Attractors for Parabolic Equations with Nonlinear Boundary Conditions, Critical Exponents, and Singular Initial Data

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1. INTRODUCTION

In this paper we study the asymptotic behavior of the solutions of the following reaction-diffusion equation with nonlinear boundary conditions,

\[ u_t - d \Delta u + f(u) = 0 \quad \text{in } \Omega \]
\[ d \frac{\partial u}{\partial n} + g(u) = 0 \quad \text{on } \Gamma = \partial \Omega \]

(1.1)

\[ u(0, x) = u_0(x), \]

where \( \Omega \subset \mathbb{R}^N, N \geq 1 \), is a bounded smooth set and \( d > 0 \). We assume that the initial data is singular in the sense that \( u_0 \in L^r(\Omega) \) for some \( 1 < r < \infty \) or \( u_0 \) is a bounded measure on \( \Omega \) or \( u_0 \in W^{1,r}(\Omega) \) for some \( 1 < r < N \). We also assume that \( f, g: \mathbb{R} \rightarrow \mathbb{R} \) are locally Lipschitz functions which grow at infinity such that they are “at most critical” as explained below.

Concerning the existence of solutions, note that if we consider only smooth initial values \( u_0 \) in (1.1), then classical results imply the existence of a locally defined and smooth solution, see for example Theorem 53, Chapter 5 in [17] or Theorem 13 (and its corollary), Chapter 7 in [14]. If the initial data is assumed to be \( u_0 \in W^{1,r}(\Omega) \), for \( r > N \), then the results in [1, 2] also imply the existence of a local solution to (1.1). All these results are obtained regardless of growth assumptions on \( f \) and \( g \) because the initial data, as well as the solution, lies in the space of continuous functions. In particular they are smooth and well defined up to the boundary.

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On the other hand, when \( f \) and \( g \) are monotonic and have the right sign, the results in [7, 8, 13], using maximal monotone operator theory, give the well posedness of (1.1) for singular initial data \( u_0 \in L'(\Omega), 1 \leq r < \infty \). A typical example of this situation corresponds to \( f(s) = |s|^{p-1}s \) and \( g(s) = |s|^{q-1}s, p, q > 1 \).

However, when no monotonicity or sign assumptions are made on \( f \) and \( g \), the results in [4, 5] show that (1.1) is well posed in \( L'(\Omega), 1 < r < \infty \), provided that \( f, g \) grow "at most critically" in \( L'(\Omega) \); that is,

\[
\limsup_{|s| \to \infty} \frac{|f'(s)|}{|s|^{p-1}}, \quad \limsup_{|s| \to \infty} \frac{|g'(s)|}{|s|^{q-1}} < \infty
\]  

(1.2)

for \( p, q > 1 \) such that

\[
p \leq 1 + \frac{2r}{N} = p_c, \quad q \leq 1 + \frac{r}{N} = q_c \quad (q < 1 + r = q_c \text{ for } N = 1).
\]  

(1.3)

The exponents \( p_c \) and \( q_c \) are denoted "critical exponents" for (1.1) in \( L'(\Omega) \). See also [9] for the case of Dirichlet boundary conditions. For measures as initial data, which formally corresponds with the case \( r = 1 \), the growth allowed for the nonlinear terms satisfies

\[
p < 1 + \frac{2}{N} = p_c \quad \text{and} \quad q < 1 + \frac{1}{N} = q_c,
\]  

(1.4)

which are the critical exponents in the space of measures, \( \mathcal{M}(\Omega) \); see also [10]. Finally, for the case of initial data in \( W^{1,r}(\Omega) \) for \( 1 < r < N \), the growth allowed for the nonlinear terms satisfies

\[
p \leq 1 + \frac{2r}{N-r} = p_c \quad \text{and} \quad q \leq 1 + \frac{r}{N-r} = q_c,
\]  

(1.5)

which are the critical exponents in \( W^{1,r}(\Omega) \). The type of solutions considered in [4, 5] is described in Section 2; see also [3].

On the other hand, note that for solutions of (1.1) to be globally defined, some sign assumptions must be made on the nonlinear terms, since otherwise solutions can blow-up in finite time; see for example [15] for \( g = 0 \) or [18, 22, 19, 23], for \( f = 0 \). When both \( f \) and \( g \) are nonzero, some balance condition between them are needed to prevent solution from blow-up; see [20] and below.

In this paper we are therefore interested in analyzing the asymptotic behavior of solutions of (1.1), with singular initial data, that is, in \( L'(\Omega), 1 < r < \infty, \mathcal{M}(\Omega) \) or in \( W^{1,r}(\Omega), 1 < r < N \), and with special attention to
the case in which the nonlinearities grow critically. Our goal is to give conditions on the nonlinear terms such that (1.1) is globally well posed and moreover has a global compact attractor.

We observe that a first crucial difference between subcritical and critical nonlinearities appears in the question of global existence. For example, for initial data in $L'(\Omega)$ and in the subcritical case, it is enough to obtain bounds on the $L'(\Omega)$-norm of the solution to prove global existence, while in the critical case stronger estimates must be obtained. More precisely, according to [5], one must obtain some suitable estimates on the solution in some suitable Bessel potential space; see Section 2.

As for the existence of attractors, in [11] the following balance condition on $f$ and $g$ was given ensuring the dissipativity of (1.1): there exists $s_0 > 0$ such that

$$f(s) \geq a_0 \quad \text{and} \quad g(s) \geq b_0 \quad \text{for} \quad |s| \geq s_0$$  \hspace{1cm} (1.6)

and the first eigenvalue of the eigenvalue problem

$$-d\Delta \varphi + a_0 \varphi = \mu \varphi, \quad \text{in} \ \Omega$$

$$d \frac{\partial \varphi}{\partial n} + b_0 \varphi = 0, \quad \text{on} \ \Gamma$$  \hspace{1cm} (1.7)

is positive. Note that no assumptions are made on the signs of $a_0$ or $b_0$ separately.

Although in [11] the initial data $u_0$ for (1.1) was taken in $H^1(\Omega)$ with subcritical nonlinear terms, conditions (1.6), (1.7) were further exploited in [6], for initial data in $L'(\Omega)$, $W^{1,r}(\Omega)$, for $1 < r < \infty$ or even in $\mathcal{M}(\Omega)$ and with even critical nonlinearities. It was then shown in [6] that, under these assumptions, (1.1) has a global attractor in these spaces possessing strong attractivity properties. Note that conditions (1.6) and (1.7) can be seen as a “linearization at infinity” condition, since they imply that for the large values of $|u|$, (1.1) is below a linear parabolic problem which is stable and therefore solutions must remain bounded.

In [20], the equation (1.1) was considered with initial data and subcritical nonlinearities in $H^1(\Omega)$. For the case in which (1.6), (1.7) fail to hold, in that paper they were given nonlinear balance conditions on $f$ and $g$ such that (1.1) is dissipative and has a global attractor in $H^1(\Omega)$. Observe that, from the results in [20], we also know that when $f$ is superlinear and $f(s) s$ is very negative for large $|s|$ then, regardless of the boundary term $g$, there always exist smooth initial data $u_0$ such that the solution of (1.1) blows up in finite time. Also, from the results in [20], if $g$ is too large with respect to $f$ and $g(s) s$ is very negative for large $|s|$, then there always exist
smooth initial data \( u_0 \) such that the solution of (1.1) blows up in finite time. Hence, some balance conditions must be actually satisfied for (1.1) to be dissipative.

Therefore, we will consider here the situation in which (1.6), (1.7) fail to hold due to a superlinear growth with bad sign on \( I' \), that is when, \( \lim \inf_{\tau \to +\infty} \frac{\tau}{\tau} = -\infty \). For such cases we will give some balance conditions on the nonlinear terms, that apply for general nonlinearities, and that will imply that (1.1) is dissipative. As said before, special attention will be payed to the case in which either \( f \) or \( g \) grow critically.

For the case of subcritical nonlinear terms and using these balance conditions, in a first step, we will derive suitable estimates on the solutions in \( L'(Q) \) which, in particular, will show the existence of an absorbing ball in this space. In a second step, using the absorbing ball, we will prove that the nonlinear semigroup defined by (1.1) is compact in \( L'(Q) \). With these, using the results in [16, 21], we get the existence of the attractor \( \mathcal{A} \). It is interesting to note that for parabolic equations, the compactness of the semigroup is usually derived from the compactness of the linear semigroup (which in turn relies on the compactness of the resolvent of the elliptic operator), the boundedness of orbits and the variation of constants formula, see for example [16, Theorem 4.2.2]. However, for (1.1) we are considering singular initial data in \( L'(Q) \), or even in \( \mathcal{M}(Q) \), and therefore the nonlinear term \( g \) is not well defined on this space. Consequently, the argument above does not go through. We have then to rely on some specific properties of the class of solutions that we are considering, that will allow us to show the existence of absorbing balls in some Bessel potential spaces which are compactly embedded in \( L'(Q) \). From this we will obtain absorbing balls in \( W^{s,q}(Q) \) for every \( s \geq 1 \) and \( 0 \leq \alpha < 1 + \frac{1}{p} \) and in \( C^\beta(\Omega) \) for every \( 0 \leq \beta < 1 \). With this, strong regularity and attractivity properties of the attractor are derived. The cases of measures or \( W^{1,q}(Q) \) initial data are considered along the same lines as above.

All along the paper we will illustrate our results for power-like nonlinearities, that is, satisfying

\[
\lim_{|s| \to \infty} \frac{f'(s)}{|s|^{p-1}} = pc_f > 0, \quad \lim_{|s| \to \infty} \frac{g'(s)}{|s|^{q-1}} = qc_g < 0 \quad (1.8)
\]

for some \( p, q > 1 \) satisfying (1.3), (1.4), or (1.5) respectively; see (3.5) below.

For critical nonlinear terms, the basic idea that we follow is that as solutions are regularized for positive times, they will enter some spaces in which the equation becomes subcritical. Therefore, if (1.1) is dissipative in this latter space, then solutions are attracted towards the attractor in this space, which describes the asymptotic behavior of solutions of (1.1). As observed before, for critical nonlinearities, stronger estimates on the solutions are
needed to obtain global existence of solutions. This extra difficulty will translate into weaker properties of the attractor than in the case of subcritical nonlinearities. In fact, in general, we will only ensure that the attractor attracts compact sets of initial data in strong norms. However for power-like nonlinearities, and in the case of initial data in $L'(\Omega)$, we will obtain again attraction of bounded sets of initial data.

