under the constraint $\mu > \delta$, is obtained.

The algorithm is based on an iterative process starting from an initial guess h for the optimal time step h^*. From this value, we consider the instants $t_k = t_0 + kh, k = 0, 1, \ldots, n$, where n is the integer part of $\frac{t_{\mathrm{f}} - t_0}{h}$. Therefore, the values $\widetilde{T}_k = \widetilde{T}(t_k)$ are obtained. From these values, an approximation Λ_3 of M_3 is computed as the maximum absolute value of quantities

$$\begin{cases} \dfrac{-5\widetilde{T}_k + 18\widetilde{T}_{k+1} - 24\widetilde{T}_{k+2} + 14\widetilde{T}_{k+3} - 3\widetilde{T}_{k+4}}{2h^3}, & k = 0, 1 \\ \dfrac{\widetilde{T}_{k+2} - 2\widetilde{T}_{k+1} + 2\widetilde{T}_{k-1} - \widetilde{T}_{k-2}}{2h^3}, & k = 2, 3, \ldots, n-2 \\ \dfrac{3\widetilde{T}_{k-4} - 14\widetilde{T}_{k-3} + 24\widetilde{T}_{k-2} - 18\widetilde{T}_{k-1} + 5\widetilde{T}_k}{2h^3}, & k = n-1, n. \end{cases} \qquad (7)$$

These formulas are based, respectively, on standard order two progressive, central and backward approximate derivative schemes of a regular function.

From Λ_3, the next value of the time step is computed (following (5)) as

$$h = \left(\frac{12(\widetilde{M} - \widetilde{m} + 2\mu)}{29\mu\Lambda_3}\, \delta \right)^{\frac{1}{3}}, \qquad (8)$$

and so on. The process stops when two consecutive values of h are close. From the final value of h, the corresponding instants t_k, interpolation $\widetilde{T}$ and the quotients

$$\widetilde{H}_k = \widetilde{u}_h(t_k) = \frac{R_h(\widetilde{T})(t_k) - \alpha P'(t_k)\widetilde{T}_k}{T^e - \widetilde{T}_k}, \qquad (9)$$

are computed. These quantities approximate the values of H in the pressures $P_k = P(t_k)$.

Algorithm

DATA
$\{\widehat{T}_k\}_{k=0}^{p}$: Measurements of $\{T(\tau_k)\}_{k=0}^{p}$.
$\widehat{\delta} > 0$: bound of measurements errors.
ε: stopping test precision.
h: guess value for h^*.
Step 1: Determine $\widetilde{T}$ and δ according to $\widehat{\delta}$ so that $\mu = \widetilde{m} - T^e > \delta$.
Step 2: While the relative error in h is greater than ε:
a) Determine the new discrete instants $\{t_k\}$ and compute $\{\widetilde{T}_k\}$.
b) Compute Λ_3 as the maximum absolute value of (7).
c) Compute the new value of h as in (8).
Step 3: Obtain the final discrete instants $\{t_k\}$ and the values $\{\widetilde{T}_k\}$.
Step 4: Compute the approximations $\widetilde{H}_k$ according to (9).

NONDIMENSIONALIZATION OF THE PROBLEM

Before performing the numerical experiments with different sets of data illustrating the behavior of the methods developed, it is convenient to nondimensionalize the problem. We want the model to involve as few dimensionless parameters as possible. Here, it suffices to consider two parameters: the pressure and a relationship between the initial and ambient temperature, as discussed below. We consider the new dimensionless variables $t^* = \frac{t-t_0}{t_f-t_0}$, $T^*(t^*) = \frac{T(t)-T^e}{T_0-T^e}$ and $P^*(t^*) = \big(P(t) - P_0\big)\alpha$. Problem (1) can be written in these new variables (see [5, pag. 57]) as

$$\begin{cases} \dfrac{dT^*}{dt^*}(t^*) = -H^*(P^*(t^*))T^*(t^*) \\ + \dfrac{dP^*}{dt^*}(t^*)\left(T^*(t^*) + T^{ea}\right), \; t^* \in (0,1) \\ T^*(0) = 1, \end{cases} \tag{10}$$

where

$$\begin{cases} H^*(s) = (t_f - t_0)H\left(\dfrac{s}{\alpha} + P_0\right) \\ T^{ea} = \dfrac{T^e}{T_0 - T^e} \end{cases}$$

(note that $H^*(P^*(t^*)) = (t_f - t_0)H(P(t))$). We use this approach to identify coefficient H^* and to find the temperature distribution for several functions P^* and several values of T^{ea}.

