Abstract

We analyse the dynamics of the non-autonomous nonlinear reaction–diffusion equation

\[ u_t - \Delta u = f(t, x, u), \]

subject to appropriate boundary conditions, proving the existence of two bounding complete trajectories, one maximal and one minimal. Our main assumption is that the nonlinear term satisfies a bound of the form \( f(t, x, u)u \leq C(t, x)|u|^2 + D(t, x)|u| \), where the linear evolution operator associated with \( \Delta + C(t, x) \) is exponentially stable. As an important step in our argument we give a detailed analysis of the exponential stability properties of the evolution operator for the non-autonomous linear problem \( u_t - \Delta u = C(t, x)u \) between different \( L^p \) spaces.

1. Introduction

In this paper we analyse the dynamics of the following non-autonomous nonlinear parabolic model problem

\[
\begin{aligned}
& u_t - \Delta u = f(t, x, u) \quad \text{in } \Omega, \ t > s, \\
& u = 0 \quad \text{on } \partial\Omega, \\
& u(s) = u_s
\end{aligned}
\]  

(1.1)
where $\Omega$ is a bounded domain in $\mathbb{R}^N$ and $f(t, x, u) : \mathbb{R} \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a suitable smooth function. We denote the solution of this equation by $u(t, s; u_s)$.

Our goal is to prove that under suitable conditions there exist two extremal complete trajectories (defined for all $t \in \mathbb{R}$), one maximal and one minimal, which give bounds for the asymptotic behaviour of solutions in an appropriate sense. We will assume that the nonlinear term satisfies

$$f(t, x, u)u \leq C(t, x)|u|^2 + D(t, x)|u| \quad \text{for all } u \in \mathbb{R},$$

for some $C \in C^\alpha(\mathbb{R}, L^p(\Omega))$ with $0 < \alpha \leq 1$ and $p > N/2$, and some function $D$ with values in $L^r(\Omega)$, $1 \leq r \leq \infty$. The crucial assumption to our analysis is that the evolution operator associated with $\Delta + C(t, x)$ is exponentially stable.

This paper makes two essentially independent contributions, which are combined in our treatment of the reaction–diffusion equation (1.1).

The first is a well-developed general theory concerning the properties (in particular the exponential stability) of the evolution operators associated with non-autonomous linear problems of the form

$$u_t - \Delta u = C(t, x)u$$

posed in $L^q(\Omega)$ ($1 \leq q < \infty$) or $C(\overline{\Omega})$, with $C \in C^\alpha(\mathbb{R}, L^p(\Omega))$, where $0 < \alpha \leq 1$ and $p > N/2$, $p > 1$. We present a complete study of the norms of the solution operator $U_C(t, s)$ between different $L^q$ spaces.

The second ingredient is a dynamical argument that, under certain natural conditions, guarantees the existence of extremal complete trajectories of a nonlinear problem. The argument makes key use of the order-preserving property of our equations, thereby deducing the existence of extremal trajectories for the nonlinear equation (1.1) from their existence for the associated linear problem (1.3). Although for the sake of simplicity we consider this problem in the phase space $C(\overline{\Omega})$, under suitable growth conditions we could also consider Sobolev spaces of initial data in $L^q(\Omega)$: the dynamical arguments would remain almost identical, but the analysis would become much more technically involved.

Also, note that in our analysis no prescribed time dependence is assumed (e.g. periodic, quasiperiodic or almost periodic).

Since our fundamental tools are comparison techniques, the results are valid for more general operators than the Laplacian and other boundary conditions provided that the problem admits a comparison principle. We also use the smoothing effect of the equations in an essential way.

The dynamical results here are the non-autonomous counterpart of those for autonomous parabolic problems established in Rodríguez-Bernal and Vidal-López [20] and Vidal-López [24].

1.1. Summary of results for the nonlinear equation

Since the initial time plays a central role in non-autonomous problems, in order to analyse the behaviour of solutions of (1.1) it is natural to make use of the notions of pullback attraction and pullback attractors. The basic idea behind these is that the relevant dynamics at the current time $t$ are those that have arisen from initial conditions long ago, i.e. we take $s \to -\infty$ in order to discount the transient behaviour (rather than taking $t \to \infty$ which is more natural in the autonomous case). The pullback attractor is then the set of possible current states, $\{A(t)\}_{t \in \mathbb{R}}$, for solutions that started arbitrarily far in the past. Of course, the more familiar concept of forwards
attraction (as $t \to +\infty$) is still relevant, although some care is needed with the definition of a ‘forwards attractor’ in non-autonomous systems.