The paper is organized as follows. In Section 2, for initial data in $L'(\Omega)$, we will describe the main properties of the solutions constructed in [4, 5] that will be used later in the paper. In Section 3 we will give balance conditions between nonlinear terms and derive some estimates on solutions of (1.1) that will help us in proving that solutions are globally defined. In Section 4 we will analyze the case of subcritical nonlinearities showing the existence of a global compact attractor that attracts bounded sets of $L'(\Omega)$ in the norm of $W^{s,a}(\Omega)$ for every $s \geq 1$ and $0 \leq a < 1 + \frac{1}{r}$ and in $C^b(\Omega)$ for every $0 \leq b < 1$. The structure of the attractor, as the unstable set of the equilibria, will also be obtained. In Section 5, we extend our results for the case of critical nonlinearities. Section 6 is devoted to extend the previous analysis for the case of measures as initial data. Section 7 deals with the case of initial data in $W^{1,r}(\Omega)$, considering both the cases $1 < r < N$ and $r \geq N$. Finally, we included a section with further generalizations and comments.

2. CRITICAL EXPONENTS AND SMOOTHNESS

Now we briefly review, he results for initial data in $L'(\Omega)$ proved in [4, 5], for which the reader is referred for details; see also [3, 9]. Note that the difficulty for solving (1.1) when $f$, $g$ grow at most critically in $L'(\Omega)$, that is, when (1.2), (1.3) are satisfied, is twofold. First, the nonlinearity can be very large: for example when $1 < r < \frac{N}{N-2}$ we have that $r < p_c$ and so, if $p$ satisfies $r < p < p_c$, $f(u)$ is not even integrable for $u \in L'(\Omega)$. On the other hand, since functions in $L'(\Omega)$ are singular, i.e. non-smooth, they have no trace and then the nonlinear term $g$ is not well defined either. Therefore, the references above introduce a class solutions that regularize in a very precise way for $t > 0$ and that solve the corresponding implicit integral formulation, that is, the variation of constants formula. These are called $\varepsilon$-regular solution; see below.

Hence, the results in [4, 5] allow to show that if $f$, $g$ grow at most critically in $L'(\Omega)$, that is, if (1.2), (1.3) are satisfied, then, for every $u_0 \in L'(\Omega)$, (1.1) has a unique local $\varepsilon$-regular solution, for some $\varepsilon > 0$, which is a classical solution for $t > 0$ and depends continuously on the initial data.

More precisely, for each $u_0 \in L'(\Omega)$, there exist $R = R(u_0) > 0$ and $\tau = \tau(u_0) > 0$ such that for any $u_i \in L'(\Omega)$ with $\|u_i - u_0\|_{L'(\Omega)} < R$ there exists
a continuous function $u: [0, \tau_0] \to L'(\Omega)$, with $u(0) = u_1$, which is the unique $\bar{\varepsilon}$-regular solution of (1.1) starting at $u_1$. In addition, this solution satisfies, for some $\gamma > \bar{\varepsilon}$ and for all $0 < \theta < \gamma$,

\[
\begin{align*}
&u \in C((0, \tau_0], H^{2\theta}_r(\Omega)), \\
&t^\theta \|u(t)\|_{H^{2\theta}_r(\Omega)} \leq M(R, \tau_0), \\
&t^\theta \|u(t)\|_{H^{2\theta}_r(\Omega)} \xrightarrow{t \to 0^+} 0,
\end{align*}
\]  

(2.1)

where $H^{2\theta}_r(\Omega)$ denotes the Bessel potential spaces in $L'(\Omega)$ of order $2\theta$. Moreover, if $u_1, v_1 \in B_{L'(\Omega)}(u_0, R)$ the following holds true:

\[
\begin{align*}
&t^\theta \|u(t, u_1) - u(t, v_1)\|_{H^{2\theta}_r(\Omega)} \\
&\leq C(\theta_0, \tau_0) \|u_1 - v_1\|_{L'(\Omega)}, \quad \text{for } t \in (0, \tau_0] \text{ and } 0 \leq \theta \leq \theta_0 < \gamma.
\end{align*}
\]  

(2.2)

It is shown in [5, Theorem 3.1] that for any $r > 1$ and $p, q$ satisfying (1.3), $\gamma$ can always be taken larger than $1/2$. This allows to perform a bootstrap argument to prove that solutions become classical for positive times. This will also be used further below. Note that (2.1) and (2.2) measure in a very precise way the smoothing of solutions starting on the ball of center $u_0$ and radius $R$. This will be crucial when analyzing the asymptotic behavior of solutions of (1.1).

If the nonlinearities are subcritical then $R$ can be taken arbitrarily large. That is, the existence time can be taken uniform on bounded sets of $L'(\Omega)$. As a consequence, and following a standard prolongation argument, when $f$ and $g$ are subcritical, if the solution exists up to a maximal time $T < \infty$ then $\lim_{t \to T} \|u(t)\|_{L'(\Omega)} = +\infty$. However, when $f$ or $g$ are critical, if $T < \infty$ then $\lim_{t \to T} \|u(t)\|_{H^{2\delta}_r(\Omega)} = +\infty$ for any $\delta > 0$, where $H^{2\delta}_r(\Omega)$ denotes the Bessel potential space in $L'(\Omega)$ of order $\delta$. Therefore, in the subcritical case to prove global existence, it is enough to obtain bounds on the $L'(\Omega)$-norm of the solution, while in the critical case stronger estimates must be obtained.

Note that if $f$ and $g$ are at most critical in $L'(\Omega)$, that is, if (1.2), (1.3) are satisfied, then for any $\bar{r} > r$, the nonlinear terms are subcritical in $L'(\Omega)$. In particular, if $u_0 \in L'_{\bar{r}}(\Omega)$, from [4, 5], there exists a unique solution of (1.1) in the sense of $L'(\Omega)$. This solution is also a solution in the sense of $L'(\Omega)$.

All along this paper when we use the term “solution”, we will refer to the $\bar{\varepsilon}$-regular solutions described above. See [4, 5] for a precise statement on $\varepsilon$-regular solutions which involves using families of interpolation spaces.
3. ESTIMATES IN $L'(\Omega)$

In this section we will derive some estimates in $L'(\Omega)$ on the solutions of (1.1) that will be used when analyzing the asymptotic behavior of solutions. As mentioned before we will consider only $\tilde{\varepsilon}$-regular solutions as described above.

For this we will make use of the following version of Poincaré’s inequality that was proved in [20].

**Lemma 3.1.** There exists a constant $c_0(\Omega)$ such that for every $\varphi \in W^{1,1}(\Omega)$

$$\left\| \varphi - \frac{1}{|\Omega|} \int_{\Omega} \varphi \right\|_{L^1(\Omega)} \leq c_0(\Omega) \left\| \nabla \varphi \right\|_{L^1(\Omega)},$$

where $\frac{1}{|\Omega|} \int_{\Omega} \varphi = \frac{1}{|\Omega|} \int_{\Omega} \varphi$.

**Proposition 3.2.** Assume that $f$ and $g$ grow at most critically in $L'(\Omega)$ and, for some $0 < \varepsilon \leq d \frac{4(r-1)}{r^2}$, the function

$$H_\varepsilon(s) = f(s) s + \frac{|\Omega|}{|\Omega|} g(s) s - \frac{c^2(\Omega)}{\varepsilon r^2} (g'(s) + (r-1)g(s))^2,$$

with $c(\Omega) = \frac{|\Omega|}{|\Omega|^2} c_0(\Omega)$, satisfies

$$\liminf_{|s| \to \infty} \frac{H_\varepsilon(s)}{|s|^2} > 0. \quad (3.1)$$

Then the following holds,

$$\frac{1}{r} \frac{d}{dt} \|u(t)\|_{L^r(\Omega)} + c_1 \int_{\Omega} |\nabla|^{r/2}|u(t)|^2 + c_2 \|u(t)\|_{L^r(\Omega)} \leq c_3, \quad (3.2)$$

where $c_1 = d(4(r-1)/r^2) - \varepsilon \geq 0$, and $c_2, c_3 > 0$ are suitable constants.

**Proof.** Multiplying (1.1) by $|u|^{r-2}u$ and integrating by parts, we get

$$\frac{1}{r} \frac{d}{dt} \|u(t)\|_{L^r(\Omega)} + d \frac{4(r-1)}{r^2} \int_{\Omega} |\nabla|^{r/2}|u(t)|^2$$

$$+ \int_{\Omega} f(u) |u|^{r-2}u + \int_{\Omega} g(u) |u|^{r-2}u = 0. \quad (3.3)$$
The last two terms can be rewritten as
\[
\int_{\Omega} \left( \frac{f(u) |u|^{r-2} u + \frac{|f'|}{|f^*|} g(u) |u|^{r-2} u}{|f'|^2} \right) - \frac{|f'|}{|f^*|} \int_{\Omega} \left( g(u) |u|^{r-2} u - \frac{\partial}{\partial r} g(u) |u|^{r-2} u \right)
\]
and Lemma 3.1 gives, with \( c(\Omega) = c_0(\Omega) \frac{|f'|}{|f^*|} \),
\[
\left| \int_{\Omega} \left( g(u) |u|^{r-2} u - \frac{\partial}{\partial r} g(u) |u|^{r-2} u \right) \right| \leq c(\Omega) \| \nabla (g(u) |u|^{r-2} u) \|_{L^1(\Omega)}.
\]
Taking derivatives and arranging terms the right hand side above can be written as
\[
c(\Omega) \left( \frac{\frac{2}{r} g'(u) u + \frac{2(r-1)}{r} g(u)}{r} \right) \| |u|^{r/2-1} \| \| \nabla |u|^{r/2} \|_{L^1(\Omega)}
\]
which can be bounded by
\[
\varepsilon \| \nabla |u|^{r/2} \|_{L^1(\Omega)}^2 + \frac{c^2(\Omega)}{4\varepsilon} \left( \frac{2}{r} g'(u) u + \frac{2(r-1)}{r} g(u) \right) \| |u|^{r/2-1} \|_{L^1(\Omega)}^2
\]
for any \( \varepsilon > 0 \). Therefore, we get
\[
\frac{1}{r} \frac{d}{dt} \| u(t) \|_{L^1(\Omega)} + \left( \frac{d \frac{4(r-1)}{r^2} - \varepsilon}{4\varepsilon} \right) \| \nabla (|u|^{r/2}) \|_{L^1(\Omega)} + \int_{\Omega} H_\varepsilon(u) |u|^{r-2} \leq 0.
\] (3.4)

Since (3.1) holds for some \( 0 < \varepsilon \leq d(4(r-1)/r^2) \), we get the result.

In particular for the case of power-like nonlinearities, that is satisfying (1.8), we have the following result that ensures that (3.1) holds true for some \( 0 < \varepsilon < d(4(r-1)/r^2) \).

**Corollary 3.3.** Assume that
\[
\lim_{|x| \to \infty} \frac{f'(s)}{|s|^{p-1}} = pc > 0, \quad \lim_{|x| \to \infty} \frac{g'(s)}{|s|^{q-1}} = qc < 0
\] (3.5)
for some \( p, q > 1 \) satisfying (1.3), and \( c(\Omega) \) is like in Proposition 3.2. If either
(i) \( p+1 > 2q \), or
(ii) \( p+1 = 2q \),
then
and

$$4(r-1) dc_f > c^2(\Omega) c^2_s (q+r-1)^2,$$  \hspace{1cm} (3.6)$$
then (3.1) holds true for some $0 < \varepsilon < d(4(r-1)/r^2)$. Moreover,

$$\frac{1}{r} \frac{d}{dt} \|u(t)\|_{L^r(\Omega)} + c_1 \int_\Omega |\nabla (|u|^{r/2})|^2 + c_2 \|u(t)\|_{L^{p+r-1}(\Omega)}^{p+r-1} \leq c_3,$$ \hspace{1cm} (3.7)$$
where $c_1, c_2, c_3 > 0$ are suitable constants.