Remark 8 : The maximum value that the dimensionless temperature T^* can reach is given by

$$T^*_{\max} = \frac{T_{ad} - T^e}{T_0 - T^e},$$

where T_{ad} is the maximum temperature that can be achieved under adiabatic conditions (i.e., when there is no heat exchange with the external environment). To determine this value, it suffices to consider the initial value problem

$$\begin{cases} T'(t) = \alpha P'(t)T(t), \; t \in (t_0, t_f) \\ T(t_0) = T_0, \end{cases}$$

whose solution is $T(t) = T_0 e^{\alpha(P(t)-P_0)}$, $t \in [t_0, t_f]$. Since P is an increasing function, $T_{ad} = T_0 e^{\alpha(P_f - P_0)} = T_0 e^{P^*(1)}$ which leads to

$$T^*_{\max} = \frac{T_{ad} - T^e}{T_0 - T^e} = (1 + T^{ea})e^{P^*(1)} - T^{ea}.$$

Remark 9 : After identifying function H^*, H can be obtained by

$$H(s) = \frac{1}{t_f - t_0} H^*\left(\alpha(s - P_0)\right), \; s \in [P_0, P_f]. \tag{11}$$

From T^* we can express temperature T as

$$T(t) = T^e + (T_0 - T^e)T^*\left(\frac{t - t_0}{t_f - t_0}\right), \; t \in [t_0, t_f].$$

Remark 10 : If the order of magnitude of function H^* is small compared to

$$\frac{dP^*}{dt^*}(t^*)\left(T^*(t^*) + T^{ea}\right),$$

this term will be dominant. Hence any function H^* of that order of magnitude would provide values of temperature with few differences. To avoid this problem we can modify the original experiment so that the new one results in a function H^* of a higher order of magnitude. If pressure in the original experiment is given by

$$P(t) = a(t - t_0) + P_0, \; t \in [t_0, t_f],$$

a slower increase of pressure (for a longer time in order to cover the same range of pressures $[P_0, P_\mathrm{f}]$) could be considered. That is, we can take

$$P_1(t) = ac(t - t_0) + P_0,\ t \in \left[t_0, t_0 + \frac{t_\mathrm{f} - t_0}{c}\right],$$

with $0 < c \leq 1$. If T_1 is the temperature obtained with this pressure, the changes of variable

$$t_1^* = \frac{c(t - t_0)}{t_\mathrm{f} - t_0}, \quad T_1^*(t_1^*) = \frac{T_1(t) - T^e}{T_0 - T^e}$$

and $P_1^*(t_1^*) = \big(P_1(t) - P_0\big)\alpha$, lead to

$$\begin{cases} \dfrac{dT_1^*}{dt_1^*}(t_1^*) = -\dfrac{1}{c} H^*\big(P_1^*(t_1^*)\big) T_1^*(t_1^*) \\ + \dfrac{dP_1^*}{dt_1^*}(t_1^*)\left(T_1^*(t_1^*) + T^{ea}\right),\ t_1^* \in (0,1) \\ T_1^*(0) = 1. \end{cases} \tag{12}$$

Since

$$\frac{dP_1^*}{dt_1^*}(t_1^*) = \alpha a(t_\mathrm{f} - t_0) = \frac{dP^*}{dt^*}(t^*),$$

the equations of problems (10) and (12) are identical, except that the new function H^* is amplified by the factor $\frac{1}{c} \geq 1$.