In this paper we are going to show (under suitable conditions on $f$, $C$, and $D$ as described above) that there exist two extremal complete trajectories for (1.1), $\psi_M(t, x)$ and $\psi_m(t, x)$, that are maximal and minimal, respectively, in the sense that any other complete trajectory $\psi(t, x)$ satisfies

$$
\psi_m(t, x) \leq \psi(t, x) \leq \psi_M(t, x) \quad \text{for all } t \in \mathbb{R}.
$$

We also prove that if $f(t, x, u)$ is $T$-periodic in time then so are $\psi_m$ and $\psi_M$.

A relatively simple argument shows that the ‘order intervals’ $[\psi_m(t), \psi_M(t)]$, consisting of all functions lying between $\psi_m$ and $\psi_M$, are positively invariant, i.e. for any $t > s$

$$
\text{if } \psi_m(s, x) \leq u_s(x) \leq \psi_M(s, x) \text{ then } \psi_m(t, x) \leq u(t, s; u_s)(x) \leq \psi_M(t, x),
$$

and also attract the dynamics of the system uniformly in the pullback sense, i.e. for every $t \in \mathbb{R}$ we have

$$
\psi_m(t, x) \leq \liminf_{x \to -\infty} u(t, s, x; u_s) \leq \limsup_{x \to -\infty} u(t, s, x; u_s) \leq \psi_M(t, x) \quad (1.4)
$$

uniformly in $x \in \overline{\Omega}$ for all $\{u_s\}$ in a bounded set of initial data $B$.

Moreover, $\psi_M(t)$ is globally asymptotically stable from above in the pullback sense, i.e. for all $v \in C_b(\mathbb{R}, X)$ such that $v \geq \psi_M$ we have

$$
\lim_{s \to -\infty} u(t, s, v)(x) = \psi_M(t, x)
$$

uniformly in $x \in \Omega$. In a similar sense, $\psi_m(t)$ is globally asymptotically stable from below.

As a consequence, there exists a pullback attractor for (6.1), denoted by $\mathcal{A} = \{\mathcal{A}(t)\}_t$, and

$$
\mathcal{A}(t) \subset [\psi_m(t), \psi_M(t)] \quad \text{for all } t \in \mathbb{R}.
$$

The two extremal trajectories lie in the pullback attractor: $\psi_m(t), \psi_M(t) \in \mathcal{A}(t)$ for all $t \in \mathbb{R}$.

A full and exact statement of these results is given in Theorem 6.1.

Observe that it is possible (and it is indeed the case in certain problems) that the extremal solutions are not uniformly bounded for all $t$. While in such a case the pullback attractor can still exist, there can be no bounded forwards attractor.

1.2. Outline of the paper

In Section 2 we recall some definitions from the theory of attractors and order-preserving dynamical systems. In particular we introduce the notion of pullback attraction in a formal way.

In Section 3 we analyse in detail the evolution operators associated with linear homogeneous non-autonomous parabolic equations. In particular we discuss questions related to regularisation and exponential stability in several function spaces. We give sufficient conditions for exponential stability, and prove its persistence under various classes of perturbation.

In Section 4 we study complete trajectories for inhomogeneous linear non-autonomous parabolic equations, giving suitable conditions for their existence and analysing their asymptotic
behaviour both as $t \to +\infty$ and as $t \to -\infty$. Further to this in Section 5 we consider asymptotically autonomous and asymptotically periodic problems: we prove that in such cases the complete trajectories inherit the properties of the underlying equation (asymptotically autonomous/periodic).

In Section 6 we prove our main result concerning extremal complete trajectories and the pullback attractor for (1.1), as outlined above. In Section 7 we analyse the case in which the extremal trajectories are bounded forward in time and give a description of the asymptotic behaviour of (1.1) starting from the pullback attractor.

In Section 8 we show how the general results from previous sections can be applied to some logistic non-autonomous model problems. Finally, in Section 9 we extend the results to some non-autonomous parabolic equations with nonlinear non-autonomous boundary conditions.

2. Some useful concepts for non-autonomous equations

Throughout the paper we will recast our equations as abstract families of (non-autonomous) evolution operators acting on an appropriate phase space.