**Proof.** From the assumptions it is clear that the leading terms for $|s| \gg 1$ are $f(s) \sim c_f |s|^{p-1} s$ and $g(s) \sim c_g |s|^{q-1} s$. Therefore, the leading terms for $H_r(s)$ are

$$c_f |s|^{p+1} + \frac{|I|}{|\Omega|} c_g |s|^{q+1} - \frac{c^2(\Omega)}{4c} \left( \frac{2}{r} q + \frac{2(r-1)}{r} \right) c^2_s |s|^2.$$ 

If $p+1 > 2q$ then clearly (3.1) holds true and the result follows. If $p+1 = 2q$ then the coefficient of the leading term in $H_r(s)$ is

$$c_f = -\frac{c^2(\Omega)}{4c} \left( \frac{2}{r} q + \frac{2(r-1)}{r} \right) c^2_s,$$

which is positive for some $0 < \varepsilon < d(4(r-1)/r^2)$ iff (3.6) is satisfied. In any case $H_r(s) |s|^{-2} \geq c_2 |s|^{p+r-1} - c_3$ and from (3.4) we get we result.  

**Remark 3.4.** Observe that the Proposition and the Corollary above, which go in the spirit of some results in [20], can be also used when $p > 1$ and $q \leq 1$ or $p = 1 = q$. In the first case, (i) of the Corollary applies and in the second a similar condition to (ii) can be derived from the proof above. Since we are mainly interested here in the case of critical exponents we do not pursue along this line.

4. GLOBAL ATTRACTORS FOR SUBCRITICAL NONLINEARITIES

In this section we consider the case of subcritical nonlinearities in $L'(\Omega)$. That is, in (1.3) both inequalities are strict.

Using the results above, we get

**Proposition 4.1.** Assume $f$ and $g$ are subcritical in $L'(\Omega)$ and (3.1) holds true.
Then for any \( u_0 \in L'(\Omega) \) the unique \( \bar{\varepsilon} \)-regular solution of (1.1) is globally defined. Even more, if in (3.1) one can take \( 0 < \varepsilon < d(4(r-1)/r^2) \), then for every \( T > 0 \)

\[
\int_0^T \int_{\Omega} |\nabla (|u|^{r/2})|^2 < \infty.
\]

Moreover for power-like nonlinearities as in Corollary 3.3, and for any \( T > 0 \),

\[
u \in L^{q+r-1}((0, T) \times \Omega) \quad \text{and} \quad u \in L^{q+r-1}((0, T) \times \Gamma).
\]

**Proof.** From (3.2) we get that \( \|u(t)\|_{L^r(\Omega)} \) is bounded for finite time and, since the nonlinearities are subcritical, then the solution is globally defined. Also, from (3.2), we obtain that the estimate on \( \nabla (|u|^{r/2}) \) in \( L^2((0, T) \times \Omega) \).

For power-like nonlinearities as in Corollary 3.3, from (3.7), we get that \( u \in L^{p+r-1}((0, T) \times \Omega) \) for any \( T > 0 \). For the estimate on the boundary, observe that integrating on \((0, T)\) in (3.3), we get

\[
-\int_0^T \int_{\Gamma} g(u) |u|^{r-2} u \leq d \frac{4(r-1)}{r^2} \int_0^T \int_{\Omega} |\nabla (|u|^{r/2})|^2 + \int_0^T \int_{\Omega} f(u) |u|^{r-2} u + \frac{1}{r} \|u(T)\|_{L^r(\Omega)}.
\]

Using (3.5), and the previous bounds, we get \( u \in L^{q+r-1}((0, T) \times \Gamma) \). \( \square \)

We can further obtain

**Theorem 4.2.** Assume \( f \) and \( g \) are subcritical in \( L'(\Omega) \) and (3.1) holds true.

Then there exists an absorbing ball in \( L'(\Omega) \). That is, there exists some constant \( M > 0 \), such that for any bounded set of initial conditions, \( B \subset L'(\Omega) \), there exists \( T = T(B) \) such that for \( t \geq T(B) \),

\[
\|u(t)\|_{L'(\Omega)} \leq M.
\]

(4.1)

Also, the orbit of any bounded set \( B \subset L'(\Omega) \) is bounded in \( L'(\Omega) \). Even more there exist absorbing balls in \( W^{s,b}(\Omega) \) for every \( s \geq 1 \) and \( 0 \leq \alpha < 1 + \frac{1}{s} \) and in \( C^\beta(\Omega) \) for every \( 0 \leq \beta < 1 \). Moreover, for power-like nonlinearities as in Corollary 3.3, there exists an absorbing ball in \( L'(\Omega) \) with entering time, \( T_0 \), independent of the initial condition.

Furthermore (1.1) has a global compact attractor in \( L'(\Omega) \) which is given by \( \mathcal{M} = W^*(E) \), that is, the unstable set of the set of equilibria of (1.1), \( E \), which is nonempty.
The attractor \( \mathcal{A} \) is compact and connected in \( W^{s, \gamma}(\Omega) \) for every \( s \geq 1 \) and \( 0 \leq \alpha < 1 + \frac{s}{N} \) and in \( C^\beta(\overline{\Omega}) \) for every \( 0 \leq \beta < 1 \) and attracts bounded sets of \( L'(\Omega) \) in the norms of \( W^{s, \gamma}(\Omega) \) and \( C^\beta(\overline{\Omega}) \).

Proof. From (3.2) we get,
\[
\frac{1}{r} \frac{d}{dt} \|u(t)\|_{L^r(\partial\Omega)} + c_1 \|u(t)\|_{L^r(\partial\Omega)} \leq c_2
\]
for \( c_1, c_2 > 0 \). Integrating this inequality we get the existence of the absorbing set and that orbits of bounded sets in \( L'(\Omega) \) are bounded in \( L'(\Omega) \). Moreover, for power-like nonlinearities as in Corollary 3.3, using (3.7) and Hölder inequality, we get that \( y(t) = \|u(t)\|_{L^r(\partial\Omega)} \) satisfies
\[
y'(t) + c_1 y(t)^{\frac{r+2}{r}} \leq c_2
\]
and since \( p + r - 1 > r \), from Lemma 5.1, Chapter 3 in [21], there exist \( \bar{M} \) and \( T_0 \) independent of the solution, such that for any \( u_0 \in L'(\Omega) \), one has \( \|u(t)\|_{L^r(\partial\Omega)} \leq \bar{M} \) for any \( t \geq T_0 \).

Therefore, it remains to prove that the nonlinear semigroup defined by (1.1) is compact in \( L'(\Omega) \). With these, using the results in [16, 21], we get the existence of the attractor \( \mathcal{A} \). As we are dealing with the \( \bar{e} \)-regular solutions constructed in [4, 5], the argument to show compactness goes as follows. We first start by observing that since the nonlinearities are sub-critical in \( L'(\Omega) \), we get from (2.1), (2.2) that for arbitrarily large \( R \) there is some \( \tau_0 > 0 \) such that for \( t \in (0, \tau_0] \),
\[
t^\theta \|u(t)\|_{H^\gamma(\Omega)} \leq M(R, \tau_0)
\]
for some \( \theta > 0 \) and for all initial values such that \( \|u_0\|_{L^\gamma(\partial\Omega)} \leq R \). As mentioned in Section 2, it was shown in [5, Theorem 3.1] that we can always choose \( \theta > 1/2 \).

Taking initial values in the absorbing ball constructed before and using the inequality above for \( t = \tau_0/2 \) we get that there exists also an absorbing ball in \( H^\gamma_{\mu}(\Omega) \). Since the inclusions \( H^\gamma_{\mu}(\Omega) \subset H^1(\Omega) \subset L'(\Omega) \) are compact, we get the compactness of the nonlinear semigroup and the existence of the global attractor, \( \mathcal{A} \).

But, at the same time, if \( r > N \), we have \( H^1(\Omega) \subset C(\overline{\Omega}) \). Therefore there exists an absorbing ball in \( C(\overline{\Omega}) \) and \( \mathcal{A} \subset C(\overline{\Omega}) \) is bounded.

If \( 1 < r \leq N \) we apply a bootstrap argument. Since \( H^1(\Omega) \subset L^m(\Omega) \) for \( m = Nr/(N-r) > r \), we then get the existence of an absorbing ball in \( L^m(\Omega) \). Since \( m > r \), then (1.1) is also subcritical in \( L^m(\Omega) \) and we can apply the argument above in \( L^m(\Omega) \) which now gives the existence of an absorbing ball in \( H^1_{\mu}(\Omega) \). Repeating the process, in a finite number of steps we
can show the existence of an absorbing ball in $H^1_m(\Omega)$, for some $m > N$. Therefore, there exists an absorbing ball in $C(\bar{\Omega})$ and $\mathcal{A} \subset C(\bar{\Omega})$ is bounded.

Once the solutions enter in the absorbing ball in $C(\bar{\Omega})$ and proceeding as in [6], by using the variations of constants formula, one obtains now that there exists an absorbing ball in $W^{\alpha,q}(\Omega)$ for every $s \geq 1$ and $0 \leq \alpha < 1 + \frac{1}{r}$.

By Sobolev embeddings we obtain an absorbing ball in $C^\beta(\bar{\Omega})$ for every $0 \leq \beta < 1$.

Therefore the attractor $\mathcal{A}$ and the $\omega$-limit sets of bounded sets of $L'(\Omega)$, which are contained in $\mathcal{A}$, are relatively compact in $W^{\alpha,q}(\Omega)$ and $C^\beta(\bar{\Omega})$. Using [16, Lemma 2.1.2], the $\omega$-limit set of any bounded set of $L'(\Omega)$ is compact, invariant and attracts in the norms of $W^{\alpha,q}(\Omega)$ and $C^\beta(\bar{\Omega})$.

To obtain the description of the attractor, we show that (1.1) is a gradient like system, see [16, Section 3.8]. Multiplying (1.1) by $u_t$ and integrating by parts we get that

$$\mathcal{F}(u) = \frac{d}{2} \int_\Omega |\nabla u|^2 + \int_\Omega F(u) + \int_\Omega G(u),$$

(4.2)

where $F(s) = \int_0^s f(r) \, dr$ and $G(s) = \int_0^s g(r) \, dr$, is a Lyapunov functional for (1.1).

