Localization on the Boundary of Blow-up for Reaction–Diffusion Equations with Nonlinear Boundary Conditions

José M. Arrieta* and Aníbal Rodríguez-Bernal
Departamento de Matemática Aplicada, Universidad Complutense de Madrid, Madrid, Spain

ABSTRACT

In this work we analyze the existence of solutions that blow-up in finite time for a reaction–diffusion equation \( u_t - \Delta u = f(x, u) \) in a smooth domain \( \Omega \) with nonlinear boundary conditions \( \partial u/\partial n = g(x, u) \). We show that, if locally around some point of the boundary, we have \( f(x, u) = -\beta u^p, \beta \geq 0 \), and \( g(x, u) = u^\theta \) then, blow-up in finite time occurs if \( 2q > p + 1 \) or if \( 2q = p + 1 \) and \( \beta < q \). Moreover, if we denote by \( T_b \) the blow-up time, we show that a proper continuation of the blowing up solutions are pinned to the value infinity for some time interval \( [T_b, \tau) \) with \( T_b \leq \tau < \infty \). On the other hand, for the case \( f(x, u) = -\beta u^p \), for all \( x \) and \( u \), with \( \beta > 0 \) and \( p > 1 \), we show that blow-up occurs only on the boundary.

Key Words: Reaction–diffusion; Blow-up; Nonlinear boundary conditions.

Mathematics Subject Classification: Primary 35K57, 35J60; Secondary 35J70.

*Correspondence: José M. Arrieta, Departamento de Matemática Aplicada, Universidad Complutense de Madrid, 28040 Madrid, Spain; E-mail: jose.arrieta@mat.ucm.es.
1. INTRODUCTION

In this article we consider the following reaction diffusion equation under nonlinear boundary conditions in a smooth domain $\Omega \subset \mathbb{R}^N$,

$$
\begin{cases}
  u_t - \Delta u = f(x,u) & \text{in } \Omega \\
  u = 0 & \text{on } \Gamma_D \\
  \frac{\partial u}{\partial n} = g(x,u) & \text{on } \Gamma_N \\
  u(0, x) = u_0(x) \geq 0 & \text{in } \Omega
\end{cases}
$$

where $\Gamma = \partial \Omega = \Gamma_D \cup \Gamma_N$ is a regular disjoint partition of the boundary of $\Omega$ and $f$ and $g$ are suitably smooth functions of $(x, u)$. The subindices $D$ and $N$ on $\Gamma$ indicate the part of the boundary with Dirichlet and Neumann type condition, respectively.

For this problem we are interested in giving conditions on the nonlinearities $f$ and $g$ for which some solutions of (1.1) blow-up due to the nonlinear flux on the boundary.

The conditions on the nonlinearities are of local nature, holding at some points close to the boundary of the domain. If the local conditions hold in fact globally near the whole boundary, we show the existence of a class of smooth initial data, with support close to the boundary, such that the corresponding solutions of (1.1) blow up in finite time.

On the other hand we show that in the case of a strongly dissipative reaction term $f(x, u)$, blow-up is localized only on the boundary, while the solution remains bounded in compact sets of $\Omega$ for all times.

Note that if $f(x, u) = \beta u^p$ with $p > 1$, $\beta > 0$, then for any flux term $g$, reaction driven blow-up is easily obtained. In fact the solutions of the Dirichlet problem

$$
\begin{cases}
  u_t - \Delta u = f(x,u) & \text{in } \Omega \\
  u = 0 & \text{on } \partial \Omega \\
  u(0, x) = u_0(x) \geq 0 & \text{in } \Omega
\end{cases}
$$

become subsolutions of problem (1.1). It is well known that we can choose appropriate initial conditions for which solutions of (1.2) become unbounded in finite time (Baras and Cohen, 1987; Friedman and McLeod, 1985; Galaktionov and Vazquez, 2002).

On the other hand if $f(x, u) = -\beta u^p$, $\beta \geq 0$, and $g(x, u) = -u^q$, solutions are globally defined and no phenomenon of blow-up in finite time occur (Brezis, 1971; Evans, 1977).

It is clear that the most interesting situation to prove a blow-up result for problem (1.1) is in the case of competing nonlinearities, with a dissipative absorption term and a nondissipative boundary flux term. That is, for nonlinearities $f$ and $g$ satisfying for sufficiently large $u \geq 0$, respectively,

$$
f(x, u) \leq 0 \quad \text{for all } x \in \Omega \quad \text{and} \quad g(x, u) \geq 0, \quad \text{for all } x \in \Gamma.
$$

(1.3)
Since we are interested in considering only nonnegative solution of (1.1), we will also assume that

\[ f(x,0) \geq 0, \quad \text{for all } x \in \Omega \quad \text{and} \quad g(x,0) \geq 0 \quad \text{for all } x \in \Gamma. \]  

(1.4)

Moreover we are interested in considering first the case in which (1.3) and (1.4) holds locally for \( x \) close to some point of the boundary of \( \Omega \).

An important particular case satisfying (1.3) and (1.4) reads, \( f(x,u) = -\beta u^p \), with \( \beta \geq 0 \), and \( g(x,u) = u^q \) which leads to the prototype problem

\[
\begin{aligned}
&u_t - \Delta u = -\beta u^p & \text{in } \Omega \\
&u = 0 & \text{on } \Gamma_D \\
&\frac{\partial u}{\partial n} = u^q & \text{on } \Gamma_N \\
u(0,x) = u_0(x) & \text{in } \Omega.
\end{aligned}
\]

(1.5)

Note that the nonlinearity in \( \Omega \), \( f(x,u) = -\beta u^p \), is dissipative while the nonlinearity on the boundary \( g(x,u) = u^q \) introduces energy into the system. We have two competing mechanisms and it turns out that the balance between \( f \) and \( g \), reflected in the relative sizes of \( p, q \) and \( \beta \), plays an essential role in the blow-up analysis of problem (1.5).

For (1.5) it has been proved in Rodríguez-Bernal and Tajdine (2001) that if \( 2q < p + 1 \) or if \( 2q = p + 1 \) and \( \beta \) is large enough, that is \( \beta > \beta_0(q,\Omega) \), then all solutions of (1.5) are globally defined in time and moreover they are globally bounded, see also Rodríguez-Bernal (2002). It was also proved there that, in the case \( \Gamma_D = \emptyset \), if \( 2q > p + 1 \) or if \( 2q = p + 1 \) and \( \beta < q \), then there exist smooth initial data such that the corresponding solution becomes unbounded in finite time. This is proved by constructing suitable subsolutions that do so. Finally observe that it was recently shown in Andreu et al. (2002) that in fact for (1.5), with \( \Gamma_D = \emptyset \), one has \( \beta_0(q,\Omega) = q \), as already known for the one dimensional case (Chipot et al., 1991); in the latter reference the critical one dimensional case for \( 2q = p + 1 \) and \( \beta = q \) was also studied.

Although the results in Rodríguez-Bernal and Tajdine (2001) apply for general nonlinearities \( f \) and \( g \), that asymptotically (for large \( u \)) behave like \( -\beta u^p \) and \( u^q \), respectively, this construction, which is essentially the same as in Andreu et al. (2002), has several drawbacks. First, the subsolutions are defined in the whole \( \Omega \) and they depend on local but also on global properties of the boundary of the domain. More precisely, they blow up at certain, suitable chosen, points of the boundary. Even more it is assumed in Rodríguez-Bernal and Tajdine (2001) that all the boundary is subjected to the nonlinear flux condition; that is, it is assumed that \( \Gamma_D = \emptyset \). In particular, the results in Andreu et al. (2002) and Rodríguez-Bernal and Tajdine (2001) do not apply for example to the case the domain is an annulus with Dirichlet boundary condition in the “outer” boundary and nonlinear Neumann one in the “inner” one. Moreover, the subsolutions have large initial data, therefore blow-up is only obtained for solutions with initial data which are large everywhere in \( \Omega \).

All these translates in some unnatural restrictions on the family of initial data that blow up in finite time and on the geometric configurations of the boundary to which the results apply.
In this article we remove all these restrictions by showing the local nature of the blow-up mechanism. Hence, we will show that if the boundary nonlinear term is dominant, locally at some part of the boundary, there will exist large classes of initial data, only determined by their values close to that part of the boundary, such that the corresponding solutions become singular in finite time. Moreover, if the local conditions hold in fact close to the whole boundary, we show the existence of a class of smooth initial data, with support close to the boundary, such that the corresponding solutions of (1.1) blow up in finite time. These results are obtained assuming only some local regularity but without any geometric requirement on the boundary.