**Definition 2.1.** Given a metric space $(X, d)$, we say that a family of mappings $\{U(t, s)\}_{t \geq s}$ is a process, a family of evolution operators or simply an evolution operator if it satisfies

1. $U(t, t) = I$ for all $t \in \mathbb{R}$,
2. $U(t, s)U(s, r)u = U(t, r)u$ for all $r \leq s \leq t$, $u \in X$, and
3. $u \mapsto U(t, r)u$ is continuous in $X$, $t > r$.

2.1. Different notions of attraction in non-autonomous problems

We begin with some useful definitions from the theory of attractors for non-autonomous systems which we will use throughout this paper (see for example Crauel, Debussche, and Flandoli [8], Kloeden and Schmalfuß [14], or Schmalfuß [22]).

We define formally the notions of attraction and absorption in both the ‘pullback’ and ‘forwards’ senses. In what follows we denote by $B$ and $K$ time-dependent families $\{B(s)\} \in \mathbb{R}$ and $\{K(s)\} \in \mathbb{R}$ of bounded sets. We begin with attraction.

**Definition 2.2.** (i) We say that $K$ attracts $B$ in the pullback sense if for each $t_0 \in \mathbb{R}$

$$\lim_{s \to -\infty} \text{dist}(U(t_0, s)B(s), K(t_0)) = 0.$$ 

(ii) We say that $K$ attracts $B$ (forwards in time) if for each $s \in \mathbb{R}$

$$\lim_{t \to \infty} \text{dist}(U(t, s)B(s), K(t)) = 0.$$ 

We say that $K$ attracts bounded sets (in whichever sense) if the above definitions hold for $B(t) \equiv B$, where $B$ is a fixed bounded set.

Stronger than this, but key to the existence results for pullback and forwards attractors, is the notion of an absorbing set.
**Definition 2.3.** (i) A bounded set \( K \) absorbs \( B \) in the pullback sense at time \( t_0 \) if there exists \( T = T(t_0, B) \leq t_0 \) such that

\[
U(t_0, s)B(s) \subset K \quad \text{for all } s \leq T \leq t_0.
\]

(ii) A bounded set \( K \subset X \) absorbs \( B \) forwards in time if for each \( s \in \mathbb{R} \) there exists \( T = T(s, B) \geq s \) such that

\[
U(t, s)B(s) \subset K \quad \text{for all } t \geq T.
\]

A time-dependent set \( \mathcal{K} \) is invariant if it is preserved under the action of \( U(t, s) \):

**Definition 2.4.** We say that \( \mathcal{K} \) is forwards invariant (with respect to \( U \)) if

\[
U(t, s)K(s) \subset K(t) \quad \text{for all } t \geq s,
\]

and that \( \mathcal{K} \) is invariant (with respect to \( U \)) if

\[
U(t, s)K(s) = K(t) \quad \text{for all } t \geq s.
\]

In the following we will fix some nonempty class \( \mathcal{D} \) of families of bounded sets of \( X \), \( \{B(s)\}_{s \in \mathbb{R}} \), as the basin of attraction. See Schmalfuß [22] for details of some of the properties required for such a class (a “universe”), but we remark here that in particular the classes that we will consider will include all time-independent bounded sets, i.e. families where \( B(t) = B \) for all \( t \in \mathbb{R} \) where \( B \subset X \) is bounded.

As a general notation used below, if an element in \( \mathcal{D} \) is of the form \( \{v(s)\}_s \) with \( v(s) \) being a single element in \( X \) then we denote it by \( v_t \).

We are now in a position to define the pullback attractor.

**Definition 2.5.** We say that a family of compact sets \( \mathcal{A} = \{A(t)\}_t \) in \( X \) is the pullback attractor (for \( U \)) with respect to \( \mathcal{D} \) if it is invariant with respect to \( U \), pullback attracts all \( B \in \mathcal{D} \), and is minimal in the sense that if \( \{K(t)\}_{t \in \mathbb{R}} \) is another pullback attracting family of closed sets then \( \mathcal{A}(t) \subset K(t) \) for all \( t \in \mathbb{R} \).

To treat the asymptotic behaviour of solutions forwards in time we define the notion of a forwards attractor.