Note that $\mathcal{F}$ is not well defined on the whole $L'(\Omega)$. However, since the solutions are smooth after positive times and we already obtained that orbits are bounded and pre-compact, using La Salle’s invariance principle, we can still obtain that the $\omega$-limit set of each initial data is included in the set of equilibria. Therefore, the description of $\mathcal{A}$ follows, see [16, Theorem 3.8.5].

Now we can obtain the same result under somehow weaker dissipative condition. In fact instead of requiring the dissipativity assumption (3.1) to be satisfied in $L'(\Omega)$, we will only need that (1.1) is dissipative in some $L^z(\Omega)$ such that the problem is still subcritical. More precisely, we have

**Theorem 4.3.** Assume $f$, $g$ are subcritical in $L'(\Omega)$. Even more assume that for some $z > 1$ such that

$$p < 1 + \frac{2z}{N} \quad \text{and} \quad q < 1 + \frac{z}{N},$$

the function

$$H_z(s) = f(s) s + \frac{|f'|}{|\Omega|} g(s) s - \frac{c^2(\Omega)}{e^{z^2}} (g'(s) s + (z-1) g(s))^2$$
satisfies

\[ \liminf_{|s| \to \infty} \frac{H_z(s)}{|s|^2} > 0 \]  

(4.3)

for some \( 0 < \varepsilon < d(4(z-1)/z^2) \).

Then (1.1) has a global attractor \( \mathcal{A} \) in \( L'(\Omega) \) which has the same properties as in Theorem 4.2.

**Proof.** Note that by a similar reasoning as in the bootstrap argument in the proof of Theorem 4.2, using (2.1) and Sobolev embeddings, we have that after a possibly very short time, the orbit of a bounded set in \( L'(\Omega) \) enters a bounded set in \( L'(\Omega) \).

Therefore, since (1.1) is subcritical in \( L'(\Omega) \), using Theorem 4.2 in \( L'(\Omega) \), we get the result.

Hence, for power-like nonlinearities, we have

**Corollary 4.4.** Assume \( f \) and \( g \) are subcritical power-like nonlinearities in \( L'(\Omega) \).

(i) Assume \( p + 1 > 2q \) or \( p + 1 = 2q \) and (3.6) holds true; that is,

\[ \frac{4dc_f c_g^2}{c^2(\Omega)} > \frac{(q + r - 1)^2}{(r - 1)} \]  

(4.4)

Then Theorem 4.2 applies.

(ii) Assume \( p + 1 = 2q \) and \( r \leq q + 1 \), or \( q + 1 < r \) and \( q < 1 + \frac{2}{N-1} \). Moreover, in either case, assume

\[ dc_f > c^2(\Omega) c_g^3. \]  

(4.5)

Then Theorem 4.3 applies with \( z = q + 1 \).

(iii) Assume \( p + 1 = 2q \), \( q + 1 < r \) and \( q \geq 1 + \frac{2}{N-1} \); then Theorem 4.3 applies for some \( q + 1 \leq z < r \), provided

\[ \frac{4dc_f c_g^2}{c^2(\Omega)} > \frac{(q - 1)^2(N + 1)^2}{(N(q - 1) - 1)}. \]  

(4.6)

**Proof.** In the first case, from Corollary 3.3, Theorem 4.2 applies. For the other cases, observe that from (3.6) in Corollary 3.3, (4.3) holds for some \( z > 1 \) provided

\[ \frac{4dc_f c_g^2}{c^2(\Omega)} > j(z) = \frac{(q + z - 1)^2}{(z - 1)}. \]
Observe that \( j(1^+) = \infty \), \( j(1) = 4q \) and (4.5) is equivalent to \( 4dc_f/c^2(\Omega) c^2_x > j(q + 1) \).

Therefore, if \( r \leq q + 1 \) and (4.5) holds true, we can take \( z = q + 1 \) and Theorem 4.3 applies in \( L^{q+1}(\Omega) \). On the other hand, if \( r > q + 1 \) and (4.5) holds true, we can still use Theorem 4.3 in \( L^{q+1}(\Omega) \) provided \( f \) and \( g \) are still subcritical in this space. From (1.3), this amounts to saying that

\[
p < 1 + \frac{2q + 1}{N} \quad \text{and} \quad q < 1 + \frac{q + 1}{N} \]

which is satisfied if \( q < 1 + \frac{q^2}{N} \).

Finally, if \( p + 1 = 2q \), \( q + 1 < r \) and \( q > 1 + \frac{2}{N} \), we can still use Theorem 4.3. in \( L^r(\Omega) \) for some \( q + 1 \leq z < r \) provided \( f \) and \( g \) are still subcritical in this space and

\[
4dc_f/c^2(\Omega) c^2_x > j(z) \]

Then, from (1.3), subcriticality implies \( z > N(q-1) \) which is still larger than \( q + 1 \) since \( q > 1 + \frac{2}{N} \).

Since \( j(z) \) is strictly decreasing on \((q + 1, \infty)\), the second condition above can be met provided

\[
4dc_f/c^2(\Omega) c^2_x > j(N(q-1)) \]

which is equivalent to (4.6).

Note that from the proof above, when \( p + 1 = 2q \), condition (4.5) is less restrictive than (4.6) which is in turn less restrictive than (4.4). However the first two can only be used in the cases stated in the corollary.

For the case \( p + 1 < 2q \) the conclusion of the corollary fails; see the comment right before Theorem 5.3.

5. GLOBAL ATTRACTORS FOR CRITICAL NONLINEARITIES

In this section we assume that either \( f \) or \( g \) is critical. That is, in (1.3), either \( p = p_c \) or \( q = q_c \).

Note that, as noted in Section 2, in case of critical nonlinearities, estimates in \( L^q(\Omega) \) do not suffice to ensure global existence and in fact stronger estimates must be obtained. Since this seems to be a difficult task, we employ an indirect argument as we now describe. The basic idea that we follow is that solutions of (1.1) are smooth for \( t > 0 \) so they enter \( L^r(\Omega) \) with \( z > r \) and so (1.1) becomes subcritical in this latter space. Therefore, if (1.1) is dissipative in \( L^r(\Omega) \), then Theorem 4.2 applies and then solutions are attracted towards the attractor in \( L^r(\Omega) \). Therefore this set describes the asymptotic behavior of solutions of (1.1) with initial data in \( L^r(\Omega) \) and critical nonlinearities.

However, we will obtain weaker properties of the attractor than in the case of subcritical nonlinearities. In fact, in general we will only ensure that the attractor attracts compact sets of \( L^r(\Omega) \) in strong norms. However for power-like nonlinearities we will obtain again attraction of bounded sets of \( L^r(\Omega) \).

More precisely, we can prove
Theorem 5.1. Assume $f$ or $g$ are critical in $L^r(\Omega)$. Then, we have

(i) Assume that the function

$$H_\varepsilon(s) = f(s)s + \frac{|I|}{|\Omega|} g(s) + \frac{c^2(\Omega)}{\varepsilon z^2} (g'(s)s + (z-1)g(s))^2$$

satisfies

$$\liminf_{|s| \to \infty} \frac{H_\varepsilon(s)}{|s|^2} > 0 \quad (5.1)$$

for some $z > r$ and $0 < \varepsilon \leq d(4(z-1)/z^2)$.

Then for any $u_0 \in L'(\Omega)$ the unique $\varepsilon$-regular solution of (1.1) is globally defined. Moreover, orbits of compact sets of $L'(\Omega)$ are compact in $L'(\Omega)$. In particular the $\alpha$-limit set of any trajectory is included in the set of equilibria of (1.1). Even more, there exists a maximal, compact, invariant and connected set $\mathcal{A}$ in $L'(\Omega)$ which attracts compact sets of $L'(\Omega)$ in the norm of $W^{\infty}(\Omega)$ for every $s \geq 1$ and $0 \leq \alpha < 1 + \frac{1}{r}$ and in $C^\beta(\bar{\Omega})$ for every $0 < \beta < 1$. Furthermore, $\mathcal{A} = W^\alpha(\mathcal{E})$ is the unstable set of the equilibria of (1.1) which is moreover compact and connected in $W^{\infty}(\Omega)$ for every $s \geq 1$ and $0 \leq \alpha < 1 + \frac{1}{r}$ and in $C^\beta(\bar{\Omega})$ for every $0 < \beta < 1$.

(ii) If additionally, (3.1) holds for some $0 < \varepsilon < d(4(r-1)/r^2)$, then for every $T > 0$,

$$\int_0^T \int_\Omega |\nabla |u|^r/2)|^2 < \infty.$$

Moreover for power-like nonlinearities as in Corollary 3.3 and for any $T > 0$,

$$u \in L^{p+r-1}((0, T) \times \Omega) \quad \text{and} \quad u \in L^{q+r-1}((0, T) \times \Gamma).$$

Furthermore, there exists an absorbing ball in $L'(\Omega)$ and the orbit of any bounded set $B \subset L'(\Omega)$ is bounded in $L'(\Omega)$. For power-like nonlinearities as in Corollary 3.3 there exists an absorbing ball in $L'(\Omega)$ with entering time, $T_0$, independent of the initial condition.

Proof. Consider first an initial condition $u_0 \in L'(\Omega)$. Since $z > r$, then (1.1) is subcritical in $L'(\Omega)$ and since (5.1) holds true, Proposition 4.1 and Theorem 4.2 apply in $L'(\Omega)$. In particular the solution is global and bounded in $L'(\Omega)$. Also, this solution is a global and bounded $\varepsilon$-regular solution in $L'(\Omega)$.

For an arbitrary initial data $u_0 \in L'(\Omega)$, from [4, 5], the solution is classical for positive times. In particular the solution enters $L'(\Omega)$ and the above applies.
Since Proposition 4.1 and Theorem 4.2 apply in $L^r(\Omega)$, denote $\mathcal{A} = W^r(E)$, the global attractor in $L^r(\Omega)$ which is compact, invariant and connected in $L^r(\Omega)$ and so in $L^r(\Omega)$.

To finish the proof, we show below that for each initial data $u_0 \in L^r(\Omega)$, the orbit of some neighborhood of $u_0$ is compact in $L^r(\Omega)$ and enters in finite time a bounded set in $L^r(\Omega)$ and therefore it is attracted by $\mathcal{A} = W^r(E)$.

For this we rely once more in the properties of $\bar{\sigma}$-regular solutions. More precisely, from (2.1), (2.2), for any $u_0 \in L^r(\Omega)$ there exist $R$ and $\tau_0 > 0$ such that for $t \in (0, \tau_0]$, and any initial data in the ball in $L^r(\Omega)$ of center $u_0$ and radius $R$,

$$t^\theta \|u(t)\|_{H^2_r(\Omega)} \leq M(R, \tau_0)$$

for some $\theta > 1/2$. Therefore a neighborhood of $u_0$ enters in finite time a bounded set of $H^2_r(\Omega)$ and so it enters a compact set in $L^r(\Omega)$.

On the other hand, with a bootstrap argument as in the proof of Theorem 4.2, we get that a neighborhood of $u_0$ enters in finite time a bounded set of $L^r(\Omega)$ and we get the result. That ends the proof of point (i).