Furthermore, we will also show that if the dissipative nonlinear term in \( \Omega, f(x,u), \) is actually present, e.g., \( f(x,u) = -\beta u^p \) with \( p > 1 \) and \( \beta > 0 \), then blow-up only occurs close to the boundary, by proving that solutions remain bounded on any compact subset of \( \Omega \), see Theorem 1.4.

These results will be again achieved by constructing suitable local subsolutions or supersolutions respectively, only assuming some local regularity of the boundary.

Note that, assuming the regularity of \( f(x,u) \) and \( g(x,u) \), we consider smooth initial data for (1.1). Hence, there exists a unique classical solution which exists as long as it remains bounded and ceases to exist when it becomes unbounded. However, it turns out that it is reasonable to consider that the solution can be continued after this time. For example, one may think, that the classical solutions ceases to exist but remains alive in a weaker sense, e.g., as a curve in some \( L^r(\Omega) \) space. This idea of continuing the solution after blow-up has been proved to be very useful for problems with Dirichlet boundary conditions, (1.2), see Baras and Cohen (1987) and Galaktionov and Vazquez (1997, 2002). In fact, for this problem, this approach has been used to discuss and analyze a great deal of possible behaviors of blowing up solutions, namely, complete or incomplete blow-up or the existence of peaking solutions (Galaktionov and Vazquez, 1997, 2002).

With this approach one constructs, for a given initial condition, the so called proper minimal solution for (1.1), which exists for all times and coincides with the classical solution as long as they remain bounded. Also, minimal proper solutions have suitable comparison and monotonicity properties. This concept becomes the most natural and flexible one to consider solutions that are continued after becoming unbounded, see Sec. 2.

Our main results are stated below in Theorem 1.1 and Theorem 1.4.

The first result shows that when the boundary term is dominant blow-up actually occurs. Moreover, it is shown that the proper minimal solution is pinned to the value infinity for some time interval after blow up, that is, it takes the value infinity in a part of the boundary for some time interval \([T, \tau] \) with \( T_b \leq T < \tau \), where \( T_b \) denotes the (classical) blow-up time, that is, the time for which the solution is classical for \( 0 < t < T_b \) and \( \lim \sup_{t \to T_b} \|u(t,\cdot)\|_{L^\infty(\Omega)} = \infty \).

**Theorem 1.1.** Let \( x_0 \in \Gamma_N \subset \partial \Omega \) be a point where the boundary is locally \( C^2 \) around \( x_0 \). More precisely, assume that there exist two constants \( \rho_0, L > 0 \), such that \( \Gamma_N \cap B(x_0, \rho_0) = \partial \Omega \cap B(x_0, \rho_0) \) can be written as a graph of a \( C^2 \) function with \( C^2 \)-norm bounded by a constant \( L \) and that for all \( x \in \Gamma_N \cap B(x_0, \rho_0) \) the segment \( \{x - \tilde{n}(x) : 0 < s < \rho_0\} \subset \Omega \), where \( \tilde{n}(x) \) is the unit outward normal to the boundary at \( x \).
Assume moreover that for all \( x \in \Omega \cap B(x_0, \rho_0) \) we have
\[
f(x, 0) \geq 0, \quad \text{and} \quad -\beta u^p \leq f(x, u) \leq 0, \quad \text{for sufficiently large } u > 0
\]
while for all \( x \in \partial\Omega \cap B(x_0, \rho_0) \) we have
\[
g(x, u) \geq 0 \quad \text{for all } u \geq 0 \quad \text{and} \quad g(x, u) \geq u^\beta \quad \text{for sufficiently large } u \geq 0
\]
with
\[
2q > p + 1 \quad \text{or} \quad 2q = p + 1 \quad \text{and} \quad \beta < q.
\]

Then there exist a positive number \( 0 < \rho < \rho_0 \), positive times \( 0 < T < \tau \), a positive smooth function \( v(t, x) \) defined for \( (t, x) \in [0, T) \times (\Omega \cap B(x_0, \rho)) \) and an initial data \( \phi_0 \in C^\infty_0(B(x_0, \rho_0)) \) such that

(i) \( \rho, T, \tau \) and \( \phi_0 \) depend only on \( L \) and \( \rho_0 \), and not on the local shape of the boundary.

(ii) The function \( v(t, x) \) is increasing in \( t \) and, as \( t \to T \),

(a) If \( p > 1 \) and \( \beta > 0 \)
\[
v(t, x) \leq v(T, x) = \frac{C}{\text{dist}(x, \partial\Omega)^{2/(p-1)}}, \quad \text{for } x \in \Omega \cap B(x_0, \rho).
\]

(b) If \( p = 1 \) or if \( \beta = 0 \), for any \( n > 0 \) we can choose the function \( v \) such that
\[
v(t, x) \leq v(T, x) = \frac{C}{\text{dist}(x, \partial\Omega)^n}, \quad \text{for } x \in \Omega \cap B(x_0, \rho).
\]

(iii) If \( u(t, x, u_0) \) is the proper minimal solution of (1.1) with initial data \( u_0 = \phi_{0|\Omega} \), then
\[
v(t, x) \leq u(t, x, u_0), \quad \text{for } (t, x) \in [0, T) \times (\Omega \cap B(x_0, \rho))
\]
and
\[
v(T, x) \leq u(t, x, u_0), \quad \text{for any } T \leq t \leq \tau \quad \text{and} \quad x \in (\Omega \cap B(x_0, \rho)).
\]

Finally, if \( \Gamma_N \) is uniformly \( C^2 \) and (1.6), (1.7) and (1.8) hold for all \( x \) in a neighborhood of the entire boundary \( \Gamma_N \) then we can choose the initial condition \( \phi_0 \in C^\infty(\Omega) \) with support concentrated around \( \Gamma_N \) and the function \( v(t, x) \)
satisfying respectively, (1.9) or (1.10), and (1.11), (1.12), with \( u_0 = \phi_0|_{\Omega} \), for all \( x \) in a neighborhood of the entire boundary \( \Gamma_N \).

**Remark 1.2.** Note that in all of the cases above we have \( v(t,x) \rightarrow \infty \) as \( t \rightarrow T^- \) uniformly for \( x \in \partial \Omega \) in a neighborhood of \( x_0 \). In particular if \( p \geq 3 \) we have that \( \|v(t,\cdot)\|_{L^{p-1/2}(\Omega)} \rightarrow \infty \) as \( t \rightarrow T^- \) and if \( p < 3 \) or if \( \beta = 0 \), we have that \( \|v(t,\cdot)\|_{L^1(\Omega)} \rightarrow \infty \) as \( t \rightarrow T^- \). In any case \( \|v(t,\cdot)\|_{L^1(\Gamma)} \rightarrow \infty \) as \( t \rightarrow T^- \).

The theorem then shows that the proper minimal solution \( u(t,x,u_0) \) ceases to exists in \( L^{p-1/2}(\Omega) \)—sense if \( p \geq 3 \) and in \( L^1(\Omega) \)—sense if \( p \leq 3 \) after time \( T \). In any case the proper minimal solution \( u(t,x,u_0) \) ceases to exists in \( L^1(\Gamma) \)—sense after time \( T \).

Note that the time \( T \) in the theorem is estimated using the blow-up properties of an appropriate subsolution. In particular, the classical solution starting at \( u_0 \) blows-up at certain time \( T_b \), with \( T_b \leq T \). The theorem above does not provide information about the behavior of the solution at time \( T_b \).

Finally, observe that the pinning property of the proper minimal solution that we mentioned right before the statement of this theorem, is obtained from (1.12). Observe that since \( v(T,x) = \infty \) for \( x \in \partial \Omega \cap B(x_0,\rho) \) then, necessarily, \( u(t,x,u_0) = 0 \) for \( T \leq t \leq \tau \) and \( x \in \partial \Omega \cap B(x_0,\rho) \).

Whether or not \( T = T_b \), or \( \tau = \infty \) remain as interesting open questions.

**Remark 1.3.** There are several arguments that indicate that the relation \( 2q = p + 1 \) is a critical relation between the growth rates of the nonlinearities. We provide two different arguments.