**Definition 2.6.** We say that a fixed compact set \( \mathcal{F} \) is the forwards attractor for \( U \) if \( \mathcal{F} \) is the minimal compact set such that for any \( s \in \mathbb{R} \) and any bounded set \( B \subset X \),

\[
\lim_{t \to \infty} \text{dist}(U(t, s)B, \mathcal{F}) = 0.
\]

Note that the notion of a pullback attractor, as introduced above, is relative to some chosen basin of attraction \( \mathcal{D} \). On the other hand the domain of attraction for the forwards attractor is restricted, as is customary, to the class of time-independent bounded sets.

The next result reproduces the standard conditions guaranteeing the existence of a pullback attractor (see Crauel et al. [8], Langa and Suárez [15], Schmalfuß [21]).
Theorem 2.7. If there exists a time-dependent compact set that is pullback absorbing for all \( B \in \mathcal{D} \) then there exists a pullback attractor with respect to \( \mathcal{D} \).

For a somewhat similar result for the case of the forwards attractor, see Section 7.

2.2. Order-preserving and exponentially stable evolution operators

One of the main tools we use in our analysis of (1.1) is the monotonicity of solutions, in various senses. To formalise these, suppose that we have an order structure on the phase space \( X \), which we will denote by \( \leq \). We will use evolution operators that preserve the order in the following sense:

**Definition 2.8.** We say that an evolution operator \( U(t, s) \) is order-preserving if there exists an order relation in \( X (\leq) \) such that
\[
    u_0 \leq v_0 \quad \Rightarrow \quad U(t, s)u_0 \leq U(t, s)v_0 \quad \text{for all } t \geq s
\]

while both solutions exist.

**Definition 2.9.** Given \( u \leq v \), the order interval defined by \( u \) and \( v \) is
\[
    [u, v] = \{ w \in X : u \leq w \leq v \}.
\]

The next definition gives us the non-autonomous analogues of the concepts of an equilibrium point and of sub- and super-solutions from the theory of autonomous problems (see Amann [1], Arnold and Chueshov [4]). In particular the notion of a complete trajectory is central to all that follows.

**Definition 2.10.** We say that a continuous map \( v : \mathbb{R} \rightarrow X \) is a complete trajectory for \( U \) if for all \( t \geq s \)
\[
    U(t, s)v(s) = v(t).
\]

We say that \( v \) is a super-trajectory for \( U \) if for all \( t \geq s \)
\[
    U(t, s)v(s) \leq v(t),
\]

and that \( v \) is a sub-trajectory for \( U \) if for all \( t \geq s \)
\[
    U(t, s)v(s) \geq v(t).
\]

Finally, we give define a concept that will be crucial in the rest of this work, namely, an exponentially stable evolution operator.

**Definition 2.11.** If \( X \) is a Banach space and \( U(t, s) \in \mathcal{L}(X) \), we say that the evolution operator \( U(t, s) \) is exponentially stable if for some \( \beta > 0 \) and \( M > 0 \)
\[
    \| U(t, s) \|_{\mathcal{L}(X)} \leq Me^{-\beta(t-s)} \quad \text{for all } t > s.
\]
2.3. Existence, uniqueness, and comparison results for our parabolic problem

We now recall some existence and uniqueness results for the nonlinear parabolic problem

\[
\begin{align*}
\begin{cases}
  u_t - \Delta u = f(t, x, u) & \text{in } \Omega, \ t > s, \\
  u = 0 & \text{on } \partial \Omega, \ t > s, \\
  u(s) = u_0,
\end{cases}
\end{align*}
\]

(2.1)

posed in \( X = L^\infty(\Omega) \).

We will assume a decomposition of the form

\[
f(t, x, u) = g(t, x) + m(t, x)u + f_0(t, x, u)
\]

(2.2)

with \( f_0 : \mathbb{R} \times \Omega \times \mathbb{R} \to \mathbb{R} \) a locally Hölder in \( t \) and locally Lipschitz function in \( u \in \mathbb{R} \) uniformly respect to \( x \in \Omega \) and \( t \in \mathbb{R} \),

\[
f_0(t, x, 0) = 0, \quad \frac{\partial}{\partial u} f_0(t, x, 0) = 0;
\]

(2.3)

\( g \) is a suitable function and \( m \in C^\sigma([\mathbb{R}, L^p(\Omega))] \), with \( 0 < \alpha \leq 1 \) and some \( p > N/2 \), \( p > 1 \).