For point (ii), since (3.1) holds for some $0 < \varepsilon < d(4(r-1)/r^2)$, the estimates on $a$ and the absorbing properties in $L^r(\Omega)$ follow now as in the proofs of proposition 4.1 and Theorem 4.2.

Note that in the argument above, since the nonlinearities are critical, then we cannot ensure that in (2.1), (2.2), $R$ can be taken arbitrarily large. Therefore, we can not conclude in general that the orbit of bounded sets in $L^r(\Omega)$ are compact in $L^r(\Omega)$ and that the nonlinear semigroup is compact. As a consequence we can not ensure in general that the set $\mathcal{A} = W^r(E)$ attracts bounded sets of $L^r(\Omega)$.

Now we particularize for power-like nonlinearities. Note that in particular, from (1.3), (1.4), or (1.5), the critical exponents satisfy

$$p_c + 1 = 2q_c.$$ 

Therefore if $f$ is critical and $g$ is subcritical, then $p = p_c$ and $p+1 = p_c + 1 > 2q$. On the other hand, if $g$ is critical then $q = q_c$ and then and $p_c + 1 = 2q_c > p + 1$. Finally, note that in dimension $N = 1$, from (1.3), $g$ is always subcritical.

**Corollary 5.2.** Assume $f$ and $g$ are power-like nonlinearities. Then we have

(i) Assume $f$ is critical and $g$ is subcritical. Then (3.1) holds true and (5.1) is satisfied for every $z > r$ and points (i) and (ii) of Theorem 5.1 apply.
Assume $f$ and $g$ are both critical, so $p+1 = p_C + 1 = 2q = 2q_C$, and (3.6) is satisfied, that is,

$$\frac{4dc_f}{c^2(\Omega) c^2_g} \geq \frac{(q + r - 1)^2}{(r - 1)}.$$ 

Then (3.1) holds true and (5.1) is satisfied for some $z > r$ and points (i) and (iii) of Theorem 5.1 apply.

(iii) Assume $f$ and $g$ are both critical, so $p+1 = p_C + 1 = 2q = 2q_C$, $r < \frac{2N}{N-1}$, and

$$dc_f > c^2(\Omega) c^2_g q_C.$$ 

Then (5.1) is satisfied for some $z > r$ and point (i) of Theorem 5.1 applies.

Proof. If $f$ is critical and $g$ is subcritical then $p+1 = p_C + 1 > 2q$ and then, from Corollary 3.3, (3.1) holds true and (5.1) is satisfied for any $z > r$.

If $f$ and $g$ are both critical, then $p+1 = p_C + 1 = 2q = 2q_C$, and as in Corollary 4.4, (5.1) holds provided

$$\frac{4dc_f}{c^2(\Omega) c^2_g} > j(z) = \frac{(q_C + z - 1)^2}{(z - 1)}$$

for some $z > r$.

Therefore, if (3.6) is satisfied, we have $4dc_f / c^2(\Omega) c^2_g > j(z)$ for $z = r$ and from Corollary 3.3, (3.1) holds true. At the same time by continuity, this is also satisfied for some $z > r$.

On the other hand, since $z = q_C + 1$ is the global minimizer of $j(z)$ and this function is strictly decreasing on $(1, q_C + 1)$, if $r < q_C + 1$ and $4dc_f / c^2(\Omega) c^2_g > j(q_C + 1) = 4q_C$, then (5.1) holds for some $z > r$. This two conditions are equivalent to $r < \frac{2N}{N-1}$ and $dc_f > c^2(\Omega) c^2_g q_C$ respectively and we get the result. 

Observe that from [20], if $p+1 < 2q$ there always exists solutions of (1.1) with smooth initial data that blow-up in finite time. Therefore, the Corollary above does not apply if $f$ is subcritical and $g$ is critical, since in this case $p+1 < p_C + 1 = 2q_C$. Therefore, the results above are optimal.

Now we show that for the case of power-like nonlinearities, a stronger result than Theorem 5.1 holds true. In fact we have

**Theorem 5.3.** Assume $f$ and $g$ are power-like nonlinearities and assume $f$ is critical and $g$ is subcritical or $f$ and $g$ are both critical, so $p+1 = p_C + 1 = 2q = 2q_C$, and (3.6) is satisfied; that is,

$$\frac{4dc_f}{c^2(\Omega) c^2_g} \geq \frac{(q + r - 1)^2}{(r - 1)}.$$
Then the nonlinear semigroup generated by (1.1) is compact in $L'(\Omega)$ and the set $\mathcal{A} = W^s(E)$, constructed in Theorem 5.1, attracts bounded sets of $L'(\Omega)$ in the norm of $W^{s, \cdot}(\Omega)$ for every $s \geq 1$ and $0 \leq \alpha < 1 + \frac{1}{s}$ and in $C^\beta(\overline{\Omega})$ for every $0 \leq \beta < 1$. Therefore, $\mathcal{A}$ is the global attractor of (1.1) in $L'(\Omega)$.

Proof. Observe that from Corollary 5.2, points (i) and (ii) of Theorem 5.1 apply and in fact (5.1) is satisfied for some $z > r$ arbitrarily close to $r$. Thus we now show that the orbit of any bounded set in $L'(\Omega)$ enters in finite time a bounded set in $L'(\Omega)$ and then we get the result since $\mathcal{A}$ is the global attractor in $L'(\Omega)$.

Consider a bounded set $B$ in $L'(\Omega)$ and let $T_0$ be the entering time in the absorbing ball in $L'(\Omega)$ given in point (ii) of Theorem 5.1. From (3.7) we obtain that there exists some $M_1 > 0$ such that for every $t > T_0$ and every solution with initial data in $B$, we have

$$\int_t^{t+1} \int_\Omega |u|^{p+r-1} \leq M_1.$$  

From here we also obtain that for every $r < z \leq p + r - 1$

$$\int_t^{t+1} \int_\Omega |u|^r \leq M_2.$$  

On the other hand, as in (3.2), we get

$$\frac{1}{z} \frac{d}{dt} \|u(t)\|_{L^z(\Omega)} + c_1 \int_\Omega |\nabla (|u|^{z/2})|^2 + c_2 \|u(t)\|_{L^z(\Omega)} \leq c_3$$

and from here we have

$$\frac{1}{z} \frac{d}{dt} \|u(t)\|_{L^z(\Omega)} + c_2 \|u(t)\|_{L^z(\Omega)} \leq c_3.$$  

Now we can use the Uniform Gronwall Lemma [21, Lemma 1.1, p. 89] to obtain that there exists some $M_3 > 0$ such that for every $t \geq T_0 + 1$ and every solution with initial data in $B$, one has

$$\|u(t)\|_{L^z(\Omega)} \leq M_3$$

and the result follows. □
6. ATTRACTORS FOR MEASURES AS INITIAL DATA

In this section we consider (1.1) when the initial data are bounded measure on $\Omega$. First we recall the existence result from [4, 5]. Assume $f$ and $g$ satisfy (1.2) with

$$p < 1 + \frac{2}{N} = p_c \quad \text{and} \quad q < 1 + \frac{1}{N} = q_c. \quad (6.1)$$

The exponents $p_c$ and $q_c$ are the critical exponents for the space of measures, $\mathcal{M}(\Omega)$. Note that $f$ and $g$ are always subcritical for this type of initial data.

The construction of the solution in [4, 5] is as follows. Choosing any $1 < r < \min\{p, q\}$ there exists $s = N/r + \delta$, with $\delta$ small enough, such that $M(\Omega) = (C(\Omega))^r \subset H_{t, r}^{-1}(\Omega)$ and for each $u_0 \in H_{t, r}^{-1}(\Omega)$, there exist $R = R(u_0) > 0$ and $\tau = \tau(u_0) > 0$ such that for any $u_1 \in H_{t, r}^{-1}(\Omega)$ with $\|u_1 - u_0\|_{H_{t, r}^{-1}(\Omega)} < R$ there exists a continuous function $u: [0, \tau] \to H_{t, r}^{-1}(\Omega)$, with $u(0) = u_1$, which is the unique $\bar{\epsilon}$-regular solution of (1.1) starting at $u_1$. In addition, this solution satisfies, instead of (2.1), (2.2), that for some $\gamma > \bar{\epsilon}$ and for every $0 < \theta < \gamma$,

$$u \in C((0, \tau_0], H_{t, r}^{-1+2\theta}(\Omega)), \quad t^\theta \|u(t)\|_{H_{t, r}^{-1+2\theta}(\Omega)} \leq M(R, \tau_0),$$

$$t^\theta \|u(t)\|_{H_{t, r}^{-1+2\theta}(\Omega)} \xrightarrow{t \to 0^+} 0. \quad (6.2)$$

Moreover, if $u_1, v_1 \in B_{H_{t, r}^{-1}(\Omega)}(u_0, R)$ the following holds:

$$t^\theta \|u(t, u_1) - u(t, v_1)\|_{H_{t, r}^{-1+2\theta}(\Omega)}$$

$$\leq C(\theta_0, \tau_0) \|u_1 - v_1\|_{H_{t, r}^{-1}(\Omega)}, \quad \text{for } t \in (0, \tau_0], \quad \text{and } 0 \leq \theta \leq \theta_0 < \gamma. \quad (6.3)$$

It is also shown in [5, Sect. 3.3] that $\gamma$ can always be taken larger than $1/2$. Also, since in this case the nonlinearities are always subcritical, the radius $R$ above can be taken arbitrarily large.

Note that choosing $r$ sufficiently close to 1 then $s$ is sufficiently small, and then $-s + 2\theta$ becomes positive. Therefore, using a bootstrap argument, the solution becomes classical for positive time.

With this, it is proved in [4, 5] that for every $u_0 \in \mathcal{M}(\Omega)$ then there exists a unique classical solution to (1.1) which satisfies

$$\int_{\Omega} u(t, x) \phi(x) \, dx \xrightarrow{t \to 0^+} \langle u_0, \phi \rangle, \quad \forall \phi \in C(\bar{\Omega}). \quad (6.4)$$
Concerning global existence and asymptotic behavior of solutions, as in Section 5 the basic idea is that since solutions are smooth for positive time, they enter $L^z(\Omega)$ with $z > 1$ and so if (1.1) is dissipative in $L^z(\Omega)$, then Theorem 4.2 applies and then solutions are attracted towards the attractor in $L^z(\Omega)$ which describes the asymptotic behavior of solution with measures as initial data. Since, according to (6.1), (1.1) is always subcritical in $\mathcal{M}(\Omega)$, we will obtain that this attractor attracts bounded sets of initial data.

Therefore, we can prove

**Theorem 6.1.** Assume $f$, $g$ satisfy the growth assumptions above. Even more assume that for some $z > 1$ the function

$$H_z(s) = f(s)s + \frac{|\Omega|}{|\Omega|} g(s)s - \frac{c^2(\Omega)}{e z^z} (g'(s)s + (z - 1)g(s))^2$$

satisfies

$$\liminf_{|s| \to \infty} \frac{H_z(s)}{|s|^2} > 0 \quad (6.5)$$

for some $0 < e < d(4(z - 1)/z^2)$.