(i) In many instances, critical powers may be detected through invariance of equations under scalings. Hence, consider for instance problem (1.1) with \( f(x,u) = -\beta u^p, \ g(x,u) = u^q, \ \Gamma_D = \emptyset \) and \( \Omega \) is a half space, that is \( \Omega = \{x \in \mathbb{R}^N, \ x_N > 0\} \). If we scale the equation by the change of variables \( v(t,x) = \lambda^{2/\alpha} u(\lambda^2 t, \lambda x), \lambda > 0 \), we obtain for \( v \) the equation

\[
v_t - \Delta v = \lambda^{2-\alpha(p-1)} v^p
\]

with boundary condition

\[
\frac{\partial v}{\partial n} = \lambda^{1-\alpha(q-1)} v^q
\]

The equation will be invariant under this transformation, as long as

\[
2 - \alpha(p - 1) = 1 - \alpha(q - 1) = 0.
\]

This is obtained for \( \alpha = 2/(p - 1) \) and \( p + 1 = 2q \).

(ii) We may also obtain the same relation through energy balances. If we denote by \( F, G \) primitives of \( f \) and \( g \), respectively, the energy associated to the
system is given by

\[ E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \int_\Omega F(u) - \int_\Gamma G(u) \]

Following the analysis done in Rodríguez-Bernal and Tajdine (2001), we can rewrite the energy as

\[ E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \int_\Omega F(u) + \frac{1}{|\Omega|} \int_\Omega \left[ G(u) - \frac{1}{|\Gamma|} \int_\Gamma G(u) \right] - \frac{1}{|\Omega|} \int_\Omega G(u) \]

But by a Poincaré type inequality

\[ \left| \int_\Omega \left[ G(u) - \frac{1}{|\Gamma|} \int_\Gamma G(u) \right] \right| \leq C \int_\Omega |\nabla G(u)| = C \int_\Omega |g(u)||\nabla u| \]

\[ \leq \varepsilon \int_\Omega |\nabla u|^2 + \frac{C}{\varepsilon} \int_\Omega |g(u)|^2 \]

which implies that

\[ E(u) \geq \left( \frac{1}{2} - \varepsilon \right) \int_\Omega |\nabla u|^2 - \int_\Omega F(u) - \frac{C}{\varepsilon} \int_\Omega |g(u)|^2 - \frac{|\Gamma|}{|\Omega|} \int_\Omega G(u) \]

Analyzing the terms in this last expression, we see that \( F(u) \sim -u^{p+1} \), \( |g(u)|^2 \sim u^{2q} \), \( G(u) \sim u^{p+1} \). Obviously, if \( q > 1 \) the dominant terms are \( F(u) \) and \( |g(u)|^2 \). We have

\[ -\int_\Omega \left( F(u) - \frac{C}{\varepsilon} |g(u)|^2 \right) \sim \int_\Omega (u^{p+1} - u^{2q}). \]

From the sign of the energy, we expect global existence if \( p + 1 > 2q \) and blow-up if \( p + 1 < 2q \). The two terms are balanced when \( p + 1 = 2q \).

The next result states that for the case of a strongly dissipative reaction term \( f(x,u) \), blow-up only occurs on the boundary.

**Theorem 1.4.** Assume

\[ f(x,u) \leq -\beta u^p, \quad \text{for all } x \in \Omega, \text{ and } u \geq 0 \text{ large enough} \]

for some \( p > 1, \beta > 0 \) and let \( u_0 \geq 0 \) a smooth initial condition.
Then given any $\Omega'$, a compact subset of $\Omega$, there exists $M = M(\Omega', \|u_0\|_{L^\infty(\Omega)})$ such that the proper minimal solution of (1.1) starting at $u_0$ satisfies

$$0 \leq u(t, x) \leq M, \quad \text{for all } (t, x) \in [0, \infty) \times \Omega'.$$

Moreover, there exists $C = C(\|u_0\|_{L^\infty(\Omega)})$ such that

$$0 \leq u(t, x) \leq \frac{C}{\text{dist}(x, \Gamma)^{2\alpha}}, \quad \text{for all } (t, x) \in [0, \infty) \times \Omega.$$

Observe that the results above call for several consequences. First of all, in the situation of Theorem 1.4, then no complete blow-up occurs; see Galaktionov and Vazquez (1997, 2002) for the Dirichlet problem. This of course, raises the question of whether there is complete blow up without the strong dissipative term, e.g., with $f(x, u) = -\beta u^p$ with $p = 1$ or $\beta = 0$. In dimension $N = 1$ a positive answer is given in Quiros et al. (2002).

Also, in Theorem 1.1, solutions are shown to be pinned for some time to the value infinity. The time $\tau$ can be taken infinity either if the solution $u(t, x)$ is monotonic in time or, more generally, it does not become very small close to the boundary; see the remark after the proof of Proposition 3.3 below. An interesting question is to elucidate if pinned solutions could in general become finite on the boundary after some time. Should this happen this would give a boundary analog of the “peaking” solutions described in Galaktionov and Vazquez (1997, 2002) for the Cauchy problem in $\mathbb{R}^N$. Another important open question is whether or not the solution remains pinned to infinity for some time interval right after $T_b$, that is, for an interval $[T_b, \tau]$.

Finally, observe that the upper and lower bounds on the spatial profile close to the boundary of the solution, obtained in Theorems 1.1 and 1.4 are the same and therefore they are sharp on the time interval $[T, \tau]$ of Theorem 1.1.

To have a flavor of the proof of Theorem 1.1 let us analyze a simple case. For this we will follow essentially the construction in Rodríguez-Bernal and Tajdine (2001), see also Andreu et al. (2002). In what follows we assume $\Gamma_D = \emptyset$ and that the boundary in a small neighborhood of $x_0 = 0$ is flat, for instance $\partial \Omega \cap B(0, \varepsilon) = \{x : x_N = 0, |x| < \varepsilon\}$, and $\Omega \subset \{x : x_N > 0\}$, that is, $\Omega$ lies completely on one side of the tangent hyperplane to the boundary at $x_0$. Also assume the diameter of $\Omega$ is small enough, for instance $\Omega \subset B(0, 2\varepsilon)$.

Assume also that we are in the situation $f(x, u) = -\beta u^p$, $g(x, u) = u^q$, with $2q = p + 1$, $q > 1$ and $\beta < q$. Then consider the solution of the ordinary differential equation $\psi' = \psi^q$, with initial condition $\psi(0) = a$, which is given explicitly by $\psi(t) = K/(T - t)^{1/(q - 1)}$ where $T = T_0 = 1/(q - 1) a^{q - 1}$ and $K$ is an appropriate constant independent of $a$. Hence, we can construct the function $v(t, x) = \psi(t - x_N)$, $t \in [0, T)$, $x \in \Omega$. Notice that $v(t, x) \geq \psi(-2\varepsilon) = K/(T + 2\varepsilon)^{1/(q - 1)}$ which can be chosen large if $\varepsilon$ is small and $\varepsilon$ is large enough. In particular let us assume that $\varepsilon$ is so small and $\varepsilon$ is chosen large enough so that we have $v^q \leq (q - \beta) v^p$. This can be achieved since $\beta < q < p$. 


By direct calculation we may show that for all \((t, x) \in [0, T] \times \Omega\)
\[ v_t - \Delta v = v^q - qv^p \leq -\beta v^p. \]

Moreover on the whole boundary we trivially have \(\partial_n v = -v^q \nabla x_N \cdot \vec{n}(x) \leq v^q\). This shows that \(v\) is a subsolution of (1.5). Notice that \(v\) satisfies (1.10) above since for this case \(2/(p-1) = 1/(q-1)\).

For more general C\(^2\) domains, and without constraint on its geometry and/or its size, we will prove Theorem 1.1 by constructing preliminary local subsolutions around a point \(x_0 \in \Gamma_N\). This subsolutions will be radially symmetric around a point outside \(\Omega\) which lies in the normal direction to \(\Omega\) at \(x_0\). Moreover this functions will be very much independent of the local geometry of \(\Omega\) around \(x_0\). This construction is performed in Sec. 3.

Later in Sec. 4 we will provide a proof of Theorem 1.1 by considering the family of all subsolutions constructed in Sec. 3 for all points of the boundary in a small neighborhood of \(x_0\). This section also contains the proof of Theorem 1.4.