Then we have the following result for which the reader is referred to [18] and [19, Theorem 1.3].

**Theorem 2.12.** Suppose that \( f \) satisfies (2.2) and (2.3) and \( g \in L^\sigma_{ loc}(\mathbb{R}, L^p(\Omega)) \) for some \( r > N/2 \) and \( \sigma = 1 \) if \( r = \infty \) or \( \sigma > \frac{2r}{2r-N} \) otherwise.

Then, for any \( u_0 \in L^\infty(\Omega) \) and \( s \in \mathbb{R} \), there exists a unique local solution of (2.1), such that for any \( 1 \leq q < \infty \) and for some \( T > 0 \)

\[
u \in C([s, s + T], L^q(\Omega)) \cap L^\infty([s, s + T], L^\infty(\Omega))
\]

and moreover \( u \in C((s, s + T), C_0(\overline{\Omega})) \). This solution is given by the variation of constants formula

\[
u(t, s; u_0) = U_m(t, s)u_0 + \int_s^t U_m(t, \tau)(g(\tau) + f_0(\tau, u(\tau))) \, d\tau
\]

(2.4)

where \( U_m(t, s) \) is the evolution operator associated to (1.3) with \( C = m \). In addition, if \( u_0 \in C_0(\overline{\Omega}) \) then \( u \in C([s, s + T], C_0(\overline{\Omega})) \).

If the solutions of (2.1) are globally defined then \( U(t, s)u_0 = u(t, s; u_0) \) defines an evolution operator in \( X = L^\infty(\Omega) \) as in Definition 2.1. Due to the smoothing effect of (2.1) we know that, for any \( t > s \), \( U(t, s) \) is a continuous and bounded map from \( L^\infty(\Omega) \) to \( C^1_0(\overline{\Omega}) \). Moreover if \( p > N \) for any \( t > s \), \( U(t, s) \) is a continuous and bounded map from \( L^\infty(\Omega) \) to \( C^1_0(\overline{\Omega}) \) (the class of \( C^1 \) functions vanishing on \( \partial \Omega \), see [19, Section 4].

If we consider the problem posed in \( L^q(\Omega) \) with \( 1 < q < \infty \) then the smoothing property of the evolution operator guarantees that all the solutions enter \( C(\overline{\Omega}) \) immediately (for \( t > s \)), and so it is sufficient to study the problem in the phase space \( C(\overline{\Omega}) \). However, notice that we need to impose some growth restrictions on \( f \) to ensure the existence of a solution of problem (2.1).
One of the main tools we will use is the following comparison principle, see [19, Section 5].

**Theorem 2.13.** Assume the hypotheses of Theorem 2.12. Assume also that \( g_0(t, x) \leq g_1(t, x) \) and \( f_0^i(t, x, u) \leq f_1^i(t, x, u) \). Then for any two initial data \( u_0, u_1 \in L^\infty(\Omega) \) and \( s \in \mathbb{R} \) in (2.1) we have

\[
\begin{align*}
\text{if} \quad u_0 &\leq u_1 \quad \text{then} \quad u_0(t, s; u_0) \leq u_1(t, s; u_1)
\end{align*}
\]

as long as they exist. On the other hand, assume

\[
\begin{align*}
g(t, x) &\geq 0, \
f_0(t, x, 0) &\geq 0.
\end{align*}
\]

Then, if \( u_0 \geq 0 \) then \( u(t, s; u_0) \geq 0 \) as long as it exists. Moreover if \( 0 \leq g_0(t, x) \leq g_1(t, x) \), \( f_0^i(t, x, u) \leq f_1^i(t, x, u) \), \( f_0^i(t, x, 0) \geq 0 \) for \( i = 0, 1 \), and \( C_0(t, x) \leq C_1(t, x) \) then for any two initial data \( u_0, u_1 \in L^\infty(\Omega) \) and \( s \in \mathbb{R} \) in (2.1) we have

\[
\begin{align*}
\text{if} \quad 0 &\leq u_0 \leq u_1 \quad \text{then} \quad u_0(t, s; u_0) \leq u_1(t, s; u_1)
\end{align*}
\]

as long as they exist.