Then for any $u_0 \in \mathcal{M}(\Omega)$ the unique $\bar{\epsilon}$-regular solution of (1.1) defined above is globally defined. Moreover, (1.1) has a global attractor $\mathcal{A}$ in $\mathcal{M}(\Omega)$ which is given by $\mathcal{A} = \mathcal{W}^*(E)$, where $E$ is the set of equilibria of (1.1), which is nonempty.

The attractor $\mathcal{A}$ is compact and connected in $W^{\infty}(\Omega)$ for every $s \geq 1$ and $0 \leq \alpha < 1 + \frac{1}{4}$ and in $C^\beta(\Omega)$ for every $0 \leq \beta < 1$ and attracts bounded sets of $L^z(\Omega)$ in the norms of $W^{\infty}(\Omega)$ and $C^{\beta}(\Omega)$.

**Proof.** As in the proof of Theorem 5.1, we consider first an initial condition $u_0 \in L^z(\Omega)$. Since $z > 1$ and (6.1) holds, then (1.1) is subcritical in $L^z(\Omega)$ and since (6.5) holds true, Proposition 4.1 and Theorem 4.2 apply in $L^z(\Omega)$. In particular the solution is global and bounded in $L^z(\Omega)$.

For an arbitrary measure as initial data $u_0 \in \mathcal{M}(\Omega)$, from [4, 5], the solution is classical for positive times. In particular the solution enters $L^z(\Omega)$ and the above applies.

Since Proposition 4.1 and Theorem 4.2 apply in $L^z(\Omega)$, denote $\mathcal{A} = \mathcal{W}^*(E)$, the global attractor in $L^z(\Omega)$ which is compact, invariant and connected in $L^z(\Omega)$ and so in $\mathcal{M}(\Omega)$.

To finish the proof, we show below that for each bounded set in $\mathcal{M}(\Omega)$ the orbit enters in finite time a bounded set in $L^z(\Omega)$ and is compact in $\mathcal{M}(\Omega)$.
From (6.2), (6.3), and for arbitrary large $R$, there exists $\tau_0 > 0$ such that for $t \in (0, \tau_0]$, and any initial data satisfying $\|u_0\|_{H^{-s+r}(\Omega)} < R$, we have

$$t^\theta \|u(t)\|_{H^{-s+r}(\Omega)} \leq M(R, \tau_0).$$

Since $-s + 2\theta > 0$, using Sobolev embeddings and with a bootstrap argument as in the proof of Theorem 4.2, we get that a the orbit of any bounded set in $\mathcal{M}(\Omega)$ enters in finite time a bounded set of $L^s(\Omega)$ and a compact set in $\mathcal{M}(\Omega)$ and we get the result.

Hence, for power-like nonlinearities, we have

**Corollary 6.2.** Assume $f$ and $g$ are power-like nonlinearities satisfying (6.1). Then if either

(i) $p+1 > 2q$

(ii) $p+1 = 2q$

and

$$dc_f > c^2(\Omega) c^2_q,$$

then the conclusion of Theorem 6.1 applies.

**Proof.** In the first case, from (3.6) in Corollary 3.3, (6.5) holds for any $z > 1$ while in the second case it holds for $z = q+1$; see also Corollary 4.4.

7. ATTRACTION IN $W^{1,r}(\Omega)$

In this section we consider (1.1) when the initial data are in $W^{1,r}(\Omega)$.

**7.1. The Singular Case $1 < r < N$**

We first start with the singular case $1 < r < N$ and so we recall the existence result from [4, 5]. Assume $f$ and $g$ satisfy (1.2) with

$$p \leq 1 + \frac{2r}{N-r} = p_c \quad \text{and} \quad q \leq 1 + \frac{r}{N-r} = q_c.$$  \hspace{1cm} (7.1)

The exponents $p_c$ and $q_c$ are the critical exponents for the space $W^{1,r}(\Omega)$, for $1 < r < N$.

Then, for each $u_0 \in W^{1,r}(\Omega)$, there exist $R = R(u_0) > 0$ and $\tau = \tau(u_0) > 0$ such that for any $u_1 \in W^{1,r}(\Omega)$ with $\|u_1 - u_0\|_{W^{1,r}(\Omega)} < R$ there exists a continuous function $u : [0, \tau_0] \to W^{1,r}(\Omega)$, with $u(0) = u_1$, which is the unique $\bar{e}$-regular solution of (1.1) starting at $u_1$. In addition, this solution satisfies, for some $\gamma > \bar{e}$ and for all $0 < \theta < \gamma$,

$$u \in C((0, \tau_0], H^{1+2\theta}(\Omega)), \quad t^\theta \|u(t)\|_{H^{1+2\theta}(\Omega)} \leq M(R, \tau_0), \quad t^\theta \|u(t)\|_{H^{1+2\theta}(\Omega)} \rightarrow_0 0,$$  \hspace{1cm} (7.2)
where $H^{1+2h}(Ω)$ denotes the Bessel potential spaces in $L^r(Ω)$ of order $1+2h$. Moreover, if $u_1, v_1 ∈ B_{W^{1,r}(Ω)}(u_0, R)$ the following holds true:

$$t^θ \|u(t, u_1) − u(t, v_1)\|_{H^{1+2h}(Ω)} ≤ C(θ_0, τ_0) \|u_1 − v_1\|_{W^{1,r}(Ω)}, \quad \text{for } t ∈ (0, τ_0], \quad 0 ≤ θ ≤ θ_0 < γ.$$  

(7.3)

As mentioned in Section 2 for the case of $L^r(Ω)$, the results in [4, 5] imply that for subcritical nonlinearities estimates in $W^{1,r}(Ω)$ are enough to obtain global existence, while for critical ones stronger estimates are needed. But note that except for the case $r = 2$, even the estimates in $W^{1,r}(Ω)$ are not easy to obtain for (1.1). However, using the previous results we will be able to overcome this difficulty in an indirect way.

In fact, as in Sections 5 and 6 the basic idea is that solutions must enter $L^z(Ω)$ for some $z > r$ such that (1.1) becomes subcritical in this latter space. Therefore, if (1.1) is dissipative in $L^r(Ω)$, then Theorem 4.2 applies and then solutions are attracted towards the attractor in $L^r(Ω)$. Therefore this set describes the asymptotic behavior of solutions of (1.1) with initial data in $W^{1,r}(Ω)$. As in the sections above, depending on the criticality or subcriticality of the nonlinear terms, we will obtain that the attractors attracts compact or bounded sets, respectively, of $W^{1,r}(Ω)$.

With this in mind observe that, denoting by $p_c(z)$ and $q_c(z)$ the critical exponents for $L^r(Ω)$ as in (1.3), and if we look for some $z$ such that the critical exponents in (7.1) are subcritical in $L^r(Ω)$, we obtain the condition

$$z > \frac{Nr}{N−r}.$$  

Then we can prove

**Theorem 7.1.** Assume $f$ and $g$ satisfy (7.1). Assume that the function

$$H_z(s) = f(s) s + |s|^z \frac{|g(s)|}{|s|^z} s − \frac{c^2(Ω)}{εz^2} (g'(s) s + (z−1) g(s))^2$$

satisfies

$$\liminf_{|s| → ∞} \frac{H_z(s)}{|s|^z} > 0$$  

(7.4)

for some $z > \frac{Nr}{N−r}$ and $0 < ε ≤ d(4(z−1)/z^2)$.

Then for any $u_0 ∈ W^{1,1}(Ω)$ the unique $ε$-regular solution of (1.1) is globally defined. Moreover, orbits of compact sets of $W^{1,1}(Ω)$ are compact in $W^{1,1}(Ω)$. In particular the $ω$-limit set of any trajectory is included in the set of equilibria of (1.1). Even more, there exists a maximal, compact, invariant and connected set $A$ in $W^{1,1}(Ω)$ which attracts compact sets of $W^{1,1}(Ω)$ in
the norm of $W^{s,t}(\Omega)$ for every $s \geq 1$ and $0 \leq \alpha < 1 + \frac{1}{r}$ and in $C^{\beta}(\Omega)$ for every $0 \leq \beta < 1$. Furthermore, $\mathcal{A} = W^{s}(E)$ is the unstable set of the equilibria of (1.1) which is compact and connected in $W^{s,t}(\Omega)$ for every $s \geq 1$ and $0 \leq \alpha < 1 + \frac{1}{r}$ and in $C^{\beta}(\Omega)$ for every $0 \leq \beta < 1$.

If moreover $f$ and $g$ are subcritical, then $\mathcal{A}$ attracts bounded sets of $W^{1,1}(\Omega)$ in the norm of $W^{s,t}(\Omega)$ for every $s \geq 1$ and $0 \leq \alpha < 1 + \frac{1}{r}$ and in $C^{\beta}(\Omega)$ for every $0 \leq \beta < 1$.

Proof. First, for an arbitrary initial data $u_0 \in W^{1,r}(\Omega)$, from [4, 5], the solution is classical for positive times. In particular the solution enters $L^\infty(\Omega)$. Now, since (1.1) is subcritical in $L^1(\Omega)$ and since (7.4) holds true, Proposition 4.1 and Theorem 4.2 apply in $L^1(\Omega)$. In particular the solution is global and remains bounded in $L^1(\Omega)$ and in $W^{1,1}(\Omega)$ for every $s \geq 1$ and $0 \leq \alpha < 1 + \frac{1}{r}$ and in $C^{\beta}(\Omega)$ for every $0 \leq \beta < 1$ for any $\tau > 0$.

In particular, the solution enters and remains bounded in $H^\theta(\Omega)$ for some $\theta > 1$ and from the results in [4, 5] we obtain that the solution is also global and bounded in $W^{1,1}(\Omega)$.

Since Proposition 4.1 and Theorem 4.2 apply in $L^1(\Omega)$, denote $\mathcal{A} = W^{s}(E)$, the global attractor in $L^1(\Omega)$ which satisfies all the regularity properties in the statement.

To finish the proof, we show below that for each initial data $u_0 \in W^{1,1}(\Omega)$, the orbit of some neighborhood of $u_0$ enters in finite time a bounded set in $L^1(\Omega)$ and therefore it is attracted by $\mathcal{A} = W^{s}(E)$. In fact from (7.2), (7.3), for any $u_0 \in W^{1,1}(\Omega)$ there exist $R$ and $\tau_0 > 0$ such that for $t \in (0, \tau_0]$, and any initial data in the ball in $W^{1,1}(\Omega)$ of center $u_0$ and radius $R$,

$$t^\theta \|u(t)\|_{H^{1,2n}(\Omega)} \leq M(R, \tau_0)$$

for some $\theta > 1/2$. Therefore a neighborhood of $u_0$ enters in finite time a bounded set of $H^{1,2n}(\Omega)$ and so it enters a compact set in $W^{1,1}(\Omega)$. Note that when the nonlinear terms are subcritical then the radius $R$ above can be taken arbitrarily large and the conclusions above applies to bounded set of initial data.