### 2. PROPER MINIMAL SOLUTIONS

As mentioned in the introduction we will work in this article with proper minimal solutions of problem (1.1) under the assumptions (1.3) and (1.4). This concept was introduced in Baras and Cohen (1987) and it has been further developed in Galaktionov and Vazquez (1997, 2002) for the case of Dirichlet boundary conditions.

Therefore in this section, we collect some of their basic properties. The main idea is the following. We consider a sequence of nonlinearities \(g_n(x, u)\) which, for fixed \(n\), \(g_n(x, \cdot)\) is a globally Lipschitz function in \(u\) uniformly in \(x \in \Gamma\), \(g_n\) is monotonically increasing as \(n \to \infty\) and \(g_n \to g\) uniformly on compact subsets of \(\Gamma \times \mathbb{R}\). Moreover we assume \(g_n(x, 0) \geq 0\) for all \(x \in \Gamma\). For a given smooth initial condition \(u_0 \geq 0\) we have a classical solution \(u_n(t, x, u_0) \geq 0\) of problem (1.1) for \(g_n\), which is defined for all \(t \geq 0\), since the interior term is dissipative. Moreover by monotonicity arguments, the sequence \(u_n\) is monotone increasing in \(n\) and converges to a function \(u(t, x, u_0)\) that may take the value infinity for certain values of \(t\) and \(x\). It can be easily proved that this function is independent of the sequence \(g_n\) and if the classical solution of (1.1) exists for \(t \in [0, \tau)\) (i.e., as long as the solution remains in \(L^\infty(\Omega)\)), then it coincides with the proper minimal solution in \([0, \tau)\). This can be easily seen by choosing the functions \(g_n\) such that \(g_n = g\) if \(|u| \leq n\) and \(x \in \Gamma\).

In particular the proper minimal solution is an extension of the classical solution beyond the time of explosion.

Since the approximants \(u_n(t, x, u_0)\) are classical solutions and they are defined for all \(t \geq 0\) we may apply comparison principles for this approximants that will in turn translate into comparison principles for their limits, the proper minimal solutions. In particular if \(0 \leq u_0 \leq v_0\) then \(u(t, x, u_0) \leq u(t, x, v_0)\) for all \(t \geq 0\), understanding that if \(u(t, x, u_0) = \infty\) at certain value of \((t, x)\) then we must have \(u(t, x, v_0) = \infty\) also. Similarly if \(g \leq \tilde{g}\) then the proper minimal solutions for \(g\) are below the proper minimal solutions for \(\tilde{g}\).
Also we have comparison with respect to the domain and the boundary conditions, in the sense that if, for instance, we consider a subset \( \Omega' \subset \Omega \) with a common boundary \( \Gamma_N' \subset \Gamma_N \) and if we consider the problem

\[
\begin{aligned}
&v_t - \Delta u = f(x, v) \quad \text{in } \Omega' \\
v = 0 \quad &\text{on } \Gamma'_D = \partial \Omega' \setminus \Gamma'_N \\
\frac{\partial v}{\partial n} = g(x, v) \quad &\text{on } \Gamma'_N \\
v(0, x) = v_0(x) \geq 0 \quad &\text{in } \Omega'
\end{aligned}
\]  

(2.1)

then their proper minimal solutions become subsolutions of (1.1). As a matter of fact we will use below that if \( z(t, x) \) is a smooth function for \( t \in [0, T) \) and \( x \in \Omega' \) and which is a subsolution of (2.1) then the proper minimal solution of (1.1) with an initial data \( u_0 \geq v_0 \), satisfies \( u(t, x, u_0) \geq z(t, x) \). Note that the last sentence can be understood either by saying that \( z(t, x) \) is extended by zero to the rest of \( \Omega \) or that the solution in the whole domain is restricted to \( \Omega' \).

In fact we can show that for large \( n, z(t, x) \leq u_n(t, x, u_0) \) for \( 0 \leq t < T \) and \( x \in \Omega \). Notice that the only difficulty to show this comes from the boundary conditions. But if we let \( T' < T \), then \( |z| \leq M \) on \( \Omega \times [0, T'] \). Then for large enough \( n \), the boundary conditions are also ordered. For this we may choose the functions \( g_n \) such that \( g_n = g \) if \( |u| \leq n \) and \( x \in \Gamma \). Hence, we have that \( z \leq u_n \) and we conclude.

For a general theory on proper minimal solutions see Galaktionov and Vazquez (1997, 2002) and references therein.

# 3. LOCAL SUB SOLUTIONS

In this section we construct local subsolutions of problem (1.1) that become unbounded in finite time. The construction is performed locally around a point \( x_0 \in \partial \Omega \) and it will be very much independent of the local geometry of the boundary. These subsolutions can be constructed if certain relations between the growth rates of \( f \) and \( g \) are satisfied, which in particular imply that the boundary term is dominant. For (1.5) this translates into relations between \( p, q \) and \( \beta \) which read \( 2q > p + 1 \) or \( 2q = p + 1 \) and \( \beta < q \).

We start by defining the following general class of domains.

**Definition 3.1.**

(i) Let \( R, \delta, L \) be positive numbers with \( \delta \) small. We will say that a domain \( D \) belongs to the class \( (R, \delta, L) \) with vertex at \( 0 \in \mathbb{R}^N \), if there exists a \( C^2 \) function \( \gamma : \mathbb{R}^{N-1} \to \mathbb{R} \) with \( \gamma(0) = 0, D\gamma(0) = 0, \|\gamma\|_{C^2(B_{R^{N-1}}(0, R+\delta))} \leq L \) such that

\[
D = \{(x', x_N) : \gamma(x') < x_N, x' \in \mathbb{R}^{N-1}, x_N \in \mathbb{R} \} \cap B_{y_R}(y_R, R + \delta)
\]

where \( y_R = (0, -R) \) and \( \partial B_{y_R}(y_R) \cap D = \{0\} \).

We will denote by \( \Gamma_0 \) the points of \( \partial D \cap \partial B_{y_R}(y_R, R + \delta) \) and by \( \Gamma_1 = \partial D \setminus \Gamma_0 \). Note that \( 0 \in \Gamma_1 \). See Fig. 1.
(ii) If \( D \) is a domain of class \( (R, \delta, L) \) with vertex at \( 0 \in \mathbb{R}^N \), then for any \( 0 < \delta' \leq \delta \) we define the domain \( D_{\delta'} = D \cap B(y_R, R + \delta') \) which is a domain of class \( (R, \delta', L') \) for some \( L' \leq L \). We also denote in this case \( \Gamma_{\delta'}^0 = \partial D_{\delta'} \cap \partial B(y_R, R + \delta') \), \( \Gamma_{\delta'}^1 = \Gamma_1 \cap D_{\delta'} \).

(iii) If \( \Omega \) is a domain with a \( C^2 \) boundary around a point, \( x_0 \in \partial \Omega \), and denoting \( y_R = x_0 + R \vec{n}(x_0) \), where \( \vec{n}(x_0) \) is the outward normal vector at \( x_0 \), then for \( R \) and \( \delta \) small the domain \( D = \Omega \cap B(y_R, R + \delta) \) belongs, up to a rigid motion that maps \( x_0 \) to 0, to the class \( (R, \delta, L) \) with vertex at \( 0 \in \mathbb{R}^N \), for some \( L = L(R, \delta) \).

We say then that \( D = \Omega \cap B(y_R, R + \delta) \) belongs to the class \( (R, \delta, L) \) with vertex at \( x_0 \).

**Remark 3.2.** Notice that if \( D \) belongs to the class \( (R, \delta, L) \) with vertex at \( 0 \in \mathbb{R}^N \) then for any \( R' \leq R \) and \( \delta' \leq \delta \), the domain \( D \cap B(y_{R'}, R' + \delta') \) belongs to the class \( (R', \delta', L') \) with vertex at \( 0 \in \mathbb{R}^N \), for some \( L' \leq L \).