3. Evolution operators for linear non-autonomous problems

We now consider the linear non-autonomous parabolic problem

\[
\begin{align*}
\begin{cases}
\frac{du}{dt} - \Delta u &= C(t, x)u, & \text{in } \Omega, \ t > s, \\
u &= 0, & \text{on } \partial \Omega, \ t > s, \\
u(s) &= u_0
\end{cases}
\end{align*}
\]

posed in \( X = L^q(\Omega) \) with \( 1 \leq q \leq \infty \). Then from the results in [19, Theorem 1.1] and [3], if \( C \in C^\alpha([0, \infty), L^p(\Omega)) \), with \( 0 < \alpha \leq 1 \) and some \( p > N/2, p > 1 \), (3.1) defines an order preserving evolution operator in \( X \). We denote this evolution operator by \( U_C(t, s) \), and \( u(t, s; u_0) = U_C(t, s)u_0 \) is the solution of (3.1). Moreover for each \( q \) and \( r \) with \( 1 \leq q \leq r \leq \infty \) the evolution operator \( U_C(t, s) \) satisfies

\[
\|U_C(t, s)u_0\|_{L^r(\Omega)} \leq M \frac{e^{\delta(t-s)}}{(t-s)^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}} \|u_0\|_{L^q(\Omega)}, \quad t > s,
\]

for some \( M > 0 \) and \( \delta \in \mathbb{R} \) (potentially depending on \( r \) and \( q \)).

3.1. Exponential stability in \( L^q \)

The following results show that in fact the exponent \( \delta \) in (3.2) is independent of \( q \) and \( r \) and is strongly related to the exponential growth of the evolution operator. In particular they show that the evolution operator is exponentially stable in \( L^q(\Omega) \) iff it is so in \( L^r(\Omega) \) for any \( 1 \leq r, q \leq \infty \). As exponential stability will be a crucial property that we will use repeatedly below, these results will be very useful in what follows.
Lemma 3.1. Assume that \( U = U_C \), as above, is an evolution operator in \( L^q(\Omega) \), \( 1 \leq q \leq \infty \), such that there exist \( M > 0 \) and \( \beta \in \mathbb{R} \) such that

\[
\| U(t, s) \|_{L^q(\Omega)} \leq Me^{\beta(t-s)} \quad \text{for all } t > s.
\] (3.3)

Then, as an operator in \( L^r(\Omega) \), with \( 1 < r \leq \infty \), \( U \) satisfies

\[
\| U(t, s) \|_{L^r(\Omega)} \leq K e^{\beta(t-s)} \quad \text{for all } t - s > 1,
\]

for some \( K \geq 1 \). In particular, the same exponent \( \beta \) in (3.3), can be used whatever the choice of the \( L^q(\Omega) \) space one makes.

Proof. First, note that from (3.2) we have

\[
\| U(t + 1, t) \|_{L^q(\Omega)} \leq C \quad \text{for all } t \in \mathbb{R}, \quad q \geq r.
\] (3.4)

\[
\| U(t + 1, t) \|_{L^q(\Omega)} \leq C \quad \text{for all } t \in \mathbb{R}, \quad q \leq r.
\] (3.5)

Now, suppose that \( r \geq q \), so that \( L^r(\Omega) \subseteq L^q(\Omega) \). Then, since \( U(t + 1, s) = U(t + 1, t)U(t, s) \),

\[
\| U(t + 1, s)u_0 \|_{L^r(\Omega)} \leq \| U(t + 1, t) \|_{L^q(\Omega)} \| U(t, s)u_0 \|_{L^q(\Omega)}.
\]

Using now (3.3) and (3.5) we have

\[
\| U(t + 1, s)u_0 \|_{L^q(\Omega)} \leq CM e^{-\beta} e^{\beta(t+1-s)} \| u_0 \|_{L^q(\Omega)} \leq C M e^{-\beta} e^{\beta(t+1-s)} \| u_0 \|_{L^r(\Omega)}.
\]

Thus

\[
\| U(t, s) \|_{L^q(\Omega)} \leq K e^{\beta(t-s)}
\]

for all \( t - s > 1 \).