Remark 11 : The dimensionless problem is governed by a different equation than the original one, and this will be taken into account in the methods we will use:

a) For the method based on the *ad hoc* experiment, it suffices to note that, in each interval where the pressure is constant, the temperature satisfies the same equation but with $T^e = 0$. We therefore consider the approximations

$$\widetilde{H}_k = \frac{1}{h} \ln\left(\frac{\widehat{T}_{2k}}{\widehat{T}_{2k+1}}\right)$$

instead of (2).

b) Concerning the iterative algorithm, we can say that the optimal step expression (8) and the quantities (7) that are used for the calculation of Λ_3 remain the same (replacing, of course, the roles of $\widetilde{T}$ and $\widetilde{T^*}$), while the approximation (9) of $\widetilde{H}_k$ becomes

$$\widetilde{H}_k = -\frac{R_h(\widetilde{T^*})(t_k^*) - \dfrac{dP^*}{dt^*}(t_k^*)(\widetilde{T_k^*} + T^{ea})}{\widetilde{T}_k}.$$

NUMERICAL RESULTS

In this section we perform a comparative study of the results obtained when using the methods considered in this paper for the identification of function H. While working on the nondimensional problem, the value of T^{ea} and the range of pressures are linked to a real situation. We use the P2 treatment data from [6], i.e., $T_0 = 313\,^\circ$K, $T^e = 295\,^\circ$K and $\alpha = 4.5045 \times 10^{-5}$ MPa^{-1}. The choice of the pressure curve is specified for each method and in both cases the range is from atmospheric pressure up to 360 MPa. Thus, the maximum value of dimensionless pressure is $a = 0.0162$ in both cases.

Given a function H, the nondimensional function H^* corresponding to the *ad hoc* experiment is, accordingly with (11), twice the function H^* corresponding to the iterative algorithm (the first method needs a time interval twice as long as the second).

In what follows, we omit the superscript *. The data for numerical tests have been obtained as follows: with a given function H, we solve the direct nondimensional problem (10), obtaining the temperature T. Then T is evaluated on an equally spaced partition of instants of time. We assume that in both experiments, the measurements have been carried out with the same time step, so that we will work in the first method with twice as many as values in the second (in particular, we take 200 in the *ad hoc* experiment and 100 in the iterative algorithm). The error measurements $\widehat{T}_k$ are built by perturbing T_k by means of random oscillations of order 1% of T_k. More precisely,

$$\widehat{T}_k = T_k\left(1 + \frac{r(t_k)}{75}\right),$$

where $r(t) = \sin(q\pi t)$ and q is a random integer between 1 and 99. Function $\widetilde{T}$ is taken as the piecewise linear interpolation of values $\widehat{T}_k$.

To allow an easy comparison, the same seven perturbations of temperature values have been generated, corresponding to the values $q = 3$, 14, 27, 42, 65, 84 and 97. Among them we selected the two which produces the smallest and the largest error in infinity norm in H, respectively.

After identifying an approximation of function H, we compute temperature T solving problem (10) and we compare it with the known solution of the direct problem. Also, different values for dimensionless parameters of the problem (the pressure curve and T^{ea}) are prescribed and the corresponding solutions are calculated. In order to analyze the quality of the identification, these solutions are compared to the exact temperature.

The different values of the parameters are generated by multiplying the original value of T^{ea} by the factors $d = 2$, $d = 1$ and $d = \frac{1}{2}$ and choosing as pressure curves the functions $P(t) = a \sin t$, $P(t) = a(e^{2t-2} - e^{-2})$ and $P(t) = \frac{a}{2}\, t(3 - t)$.

In all the figures and tables "Error" denotes the maximum norm error in H and "% Error" denotes the percentage relative error in maximum norm of T, i.e.,

$$\frac{\max_k |\widetilde{T}_k - T_k|}{\max_k |T_k|} \times 100.$$

First method: *ad hoc* experiment

For this method, the value for the first parameter of the nondimensional problem is $T^{ea} = \frac{295}{18}$, while the slope of the pressure (where it is not constant) is $0.0324\,(= 2a)$; what causes the pressure to take all values in the pressure range $[0, a]$ when time lies in $[0, 1]$. We consider the function

$$H(s) = 4 \exp\left(\frac{s}{a}\right).$$

Figure 1 shows the identified function H (and corresponding computed temperature) for the smallest and largest error in H.

Table 1 shows the percentage relative error in temperature (in maximum norm) for each of the nine data sets considered, both for the smallest and largest error in H.

Table 1: *Ad hoc* experiment. Temperature error (%). Smallest (above) and largest (below) error in identified H.