We pose the following auxiliary problem

\[
\begin{cases}
  u_t - \Delta u = f(x, u) & \text{in } D \\
  u = 0 & \text{on } \Gamma_0 \\
  \frac{\partial u}{\partial n} = g(x, u) & \text{on } \Gamma_1 \\
  u(0, x) = u_0(x) & \text{in } D,
\end{cases}
\]

where \( D \) belongs to the class \( (R, \delta, L) \) with vertex at \( 0 \in \mathbb{R}^N \).
In the following proposition we will show, under certain conditions for \( f \) and \( g \), the existence of initial data \( u_0 \geq 0 \) for which an appropriate subsolution of (3.1) that becomes unbounded at certain time can be constructed. Moreover the initial condition and the time of blow-up can be obtained uniformly on domains of the class \((R, \delta, L)\) with vertex at \( 0 \in \mathbb{R}^N \). Also note that this construction can be carried over, by means of a rigid motion to any domain \( D \) which belongs to the class \((R, \delta, L)\) with vertex at a given point \( x_0 \in \mathbb{R}^N \).

Observe that below we implicitly assume that \( f(x, u) \) (respectively \( g(x, u) \)) is somehow defined for \( u \geq 0 \) and for all \( x \) in every domain \( D \) of the class \((R, d, L)\) with vertex at \( 0 \in \mathbb{R}^N \) (respectively, on the part \( \Gamma_1 \) of its boundary).

Thus, we have

**Proposition 3.3.** Let \( R, \delta, L \) be positive numbers with \( \delta \) small enough.

Assume that for sufficiently large \( u \geq 0 \) and for any domain \( D \) of class \((R, \delta, L)\) with vertex at \( 0 \in \mathbb{R}^N \) we have

\[
0 \geq f(x, u) \geq -\beta u^p, \quad g(x, u) \geq u^q
\]

with \( 2q > p + 1 \) or \( 2q = p + 1 \) and \( \beta < q \). Moreover assume \( f(x, 0) \geq 0 \) and \( g(x, u) \geq 0 \) for all \( u \geq 0 \) and \( x \in D \).

Then there exist \( 0 < \delta' < \delta \), times \( 0 < T < \tau \), an initial data \( \phi_0 \in C_0^\infty(B(y_R, R + \delta)) \) and a positive smooth function \( z(t, x) \), defined for \( t \in [0, T) \) and \( x \in \mathbb{R}^N \setminus B(y_R, R) \), such that for any domain \( D \) of class \((R, \delta, L)\) with vertex at \( 0 \in \mathbb{R}^N \) the proper minimal solution of (3.1) with initial condition \( u_0 = \phi_0|_D \) satisfies

\[
\int_{\mathbb{R}^N} u(t, x) \geq z(t, x) \quad \text{for } (t, x) \in [0, T) \times D_{\delta'}
\]

and

\[
\int_{\mathbb{R}^N} u(t, x) \geq z(T, x) \quad \text{for } (t, x) \in [T, \tau] \times D_{\delta'}.
\]

Moreover for fixed \( x \in \mathbb{R}^N \setminus B(y_R, R) \) the function \( z(\cdot, x) \) is increasing in \( t \) and

(i) If \( p > 1 \) then

\[
z(t, x) \to z(T, x) = \frac{C}{(|x - y_R| - R)^{2/(p-1)}}, \quad \text{as } t \to T^-
\]

uniformly on compact sets of \( \mathbb{R}^N \setminus B(y_R, R) \).

(ii) If \( p = 1 \) or if \( \beta = 0 \), then for any arbitrary large positive number \( n \) we can choose the function \( z \) such that

\[
z(t, x) \to z(T, x) = \frac{C}{(|x - y_R| - R)^n}, \quad \text{as } t \to T^-
\]

uniformly on compact sets of \( \mathbb{R}^N \setminus B(y_R, R) \).
Remark 3.4. Notice that the initial condition \( \phi_0 \) and the times \( T < \tau \) can be chosen uniformly for any domain \( D \) of class \((R, \delta, L)\) with vertex at \( 0 \in \mathbb{R}^N \).

Before proving this proposition we will prove several technical lemmas which show certain non-degeneracy of solutions of elliptic and parabolic problems in domains of class \((R, \delta, L)\). The first lemma states that the first eigenfunction of a Dirichlet–Neumann problem remains positive away from the Dirichlet boundary, uniformly for domains of class \((R, \delta, L)\).

Lemma 3.5. For any domain \( D \) of class \((R, \delta, L)\), \( \delta \) small enough, with vertex at \( 0 \in \mathbb{R}^N \), we denote by \( \phi^D \) and \( \mu^D \), respectively, the first positive eigenfunction and eigenvalue of the problem

\[
\begin{aligned}
-\Delta \phi &= \mu \phi \quad \text{in } D \\
\phi &= 0 \quad \text{on } \Gamma_0 \\
\frac{\partial \phi}{\partial n} &= 0 \quad \text{on } \Gamma_1
\end{aligned}
\]

(3.2)

with \( \|\phi^D\|_{L^\infty(D)} = 1 \). Then for any fixed \( \delta' \), with \( 0 < \delta' < \delta \), we have

\[
\inf \{ \phi^D(x); \ x \in D_{\delta'}; D \text{ domain of class } (R, \delta, L) \text{ with vertex at } 0 \in \mathbb{R}^N \} > 0
\]

and

\[
\sup \{ \mu^D; D \text{ domain of class } (R, \delta, L) \text{ with vertex at } 0 \in \mathbb{R}^N \} < \infty.
\]

Proof. Notice first that since the boundary \( \Gamma_1 \) is uniformly bounded in \( C^2 \), there will exist a fixed \( \epsilon > 0 \) such that if \( x_\epsilon = (0, \ldots, 0, \epsilon) \) then \( B(x_\epsilon, \epsilon) \subset D \) for any domain of class \((R, \delta, L)\). This implies that \( \mu^D \leq \lambda \) the first eigenvalue of the Laplace operator in \( B(x_\epsilon, \epsilon) \) with Dirichlet boundary conditions. This implies the second assertion.

Notice also that from the regularity of the boundary \( \Gamma_1 \), uniform for all domains \( D \) of class \((R, \delta, L)\), we can construct extension operators \( W^{1,q}(D) \rightarrow W^{1,q}(\mathbb{R}^N) \), \( L^q(D) \rightarrow L^q(\mathbb{R}^N) \), \( 1 \leq q \leq \infty \) with norms uniformly bounded with respect to \( D \). This will imply that the constant of the standard Sobolev embeddings and the norm of the trace operators \( T_D : W^{1,q}(D) \rightarrow L^q(\Gamma_1) \) are also uniformly bounded with respect to \( D \). Taking into account that \( \mu^D \) is uniformly bounded and \( \|\phi^D\|_{L^\infty(D)} = 1 \), using elliptic regularity theory applied to problem (3.2), we can obtain that the family of functions \( \phi^D \) is Hölder continuous in \( D_{\delta'} \) with norm uniformly bounded with respect to \( D \), that is, there exist constants \( 0 < \alpha < 1 \), and \( C > 0 \) such that \( \|\phi^D\|_{C^\alpha(D_{\delta'})} \leq C \) for any domain \( D \) of class \((R, \delta, L)\) and \( \delta' < \delta \) fixed. (This can be achieved, for example, by using Lemma B.1(v) in Arrieta et al. (2000) and noting that the constant appearing in this lemma depends on \( \Omega \) through the constant of the Sobolev embeddings and trace operators norms, which in our case can be chosen independent of the domain).

Hence, if the first assertion of the lemma is not true then there will exist \( 0 < \delta' < \delta \) and a sequence of domains \( D_n \) of class \((R, \delta, L)\) with vertex at \( 0 \in \mathbb{R}^N \), such that \( \inf \{ \phi^D_n(x); x \in (D_n)_{\delta'} \} \rightarrow 0 \) as \( n \rightarrow \infty \). But since the boundaries \( (\Gamma_n)_l \) are given by \( C^2 \) functions which are uniformly bounded in this topology, we can
extract a subsequence of $D_n$, that we still denote $D_n$, such that the boundaries $(\Gamma_n)_1$ converge in the $C^{1,\eta}$ topology, $0 < \eta \leq 1$, to the boundary $\Gamma_1$ of a limiting domain $D$. Under this kind of smooth perturbation of the domain we have $\phi^{D_n} \rightarrow \phi^D$ in $L^2$ as $n \rightarrow \infty$, see Courant and Hilbert (1953).

The fact that the sequence $\phi^{D_n}$ is uniformly bounded in $C^a((D_n)_{d'})$, that $\phi^D > 0$ in $D_{d'}$ and that $\phi^{D_n} \rightarrow \phi^D$ in $L^2$ implies that $\inf \{ \phi^{D_n}(x); x \in (D_n)_{d'} \} > 0$ which is a contradiction.