But at the same time, after solutions in the neighborhood of $u_0$ above enter $H^{1,2n}(\Omega)$ this space is contained in $W^{s,t}(\Omega)$ for some $s > r$ and (1.1) becomes subcritical in this space. Repeating the argument above, now in $W^{1,1}(\Omega)$, and similarly to the bootstrap argument as in the proof of Theorem 4.2, we obtain that the solutions in the neighborhood of $u_0$ enter in finite time a bounded set of $L^1(\Omega)$ and we get the result. Finally, note that for subcritical nonlinearities the neighborhood of $u_0$ above is arbitrarily large and we get the attraction of bounded sets of $W^{1,1}(\Omega)$ towards $\mathcal{A}$.
Hence, for power-like nonlinearities, we have

**Corollary 7.2.** Assume $f$ and $g$ are power-like nonlinearities satisfying (7.1). Then if either

(i) $p+1 > 2q$

(ii) $p+1 = 2q$, $r < \frac{N(q+1)}{N+q+1}$ and

\[ dc_f > c^2(\Omega) c^2_q \]

or

(iii) $p+1 = 2q$, $\frac{N(q+1)}{N+q+1} \leq r < N$ and

\[ \frac{4dc_f}{c^2(\Omega) c^2_q} > \frac{(q-1)(N-r)+Nr}{(N-r)(Nr-1)} \]

then the conclusion of Theorem 7.1 applies.

**Proof.** In the first case, from (3.6) in Corollary 3.3, (7.4) holds for any $z > 1$, while in the second case we can take $z = q+1 > \frac{N}{N-r}$. For the third case, we have $q+1 \leq \frac{N}{N-r}$ and so (7.4) holds provided

\[ \frac{4dc_f}{c^2(\Omega) c^2_q} > j \left( \frac{Nr}{N-r} \right), \]

where, as in Corollary 4.4, $j(z) = (q+z-1)^2/(z-1)$, and we get the result. \( \Box \)

Observe that in [20] the following dissipative conditions were given for (1.1) in $H^1(\Omega)$ with subcritical nonlinear terms: for some $\varepsilon \in (0, \frac{d}{4})$

\[ \liminf_{|s| \to \infty} \frac{F(s) + \frac{|f'|}{|\Omega|} g(s) - \frac{c^2(\Omega)}{4\varepsilon} g^2(s)}{|s|} > 0, \quad (7.5) \]

where $c(\Omega)$ is as before, and for some $\varepsilon \in (0, d)$

\[ \liminf_{|s| \to \infty} \frac{f(s) + \frac{|f'|}{|\Omega|} g(s) s - \frac{c^2(\Omega)}{4\varepsilon} (g'(s) s + g(s))^2}{|s|} > 0, \quad (7.6) \]

where $F$ and $G$ represent, respectively, the primitives of $f$ and $g$. The first of these conditions ensures that the natural energy of (1.1) given in (4.2) is bounded below, while the second one ensures that the set of equilibrium is bounded. Note that the latter corresponds to (3.1) with $r = 2$ but with the
square power reduced to one in the denominator. Also, this conditions reduce to \( p + 1 > 2q \) or \( p + 1 = 2q \) and

\[
d c_j > c^2(\Omega) c_j^2 q
\]

in the case of power-like nonlinearities (indeed some extra work was needed in [20] to reduce the balance of the coefficients to that single condition).

As can be seen from Theorem 7.1 and Corollary 7.2 we now obtain similar but slightly different balance conditions that ensure the dissipativity of (1.1) in \( W^{1,1}(\Omega) \), which can be used even in the case of critical nonlinearities.

### 7.2. The Nonsingular Case \( r \geq N \)

Now we consider the case of initial data in \( W^{1,1}(\Omega) \) for \( r \geq N \). Observe that as mentioned before, for \( r > N \) no restrictions are required in the growth of \( f \) and \( g \) to obtain local solutions of (1.1). This can also be derived from the results in [4, 5] which moreover, for the case \( r = N \) allows a much faster growth than just polynomial as in (1.2). Indeed these cases should not be considered as of singular initial data since in fact initial data in these spaces are smooth and well defined up to the boundary.

However the techniques developed in previous sections will allow us to obtain some balance conditions implying dissipativity if we assume some polynomial growth on \( f \) and \( g \) as in (1.2), but with no upper bound for \( p \) or \( q \). In such a case, according to [4, 5], (1.1) is always subcritical in \( W^{1,1}(\Omega) \) for \( r \geq N \).

Since the \( \bar{\epsilon} \)-regular solutions constructed in [4, 5] behave in a similar fashion as in the previous cases that we have considered, we just outline the main arguments that we use.

Indeed, solutions are classical for positive times and so they enter in \( L^z(\Omega) \) for \( z \) such that

\[
z > \max \left\{ \frac{(p - 1)N}{2}, (q - 1)N \right\}
\]

and so (1.1) becomes subcritical in \( L^z(\Omega) \). Therefore, if (1.1) is dissipative in this latter space we conclude the existence of an attractor. More precisely, we have

**Theorem 7.3.** Assume \( f \) and \( g \) satisfy

\[
\limsup_{|s| \to \infty} \frac{|f'(s)|}{|s|^{p - 1}}, \quad \limsup_{|s| \to \infty} \frac{|g'(s)|}{|s|^{q - 1}} < \infty
\]
for \( p, q \geq 1 \) and the function
\[
H_z(s) = f(s) s + \left| \frac{f'}{f} \right| g(s) s - \frac{c^2}{\varepsilon z^2} (g'(s) s + (z-1) g(s))^2
\]
satisfies
\[
\liminf_{|s| \to \infty} \frac{H_z(s)}{|s|^2} > 0
\]
for some
\[
z > \max \left\{ \frac{(p-1)N}{2}, (q-1)N \right\}
\]
and \( 0 < \varepsilon \leq d(4(z-1)/z^2) \).

Then for any \( u_0 \in W^{1,\prime}(\Omega) \) the unique \( \varepsilon \)-regular solution of (1.1) is globally defined. Moreover, orbits of bounded sets of \( W^{1,\prime}(\Omega) \) are compact in \( W^{1,\prime}(\Omega) \) and there exists a maximal, compact, invariant and connected set \( \mathcal{A} \) in \( W^{1,\prime}(\Omega) \) which attracts bounded sets of \( W^{1,\prime}(\Omega) \) in the norm of \( W^{s,\prime}(\Omega) \) for every \( s \geq 1 \) and \( 0 < \alpha < 1 + \frac{1}{q} \) and in \( C^{\beta}(\bar{\Omega}) \) for every \( 0 < \beta < 1 \). Furthermore, \( \mathcal{A} = W^s(E) \) is the unstable set of the equilibria of (1.1) which is compact and connected in \( W^{s,\prime}(\Omega) \) for every \( s \geq 1 \) and \( 0 < \alpha < 1 + \frac{1}{q} \) and in \( C^{\beta}(\bar{\Omega}) \) for every \( 0 < \beta < 1 \).

For the case of power-like nonlinearities, (7.7) is satisfied provided either one of the following holds

(i) \( p+1 > 2q \)

(ii) \( p+1 = 2q \) and \( q < 1 + \frac{2}{p-1} \) and
\[
dc f > c^2(\Omega) c^2_\varepsilon q
\]
which is the same as (4.5), or

(iii) \( p+1 = 2q, q > 1 + \frac{2}{p-1} \) and
\[
\frac{4dc f}{c^2(\Omega) c^2_\varepsilon} > (q-1)^2 (N+1)^2 (N(q-1)-1)
\]
which is the same as (4.6).

Just note that for power-like nonlinearities the first case is clear as in previous cases. The second corresponds to the case in which we can take \( z = q+1 > (q-1)N = \frac{(p-1)N}{2} \) and the last one corresponds to taking \( z > (q-1)N > q+1 \).
8. FINAL REMARKS

Observe that the main balance condition between nonlinear terms that we have used in this paper is (3.1) in Proposition 3.2, a condition that reflects a competition among diffusion, reaction and boundary flux. As this condition can be seen as a generalization of conditions (1.6)–(1.7), we give some comparison between them. For that we will consider the case of power-like nonlinearities, that is satisfying (1.8). When both nonlinear terms cooperate, that is when $c_f$ and $c_g$ have the same sign, (1.6)–(1.7) apply only in the case of internal and boundary dissipation, that is, $c_f, c_g > 0$, provided $p$ or $q$ are strictly larger than 1. For the case of competing nonlinearities, that is, when the signs of $c_f$ and $c_g$ are different, observe that (1.6)–(1.7) apply for the case of internal dissipation, that is, $c_f > 0$, if $p > 1$ and $q \leq 1$ or $p = 1$ and $q < 1$. For the case of boundary dissipation, that is, $c_g > 0$, (1.6)–(1.7) apply provided $q \geq 1$ and $p < 1$. In both situations, the case $p = q = 1$ is covered by (1.6)–(1.7) depending on the values of $c_f$ and $c_g$. However (1.6)–(1.7) will fail to hold in the case of superlinear and nondissipative terms on either $\Omega$ or $\Gamma$, that is provided $p > 1$ and $c_f < 0$, or $q > 1$ and $c_g < 0$, and in the case of sublinear nonlinearities, that is, $p, q < 1$. Since the results in [20] imply that when $f$ is superlinear and non dissipative, regardless of $g$ there are always solutions that blow-up in finite time, condition (3.1) allows us to cover the case of internal but not boundary dissipation, obtaining the balance according to the condition $p + 1 > 2q$ or, in case of equal sign, depending on the values of $c_f$ and $c_g$. The optimality of this condition is discussed below. From (3.1) it could also be obtained the dissipativity of (1.1) in the case of sublinear nonlinearities. As we were mainly concerned with critical nonlinearities we have not pursued this line here; see for example [20].