Pressure	**Factor over parameter T^{ea}**		
	$d = 2$	$d = 1$	$d = 0.5$
sinusoidal	0.48	0.48	0.47
exponential	0.87	0.86	0.85
quadratic	0.36	0.36	0.36
sinusoidal	1.37	1.36	3.84
exponential	5.11	5.10	5.09
quadratic	1.06	1.97	2.45

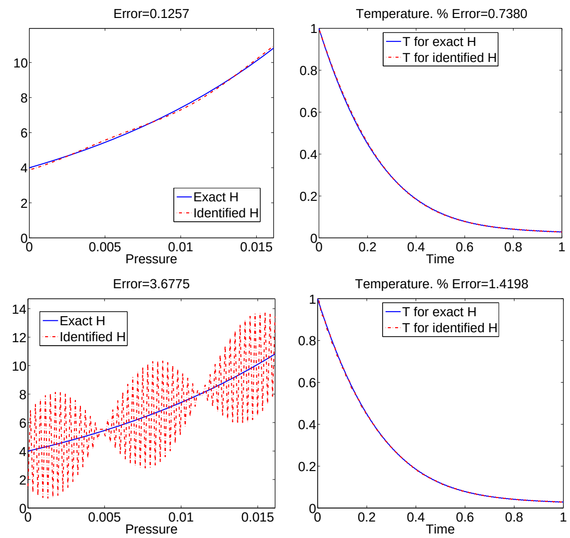

Figure 1: *Ad hoc* algorithm (Top: smallest error in H. Bottom: largest error in H)

The error in this method (we remind that it provides exact values when there is no measurement error) increases with the frequency of the oscillatory perturbation: the error in H grows with the value of q, being smaller for $q = 3$ (the smoother perturbation) and larger for $q = 97$ (more oscillatory perturbation).

Second method: iterative algorithm

Now, the value $T^{ea} = \frac{295}{18}$ is the same as before, but the pressure increase changes (since now there is no constant steps); in fact, $P(t) = 0.0162t$.

As already has been mentioned, H must be a half of the chosen in the previous method, i.e.,

$$H(s) = 2 \exp\left(\frac{s}{a}\right).$$

Figure 2 shows the identified function H (and corresponding computed temperature) for the smallest and largest error in H.

Table 2 shows the percentage relative error in temperature (in maximum norm) for each of the nine data sets considered, both for the smallest and largest error in H.

This algorithm uses interpolation of approximate values of T at instants that are not from the original partition. Therefore, their behavior

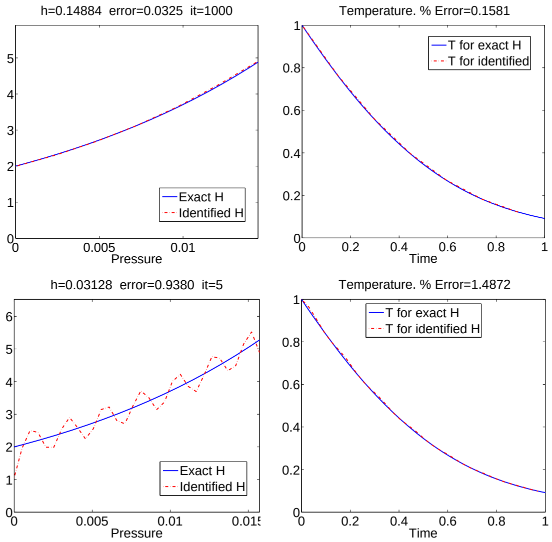

Figure 2: Iterative algorithm (Top: smallest error in H. Bottom: largest error in H)

Table 2: Iterative algorithm. Temperature error (%). Smallest (above) and largest (below) error in identified H.

Pressure	**Factor over parameter T^{ea}**		
	$d = 2$	$d = 1$	$d = 0.5$
sinusoidal	0.17	0.16	0.15
exponential	0.17	0.17	0.17
quadratic	0.14	0.13	0.13
sinusoidal	1.49	1.49	1.49
exponential	4.46	4.44	4.44
quadratic	0.94	0.94	0.94

is not directly linked to frequency of oscillatory perturbations.

In conclusion, although the size of the error in H is moderate for both methods, temperatures calculated from approximate identifications are quite accurate (the error is always of the order of the measurement error). The first method usually provides a better approximation of the temperature when solving for the initial parameters. However, when the identified temperature for the nine data sets is considered, the second method is generally more accurate in the case of largest error in H and therefore it can be considered more robust.

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