Now, the second lemma states that the solutions of some dissipative parabolic equation remain positive, away from the Dirichlet boundary, uniformly for domains of class $(R, \delta, L)$. For this, we will compare these solutions with solutions of a linear problem and then use the previous lemma.

Lemma 3.6. Let $R, \delta, L$ be all positive numbers with $\delta$ small enough and let $0 < \delta' < \delta$.

Assume that there exists $M_0 \geq 0$, such that for any domain $D$ of class $(R, \delta, L)$ with vertex at $0 \in \mathbb{R}^N$, and for all $x \in D$

$$f(x, 0) \geq 0, \quad f(x, u) \leq 0, \quad \text{for } u \geq M_0.$$  

Then for each $\mu > 0$ there exist $\tau > 0$ and $M \geq M_0$ such that for any domain $D$ of class $(R, \delta, L)$ with vertex at $0 \in \mathbb{R}^N$, and any smooth initial condition $w_0$, defined in $D$ and satisfying $w_0(x) \geq M$ for $x \in D_{d'}$, the solution of

$$\begin{cases}
    w_t - \Delta w = f(x, w) & \text{in } D \\
    w = 0 & \text{on } \Gamma_0 \\
    \frac{\partial w}{\partial n} = 0 & \text{on } \Gamma_1 \\
    w(0, x) = w_0(x) \geq 0 & \text{in } D,
\end{cases}$$  

(3.3)

satisfies $w(t, x, w_0) \geq \mu$ for $x \in D_{\delta/2}$ and $0 \leq t \leq \tau$.

Proof. Note that the solution of problem (3.3) is defined for all $t > 0$ and remains bounded as $t \rightarrow \infty$ uniformly in $D$.

Notice now that, with the notations in Lemma 3.5, since $w_0(x) \geq M\phi^{D_{\delta'}}(x)$ for $x \in D_{\delta'}$ then by comparison $w(t, x, w_0) \geq v(t, x, M\phi^{D_{\delta'}})$ for $x \in D_{\delta'}$ where $v$ is the solution of

$$\begin{cases}
    v_t - \Delta v = f(x, v) & \text{in } D_{\delta'} \\
    v = 0 & \text{on } \Gamma_{0}^{\delta'} \\
    \frac{\partial v}{\partial n} = 0 & \text{on } \Gamma_{1}^{\delta'} \\
    v(0, x) = M\phi^{D_{\delta'}}(x) & \text{in } D_{\delta'}.
\end{cases}$$  

(3.4)
Also, by the maximum principle we have that, if \( M \geq M_0 \) then, \( v(t, x, M\phi^{D_v}) \leq M \) for all \( t \geq 0 \). Hence \( v(t, x, M\phi^{D_v}) \geq V(t, x, M\phi^{D_v}) \) where \( V \) is the solution of

\[
\begin{cases}
V_t - \Delta V = \tilde{M}V & \text{in } D_\delta \\
V = 0 & \text{on } \Gamma^0_{D_v} \\
\frac{\partial V}{\partial n} = 0 & \text{on } \Gamma^1_{D_v} \\
V(0, x) = M\phi^{D_v}(x) & \text{in } D_\delta.
\end{cases}
\]

(3.5)

where \( \tilde{M} \leq 0 \) is chosen such that \( f(x, v) \geq \tilde{M}v \) for all \( 0 \leq v \leq M \) and \( x \in D_{D_v} \). Hence the solution of (3.5) is given by \( V(t, x, M\phi^{D_v}) = e^{(-\mu^{D_v} + \tilde{M})t}M\phi^{D_v}(x) \).

Therefore, from Lemma 3.5, given \( L > 0 \) there exist \( \tau \) and \( M \) such that

\[
\inf\{V(\tau, x, M\phi^{D_v}), \ x \in D_{\delta/2}; \ D \text{ domain of class } (R, \delta, L)\}
\]

with vertex at \( 0 \in \mathbb{R}^N \) \( \geq \mu \)

and the result is proved. \( \square \)

**Proof of the Proposition.** In order to construct the function \( z(t, x) \) we need to analyze the behavior of the solutions of the ordinary differential equation

\[
\begin{cases}
\psi'(t) = \eta \psi^m(t) \\
\psi(0) = a \geq 1
\end{cases}
\]

(3.6)

for certain values of \( m \in \mathbb{R}, m > 1 \) and \( \eta > 0 \). The values of these parameters will be chosen later according to the different values of \( p, q \) and \( \beta \). Obviously the solution of (3.6) depends on \( m, \eta, a \), although we will only make explicit the dependence on \( a \).

**Step 1.** By direct computation \( \psi_a(t) = E/(T_a - t)^{1/(m-1)} \) for \( -\infty < t < T_a \) with

\[
E = \frac{1}{(\eta(m-1))^{1/(m-1)}} \quad \text{and} \quad T_a = \frac{1}{\eta(m-1)a^{m-1}}.
\]

Observe that, since \( a \geq 1 \) and \( m > 1 \), \( T_a \leq 1/\eta(m-1) \) and that \( T_a \to 0 \) as \( a \to +\infty \). Notice also that \( \psi_a(t) \leq E/(-t)^{1/(m-1)} \) for any \( t < 0 \) and any \( a \geq 1 \).

Let us define the function \( h(x) = R - |x - y_R|, \) which is negative in \( \mathbb{R}^N \setminus B(y_R, R) \) and \( h(0) = 0 \).

Then we define \( z_a(t, x) = \psi(t + h(x)), \) for \( x \in \mathbb{R}^N \setminus B(y_R, R) \) and \( t \in [0, T_a] \).

Clearly \( z_a(t, x) \) is positive, increasing in \( t \) and

\[
z_a(t, x) \to z_a(T_a, x) = \frac{E}{(|x - y_R| - R)^{1/(m-1)}}, \quad \text{as } t \to T_a^-,
\]

uniformly on compact sets of \( \mathbb{R}^N \setminus B(y_R, R) \).
Now we consider any domain \( D \) of class \((R, \delta, L)\) with vertex at \( 0 \in \mathbb{R}^N \). We are going to show below, by a direct computation, that

\[
\begin{aligned}
\frac{\partial z_a}{\partial t} - \Delta z_a &\leq (1 + \frac{N - 1}{R} - \eta m z_a^{m-1}) \eta z_a^m, & x \in \mathbb{R}^N \setminus B(y_R, R), & t \in (0, T_a) \\
\frac{\partial z_a}{\partial n} &\leq \eta z_a^m, & x \in \Gamma_1, & t \in (0, T_a)
\end{aligned}
\tag{3.7}
\]

For the first inequality, using that \( z_a(t, \cdot) \) is radially symmetric around the point \( y_R \) and (3.6), we have that if \( r = |x - y_R| \), then \( \partial z_a / \partial r = -\psi'(t + R - r) = -\eta z_a^m \) and \( \partial^2 z_a / \partial r^2 = \eta m z_a^{m-1} \partial z_a / \partial r \). With the expression of the Laplace operator for radially symmetric functions we obtain the first inequality in (3.7).

For the second, note that on \( \Gamma_1 \) we have

\[
\frac{\partial z_a}{\partial n} = \nabla z_a \cdot \vec{n}(x) = \psi'(t + h(x)) \nabla h(x) \cdot \vec{n}(x) \leq \eta z_a^m
\]

since \( \nabla h(x) \cdot \vec{n}(x) \leq 1 \). Hence (3.7) is satisfied.