On the other hand, observe that, since we were mainly interested in critically growing nonlinearities, we have considered with some extension the case of power-like nonlinearities. However most of the analysis above holds for nonlinearities $f$ and $g$ that grow at most critically and, for some $z > 1$ and $0 < \varepsilon < d(4(z−1)/z^2)$, the function

$$H_z(s) = f(s) s + \frac{|f|}{|\Omega|} g(s) s - \frac{c^2(\Omega)}{\varepsilon z^2} \left( g'(s) s + (z−1) g(s) \right)^2$$

with $c(\Omega) = \frac{|\Gamma|}{|\Omega|} \delta_0(\Omega)$ satisfies for some $\delta > 0$

$$\liminf_{|s| \to \infty} \frac{H_z(s)}{|s|^{z+\delta}} > 0. \quad (8.1)$$
If the above holds for $z = r$, we get from (3.4) and instead of (3.2) or (3.7),

$$\frac{1}{r} \frac{d}{dt} \|u(t)\|_{L^r(A)} + c_1 \int_A |V(|u|^{r/2})|^2 + c_2 \|u(t)\|_{L^2(A)}^{4/r} \leq c_3,$$  \hspace{1cm} (8.2)

where $c_1 = d(4(r-1)/r^2) - \varepsilon > 0$, and $c_2, c_3 > 0$ are suitable constants. This is satisfied if, for example, for $\alpha < p_C$, $\beta < q_C$ and $\alpha > 2\beta > 0$, one has

$$\lim_{|s| \to \infty} f'(s) = c_4 > 0, \quad \lim_{|s| \to \infty} g'(s) = c_5 < 0$$

and so

$$\lim_{|s| \to \infty} c_4 > 0, \quad \lim_{|s| \to \infty} c_5 < 0.$$

The case $\alpha = 2\beta > 0$ can also be considered by imposing some condition on $c_4, c_5$ similar to (3.6).

With assumption (8.1) all the results that we proved above for the case of power-like nonlinearities hold true.

Also note that in all cases in which our results apply we have, as a consequence of the existence of the attractor, that there exists some equilibrium solution of (1.1), [16]. If $f$ and $g$ have no common real zero, this equilibrium must be nontrivial, that is, nonconstant in $\mathcal{W}$.

Observe also that the restrictions on the nonlinear terms used to ensure the existence of solutions are imposed on the derivatives. On the other hand, dissipative conditions are typically imposed on the nonlinear terms themselves, see (1.6) and (7.5). However in our analysis we have found dissipative conditions in which the phase space, $f$, $g$ and $g'$ appear in terms of the function $H_z(s)$ above, see (3.1), (4.3), (5.1), (6.5), (7.4) and (7.7).

This leads to some delicate cases in which some arguments, previous to our analysis, must indeed be used in order to obtain optimal results. For example consider

$$f(s) = |s|^{\alpha - 1} s \quad \text{and} \quad g(s) = -|s|^\beta - 1 s + \sin(|s|^{-1} s)$$

with $\alpha < p_C$, $1 < \beta < \gamma < q_C$. Since $\beta > 1$ then (1.6) is not satisfied. But on the other hand, since $g'(s)$ grows like $|s|^{\alpha - 1}$, in order to use any of the dissipativity conditions in this paper, it would suffice that $\alpha + 1 > 2\gamma$.

Now we show that indeed the condition $\alpha + 1 > 2\beta$ is enough to ensure the dissipativity of (1.1). For this note that

$$g(s) = -|s|^\beta - 1 s - 1 \leq g(s) \leq g(s) = -|s|^\beta - 1 s + 1.$$
Since it was shown in [6] that there exists monotonicity of $\varepsilon$-regular solutions of (1.1) with respect to nonlinearities, we obtain that for any initial data $u_0$ we have

$$u(t, u_0, f, g_0) \leq u(t, u_0, f, g) \leq u(t, u_0, f, g_1)$$

for as long as the solutions exist. Therefore, the conditions in this paper imply that if $\alpha + 1 > 2\beta$ then $u(t, u_0, f, g_0)$ and $u(t, u_0, f, g_1)$ enter in absorbing balls on regular spaces. This implies that the solutions $u(t, u_0, f, g)$ are globally defined and enter in an absorbing ball, say of $C(\bar{\Omega})$. The arguments in previous sections show then that (1.1), with nonlinearities $f$ and $g$, has also an attractor.

Concerning the balance conditions that we have found, which are of the form (4.3) for different values of $z$ depending on the different cases we have considered, it is quite important to note that they can also be applied when there is an exact balance between $f$ and $g$, which for power-like nonlinearities corresponds to $p + 1 = 2q$. In fact all the cases that we presented as illustrations of this situation, e.g. (3.6), arise from imposing that although both nonlinear terms are of the same size, the coefficient of the leading term in the balance is positive. However the dissipative conditions we gave above permit, in the case that this coefficient is zero, to establish the balance by looking at the second or successive leading term in $f(s)$ and $g(s)$ as $|s| \to \infty$. For simplicity of the exposition we have not considered this kind of examples explicitly.

Note that, as mentioned before, for the case of power-like nonlinearities it was proven in [20] that if either

$$p + 1 < 2q \quad \text{or} \quad p + 1 = 2q \quad \text{and} \quad dc_f < c_g^2q$$

there always exists smooth enough initial data $u_0$ such that the solution of (1.1) blows-up in finite time. Observe that in the case $p + 1 = 2q$, the condition above on $c_f, c_g$ can be written as

$$\frac{4dc_f}{c^2(\Omega) c_g^2} < \frac{4q}{c^2(\Omega)}$$

while the conditions we found on the coefficients for dissipativity read

$$\frac{4dc_f}{c^2(\Omega) c_g^2} > j(z) = \frac{(q + z - 1)^2}{(z - 1)}$$

for several different values of $z$, depending on the different cases we considered before. Since $j(z)$ has a global minima at $z = q + 1$ and $j(q + 1) = 4q$, the less restrictive of such conditions, when it applies reads

$$\frac{4dc_f}{c^2(\Omega) c_g^2} > 4q.$$
Also, according to [20], in dimension $N = 1$, $c(\Omega) = 1$ so the two conditions above cover all cases but $4d_\mathcal{C}/c^2(\Omega) c_\mathcal{C}^2 = 4q$ for which the reader is referred to [12].

Therefore in higher dimensions there still exists a gap among the conditions on $c_f, c_g$ that imply dissipativity and the ones that imply blow-up. The behavior of (1.1) for such cases of the coefficients is of great interest and will be further studied elsewhere.

Although we have considered constant diffusion and space-independent nonlinear terms in (1.1), our arguments work equally well for the case of

$$
\begin{align*}
  u_t - \text{Div}(a(x) \nabla u) + f(x, u) &= 0 & \text{in } \Omega \\
  a(x) \frac{\partial u}{\partial n} + g(x, u) &= 0 & \text{on } \Gamma = \partial \Omega \\
  u(0, x) &= u_0(x),
\end{align*}
$$

where $a(x) > a_0 > 0$ is a smooth diffusion coefficient. In fact in such a case in Proposition 3.2, we obtain instead of (3.3)

$$
\begin{align*}
  \frac{1}{r} \frac{d}{dt} \|u(t)\|_{L^r(\omega)} + \frac{4(r-1)}{r^2} \int_\omega a(x) |\nabla (|u|^{r/2})|^2 \\
  + \int_\omega f(x, u) |u|^{r-2} u + \int_\omega g(x, u) |u|^{r-2} u = 0.
\end{align*}
$$

Assuming that $g$ can be extended to a function in $\tilde{\Omega} \times \mathbb{R}$, the argument runs as in the proposition with the only change that, after using Poincaré’s inequality, we obtain an upper bound of the form

$$
\begin{align*}
  c(\Omega) \left( \frac{2}{r} g_\omega(\cdot, u) u + \frac{2(r-1)}{r} g(\cdot, u) \right) |u|^{r/2-1} |\nabla |u|^{r/2}|_{L^r(\omega)} \\
  + \|g_\omega(\cdot, u)\|_{L^1(\omega)} |u|^{r-1} \right). \quad (8.3)
\end{align*}
$$

Therefore we obtain, instead of (3.4) and for $\varepsilon \leq \alpha_0 (4(r-1)/r^2)$,

$$
\begin{align*}
  \frac{1}{r} \frac{d}{dt} \|u(t)\|_{L^r(\omega)} + \int_\omega \left( a(x) \frac{4(r-1)}{r^2} - \varepsilon \right) |\nabla (|u|^{r/2})|^2 \\
  + \int_\omega H_r(x, u) |u|^{r-2} \leq 0 \quad (8.4)
\end{align*}
$$

with

$$
\begin{align*}
  H_r(x, s) &= f(x, s) s + \frac{\int_\Gamma}{|\mathcal{Q}|} g(x, s) s - \frac{c^2(\Omega)}{\varepsilon r^2} (g_\omega(x, s) s + (r-1) g(x, s))^2 \\
  &- |g_\omega(x, s)| |s|
\end{align*}
$$

and $\mathcal{Q}$ being the set of singular points of $\Omega$. The function $H_r(x, s)$ is continuous in $(x, s) \in \tilde{\Omega} \times \mathbb{R}$.

In particular, if $e_0 < 4(r-1)/r^2$, we have a sharp bound for the solutions of (8.4).

We assume that $g$ is extended to a function $\tilde{g} \in L^1(\tilde{\Omega} \times \mathbb{R})$.
for \( x \in \Omega \) and \( s \in \mathbb{R} \). Therefore (3.1) can be replaced for example with

\[
H_r(x, s) \geq C_0(x) |s|^2 - C_1(x)
\]  

(8.5)

for \( x \in \Omega \) and \( s \in \mathbb{R} \), where \( C_0, C_1 \) are nonnegative functions in \( \Omega \), such that \( C_0(x) \geq \delta > 0 \) and \( C_1 \) is bounded for \( 1 < r < 2 \) or in \( L^{r/2}(\Omega) \) for \( r \geq 2 \).

In fact, if in the step before (8.4) one uses Young inequality inside the integral of the first term we get, instead of (8.4) and for \( 0 < \sigma(x) \leq a(x) 4(r−1)/r^2 \) for each \( x \in \Omega \),

\[
\frac{1}{r^2} \frac{d}{dt} \|u(t)\|_{L_r(\Omega)} + \int_{\Omega} \left( a(x) \frac{4(r−1)}{r^2} - \sigma(x) \right) \|u(t)\|_{L_r(\Omega)}^r + \int_{\Omega} H_r(x, u) |u|^{r−2} \leq 0,
\]

(8.6)

where now

\[
H_r(x, s) = f(x, s) s + \frac{|f'|}{|\Omega|} g(x, s) s - \frac{c^2(\Omega)}{\sigma(x) r^2} (g_s(x, s) s + (r−1) g(x, s))^2
\]

\[-|g_s(x, s)| |s|
\]

for \( x \in \Omega \) and \( s \in \mathbb{R} \). This function must then satisfy (8.5) for dissipativeness.

Note that these conditions do not change the balance in the case of power-like nonlinearities, for which \( g(x, s) \sim c_j(x) |s|^{p−1} s \) for large \( |s| \) and for a smooth function \( c_j(x) \). In this case the extra term in \( H_r(x, s) \), \( |g_s(x, s)| |s| \), behaves as \( |c_j c_p(x)| |s|^p+1 \) which is not of the highest order. In fact if \( f(x, s) \sim c_j(x) |s|^{p−1} s \) for large \( |s| \), then (8.5) is satisfied whenever \( p+1 > 2q \) or \( p+1 = 2q \) and

\[
4a(x) c_j(x) \geq c^2(\Omega) c_j^2(x) \frac{(q+r−1)^2}{r−1} + \delta_0
\]

for all \( x \in \Omega \) and some \( \delta_0 > 0 \).

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**REFERENCES**