**Step 2.** In this step we will find \( a_0 \), large enough, and \( \delta' \), with \( 0 < \delta' < \delta \), such that for any \( a \geq a_0 \), the function \( z_a \) satisfies

\[
\begin{aligned}
\frac{\partial z_a}{\partial t} - \Delta z_a &\leq -\beta z_a^p \leq f(x, z_a), & x \in D_{\delta'/2}, & t \in (0, T_a) \\
\frac{\partial z_a}{\partial n} &\leq z_a^q \leq g(x, z_a), & x \in \Gamma^1_{\delta'/2}, & t \in (0, T_a)
\end{aligned}
\tag{3.8}
\]

Observe first that for any \( 0 < \delta' < \delta \), we have \( z_a(t, x) \geq \psi(-\delta') = \frac{1}{(q(m-1)(\tau_0 + \delta'))^{1/(m-1)}} \) for each \( t \in (0, T_a) \) and \( x \in B(y_R, R + \delta') \setminus B(y_R, R) \). Moreover, we know that \( T_a \to 0 \) as \( a \to \infty \). Hence, for any \( 0 < \varepsilon < m \), we can choose \( a_0 \) large enough and \( 0 < \delta' < \delta \) small enough so that, for any \( a \geq a_0 \), \( z_a(t, x) \) becomes so large that \( 1 + (N - 1)/(R) - \eta m z_a^{m-1} \leq 0 \), \( -\beta z_a^p \leq f(x, z_a) \) and \( z_a^q \leq g(x, z_a) \) for \( x \in B(y_R, R + \delta') \setminus B(y_R, R) \) and \( t \in (0, T_a) \). This implies that

\[
\begin{aligned}
\frac{\partial z_a}{\partial t} - \Delta z_a &\leq -\eta^2 (m - \varepsilon) z_a^{2m-1}, & x \in B(y_R, R + \delta') \setminus B(y_R, R), & t \in (0, T_a) \\
\frac{\partial z_a}{\partial n} &\leq \eta z_a^m, & x \in \Gamma_1, & t \in (0, T_a)
\end{aligned}
\tag{3.9}
\]

Now, since \( z_a \) is large enough on \( B(y_R, R + \delta') \setminus B(y_R, R) \times [0, T_a] \), it is clear that taking \( m \) such that \( 2m - 1 \geq p \) and \( m \leq q \), the equations and boundary conditions in (3.1) and (3.9) can be compared. In fact, we now consider different cases:

(i) If \( 2q = p + 1 \) and \( \beta < q \) let us choose \( m = q = (p + 1)/2, \eta = 1 \) and \( \varepsilon \) small enough so that \( \beta < q - \varepsilon \). Then the function \( z_a \) satisfies (3.8).

(ii) If \( 2q > p + 1 \) and \( p > 1 \) choose \( m = (p + 1)/2, \varepsilon = 1/2 \) (the choice of \( 0 < \varepsilon < m \) is irrelevant in this case) and \( \eta \) large enough so that \( \eta^2 (m - \varepsilon) > \beta \). Choosing now \( a \) large enough we can obtain that \( z_a(t, x) \) is large enough for \( x \in \Gamma^1_{\delta'/2} \) and...
reaction–diffusion equations 1143

\[ t \in (0, T), \text{ so that } \eta z_a^m = \eta z_a^{(p+1)/2} \leq z_a^q \text{ for } x \in \Gamma^{1}_{\partial/2} \text{ and } t \in (0, T). \] Hence \( z_a \) satisfies (3.8).

(iii) If \( 2q > p + 1 \) and \( p = 1 \) or if \( \beta = 0 \), we can proceed as in (ii) with any \( m \) satisfying \( q > m > 1 \) and again \( z_a \) satisfies (3.8).

**Step 3.** We define \( \mu = E(\delta'/2)^{-1/(m-1)} \) and observe that \( \mu \) is independent of \( a \). Then, for \( 0 \leq t < T_a \) and \( x \in \Gamma^{0}_{\partial/2} \) we have, for all \( a \geq a_0 \),

\[ z_a(t, x) \leq \psi(t - \delta'/2) \leq \psi(T_a - \delta'/2) = E(\delta'/2)^{-1/(m-1)} = \mu. \]

Moreover, for this \( \mu \), from Lemma 3.6 there exist \( \tau > 0 \) and \( M > \mu \) such that for any smooth initial condition \( w_0 \), defined in \( D \) and satisfying \( w_0(x) \geq M \) for \( x \in \partial D \), the solution of (3.3) satisfies \( w(t, x, w_0) \geq \mu \) for \( x \in \partial D \) and \( 0 \leq t \leq \tau \).

We fix now the parameter \( a \geq a_0 \), with the property that \( T_a < \tau \). We also define \( T = T_a \) and \( z(t, x) = z_a(t, x) \). Obviously, for this choice, the function \( z \), verifies (3.8) for \( 0 \leq t < T \) and \( z(t, x) \leq \mu \), for \( x \in \Gamma^{1}_{\partial/2}, 0 \leq t < T \).

**Step 4.** We choose now the initial condition \( \phi_0 \). Let \( \phi_0 \in C^\infty_0(B(y_R, R + \delta)) \) such that \( \phi_0 \geq M \) and \( \phi_0(x) \geq z(0, x), \) in \( \partial D \).

To actually conclude that \( z \) is a subsolution of (3.1), we show now that on the piece of the boundary \( \Gamma^{0}_{\partial/2} \), we have the estimate \( z(t, x) \leq u(t, x, \phi_0) \), for \( 0 \leq t < T \). Observe that from Step 3, \( z(t, x) \leq \mu \), \( x \in \Gamma^{1}_{\partial/2}, 0 \leq t < T \) and, from Lemma 3.6, \( \mu \leq w(t, x, \phi_0), x \in \Gamma^{1}_{\partial/2}, 0 \leq t < T \). Moreover, by comparison we have \( w(t, x, \phi_0) \leq u(t, x, \phi_0) \) in \( \overline{D} \times [0, \infty) \).

In particular, we have

\[
\begin{align*}
\frac{\partial z}{\partial t} - \Delta z \leq f(x, z), \quad x \in \Gamma^{1}_{\partial/2}, & \quad t \in (0, T) \\
z(t, x) \leq u(t, x, \phi_0), \quad x \in \Gamma^{1}_{\partial/2}, & \quad t \in [0, T) \\
\frac{\partial z}{\partial t} \leq g(x, z), \quad x \in \Gamma^{1}_{\partial/2}, & \quad t \in (0, T) \\
z(0, x) \leq \phi_0(x), \quad x \in \partial D. & 
\end{align*}
\]

(3.10)

which, by comparison, implies that \( z(t, x) \leq u(t, x, \phi_0) \) for \( 0 \leq t < T \) and \( x \in \partial D \).

Taking into account that

\[ z(T, x) = \frac{E}{(|x - y_R| - R)^{m-1}}, \quad x \in \mathbb{R}^N \setminus B(y_R, R), \]

where \( m \) has been chosen before, we get the result for the interval \( [0, T) \).

**Step 5.** For \( T \leq t < \tau \), note that we have that for any \( 0 < \varepsilon \leq T \), such that \( T + \varepsilon \leq \tau \), the function \( z(t, x) = z(t - \varepsilon, x) \) also satisfies the first three inequalities in (3.10) for \( \varepsilon \leq t \leq T + \varepsilon \). On the other hand at time \( t = \varepsilon \) we have \( z(\varepsilon, x) = z(0, x) \leq z(\varepsilon, x) \leq u(\varepsilon, x) \) since \( z \) is increasing in time. Hence \( u(t, x) \geq z(t, x) \) for
\[ \varepsilon \leq t < T + \varepsilon \quad \text{and} \quad x \in D_{\delta/2}. \] In particular \( u(T + \varepsilon, x) \geq z(T, x) \) for \( x \in D_{\delta/2} \).

Repeating the argument a finite number of steps the proof is complete.

**Remark 3.7.** Observe that the time \( \tau \) in the Proposition above can be any time for which

\[ u(t, x) \geq \mu \quad \text{for} \quad T \leq t \leq \tau \quad \text{and} \quad x \in \Gamma_{\delta/2}^0. \]

In particular we can take \( \tau = \infty \) provided \( u(t, x) \) is monotonically increasing in time or, more generally, if \( u \) does not become very small close to the boundary in the sense that

\[ u(t, x) \geq \mu \quad \text{for} \quad t \geq T \quad \text{and} \quad x \in \Gamma_{\delta/2}^0. \]

In this case the solution is pinned to infinity for all times after time \( T \).

**Remark 3.8.** The statement and proof of the proposition above has been written in terms of power-like nonlinearities. However a close look to the arguments above shows that it can be easily modified to cover some other situations. For example we can assume that \( c \) satisfies, instead of (3.6),

\[
\begin{cases}
\psi'(t) = H(\psi(t)) \\
\psi(0) = a \geq 1
\end{cases}
\]

for a positive nondecreasing function \( H \) such that \( \int_0^\infty ds/H(s) < \infty \). Hence, \( \psi \) blows up in a time \( T(a) \to 0 \) as \( a \to \infty \).

Now we can define as before \( z(t, x) = \psi(t + h(x)) \). Since for \( x \in B(y_R, R + \delta') \setminus B(y_R, R) \) and \( t \in [0, T) \) we have \( z(t, x) \geq \psi(-\delta') \), which can be made arbitrarily large for \( a \) large and \( \delta' \) small, we have that (3.8) is satisfied provided

\[
H(u) \leq g(x, u), \quad -f(x, u) \leq H(u)H'(u)
\]

for sufficiently large \( u \geq 0 \) and locally in \( x \) around 0.

The rest of the argument runs as above. We define \( \mu = \psi(T - \delta'/2) \), which is independent of \( a \). Moreover, for \( 0 \leq t < T \) and \( x \in \Gamma_{\delta/2}^0 \) we have

\[
z(t, x) \leq \psi(t - \delta'/2) \leq \psi(T - \delta'/2) = \mu
\]

so (3.10) can be met again.

Hence, local subsolutions that blow-up in finite time can be constructed. Note that this situation allows to consider nonlinear terms of the form

\[
f(x, u) = -\beta u \log^p(u), \quad g(x, u) = u \log^q(u)
\]

for large values of \( u \) and locally in \( x \) around 0, with \( q > 1 \) and \( 2q > p \) or \( 2q = p \) and \( 0 \leq \beta < 1 \).
4. PROOF OF THE MAIN RESULTS

We can proceed now to give a proof of the main results. The basic idea that we follow below is that once we have constructed a subsolution, as in Proposition 3.3, around any point of the boundary, we can consider the envelope of a family of subsolutions as the vertex of the reference domain moves in a small piece of the boundary of \( \Omega \). In fact we have

**Proof of Theorem 1.1.** Let us choose \( \rho' \leq \rho_0 \) small enough such that the map \((\partial \Omega \cap B(x_0, \rho')) \times (-\rho', \rho') \rightarrow \mathbb{R}^N\), given by \((x, s) \rightarrow x + s\mathbf{n}(x)\) is a well defined diffeomorphism. Notice that \( \rho' \) depends only on \( L \) and \( \rho_0 \). We can choose now, \( R \) and \( \delta < \rho \) such that for all \( \xi \in \partial \Omega \cap B(x_0, \rho'/4)\), if \( y_R(\xi) = \xi + R\mathbf{n}(\xi)\), then \( D_\xi = B(y_R(\xi), R + \delta) \cap \Omega \) is, up to a rigid motion, an \((R, \delta, L)\) domain with vertex at \( 0 \in \mathbb{R}^N \). Moreover \( D_\xi \subset B(x_0, \rho') \) for all \( \xi \in \partial \Omega \cap B(x_0, \rho'/4) \).

We apply now Proposition 3.3 and obtain the number \( \delta \) and the functions \( \phi_0^\xi \in C_\infty^c(B(y_R(\xi), R + \delta)) \) and \( z^\xi(t, x) \) for any \( \xi \in \partial \Omega \cap B(x_0, \rho'/4) \). Notice that the functions \( \phi_0^\xi \) and \( z^\xi(t, x) \) coincide up to a rigid motion.

If the initial data \( u_0 \in C_\infty^c(B(x_0, \rho_0)) \) is chosen large enough around \( x_0 \) so that \( u_0 \geq \phi_0^\xi \) in \( D_\xi \), then we will have that \( u(t, x, u_0) \geq z^\xi(t, x) \). In particular \( u(t, x, u_0) \geq \sup\{z^\xi(t, x); \xi \in \partial \Omega \cap B(x_0, \rho'/4)\} \equiv v(t, x) \). It is not difficult to see now that \( v(t, x) \) satisfies the statements of the theorem with \( \rho = \delta \).

If the boundary \( \Gamma_N \) is uniformly \( C^2 \) the construction performed can be done uniformly for all \( x_0 \in \Gamma_N \).

Now we turn into the proof of Theorem 1.4. For this we will rely on a priori estimates obtained via the construction of a solution of the problem \(-\Delta z = -\beta z^p \) in a ball that takes the value infinity in the boundary of the ball. The construction of this solution is already well known and can be found in Keller (1957), Osserman (1957), Veron (1992) and García-Melián et al. (2001, 1998). We state this construction in the following lemma.

**Lemma 4.1.** Assume \( p > 1 \) and \( \beta > 0 \) and consider a ball in \( \mathbb{R}^N \) of radius \( a > 0 \) and the following singular Dirichlet problem

\[
\begin{cases}
-\Delta z = -\beta z^p & \text{in } B(0, a) \\
z = \infty & \text{on } \partial B(0, a).
\end{cases}
\]

Then

(i) There exists a unique positive radial solution, \( z_a(x) \).

(ii) As a function of \( a \) the solution satisfies

\[
z_a(x) = \frac{1}{a^{\frac{N-1}{p}}} z_1\left(\frac{x}{a}\right).
\]

(iii) The solution satisfies

\[
m(a) = \inf_{B(0, a)} z_a(x) = C \frac{1}{a^{\frac{N-1}{p}}} \rightarrow \infty \quad \text{as } a \rightarrow 0.
\]
Proof. Part (i) follows easily from the references given above (e.g., García-Melián et al., 2001, Lemma 3; García-Melián et al., 1998, Lemma 6.2). Note in fact that \( z_a \) is the proper minimal solution to the Dirichlet problem above, since it is obtained as the monotone pointwise limit, as \( M \to \infty \), of the unique positive radial solutions of the solutions of the elliptic equation above with Dirichlet data \( z = M \) on the boundary of the ball.

Part (ii) follows easily by rescaling and part (iii) is an obvious consequence.

Now, using the functions above as local interior supersolutions, we can prove Theorem 1.4.

Proof of Theorem 1.4. Take a point \( x_0 \in \Omega \), which without loss of generality we can assume \( x_0 = 0 \) and \( a > 0 \) such that \( B(x_0, a) \subset \Omega \). Given a smooth initial data \( u_0 \) of (1.1), we can always assume \( a \) is small enough such that \( 0 \leq u_0 \leq m(a) \) on \( B(x_0, a) \) and \( f(x, u) \leq -\beta u^p \) for all \( u \geq m(a) \) and \( x \in \Omega \), with \( m(a) \) as in the Lemma above. Then we claim that in fact

\[
0 \leq u(t, x) \leq z_a(x) \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad x \in B(x_0, a).
\]

In particular, this proves the first statement of Theorem 1.4.

In fact consider the sequence of positive, globally defined classical solutions \( u_n(t, x) \) that approach the proper minimal solution of (1.1), as explained in Sec. 2. Hence for \( n \) and \( T > 0 \) fixed, \( 0 \leq u_n(t, x) \leq C(n, T) \) for all \( 0 \leq t \leq T \) and \( x \in \Omega \). Hence for sufficiently large \( M \) consider, as indicated in the Lemma above, \( z_M(x) \) the unique positive radial solutions of the solutions of the elliptic equation \( -\Delta z = -\beta z^p \) in \( B(0, a) \) with Dirichlet data \( z = M \) on the boundary.

Hence by the assumption on \( f(x, u) \) and the parabolic maximum principle, we have

\[
0 \leq u_n(t, x) \leq z_M(x) \quad \text{for all} \quad t \in [0, T] \quad \text{and} \quad x \in B(x_0, a).
\]

Taking the limit \( M \to \infty \) we get

\[
0 \leq u_n(t, x) \leq z_a(x) \quad \text{for all} \quad t \in [0, T] \quad \text{and} \quad x \in B(x_0, a).
\]

Note that the above inequality now holds for any \( T > 0 \). Now we take the limit \( n \to \infty \) and the claim is proved.

Now take \( a \) small enough and define \( k(a) = \max_{B(0, a/2)} z_a(x) = z_a(a/2) = \frac{C}{a^{n+1}} \). Then by using translations of \( z_a(x) \) and the claim above we get that

\[
0 \leq u(t, x) \leq k(a) \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad x \in \Omega_a
\]

where \( \Omega_a = \{ x \in \Omega, \text{dist}(x, \Gamma) \geq a \} \). This and the expression for \( k(a) \) gives the second statement of Theorem 1.4.
ACKNOWLEDGMENT

This research was partially supported by DGES, BFM2000-0798, Spain.

REFERENCES


Received April 2003
Revised April 2004