Suppose now that \( 1 \leq r < q \), and therefore \( L^q(\Omega) \subseteq L^r(\Omega) \). Now, we remark that \( U(t + 1, s) = U(t + 1, s + 1)U(s + 1, s) \). So, using (3.3) and (3.4)

\[
\| U(t + 1, s)u_0 \|_{L^r(\Omega)} \leq C \| U(t + 1, s + 1) \|_{L^q(\Omega)} \| U(s + 1, s) \|_{L^q(\Omega)}
\]

\[
\leq C M e^{-\beta} e^{\beta(t+1-s)} \| u_0 \|_{L^r(\Omega)}.
\]

Thus,

\[
\| U(t, s) \|_{L^r(\Omega)} \leq K e^{\beta(t-s)}
\]

for all \( t - s > 1 \). \( \Box \)

We also have the following estimate between different Lebesgue spaces:
Lemma 3.2. Suppose that $U(t, s)$ satisfies (3.3). Then, for $1 \leq q \leq r \leq \infty$

$$
\|U(t, s)\|_{L^q(\Omega), L^r(\Omega')} \leq \begin{cases} 
K(t - s)^{-\frac{q}{2}(\frac{1}{q} - \frac{1}{r})} & \text{if } t - s \leq 2, \\
K e^{\beta(t-s)} & \text{if } t - s > 2,
\end{cases}
$$

(3.6)

for some constant $K$. In particular, for all $\epsilon > 0$ there exists $M_\epsilon > 0$ such that

$$
\|U(t, s)\|_{L^q(\Omega), L^r(\Omega')} \leq M_\epsilon \frac{e^{(\beta+\epsilon)(t-s) - (t-s)^2 \frac{q}{2}(\frac{1}{q} - \frac{1}{r})}}{(t-s)^{\frac{q}{2}(\frac{1}{q} - \frac{1}{r})}} \text{ for all } t > s.
$$

Proof. From (3.2) for $t - s \leq 2$, there exists a constant $K_1$ such that

$$
\|U(t, s)\|_{L^q(\Omega), L^r(\Omega')} \leq K_1(t - s)^{-\frac{q}{2}(\frac{1}{q} - \frac{1}{r})}
$$

and, for $t - s > 2$, from (3.5), there exists a constant $K_2$ such that

$$
\|U(t, s)\|_{L^q(\Omega), L^r(\Omega')} \leq \|U(t, t-1)\|_{L^q(\Omega), L^r(\Omega')} \leq K_2 e^{\beta(t-s)}.
$$

Thus, (3.6) holds for some $K \geq 1$. The rest follows easily. \qed

3.2. Sufficient conditions for exponential stability

Now we give sufficient conditions for the exponential stability of an evolution operator $U(t, s) = U(t, s)$, for which we will make use of the Hilbert structure of the space $L^2(\Omega)$ and Lemma 3.1. We therefore consider

$$
\begin{align*}
\begin{cases}
    u_t - \Delta u = C(t, x)u & \text{in } \Omega, \ t > s, \\
    u = 0 & \text{on } \partial \Omega, \ t > s, \\
    u(s) = u_0.
\end{cases}
\end{align*}
$$

(3.7)

In the simplest case, when $C$ does not depend on $t$, i.e. $C(t, x) = C(x)$ and the operator $\Delta + C(x)$ does not depend on time, we know that the semigroup associated with $\Delta + C(x)$ is exponentially stable if and only if the first eigenvalue of

$$
\begin{align*}
\begin{cases}
    -(\Delta + C(x))u = \lambda u & \text{in } \Omega, \\
    u = 0 & \text{on } \partial \Omega
\end{cases}
\end{align*}
$$

is positive.

To treat the time-dependent case, we therefore take $X = L^2(\Omega)$ and for any fixed $t \in \mathbb{R}$, consider the first eigenvalue of

$$
\begin{align*}
\begin{cases}
    -\Delta u - C(t, x)u = \lambda(t)u & \text{in } \Omega, \\
    u = 0 & \text{on } \partial \Omega
\end{cases}
\end{align*}
$$
which satisfies
\[
\int_{\Omega} (|\nabla \varphi|^2 - C(t, x)|\varphi|^2) \, dx \geq \lambda_1(t) \|\varphi\|^2.
\] (3.8)
for all smooth functions $\varphi$ vanishing on $\partial \Omega$, where we have denoted by $\| \cdot \|$ the norm in $L^2(\Omega)$.

Multiplying the first equation in (3.7) by $u(t)$ and integrating in $\Omega$, we have
\[
\frac{d}{dt} \|u(t)\|^2 + \int_{\Omega} (|\nabla u|^2 - C(t, x)|u|^2) \, dx = 0.
\]
By (3.8) we have
\[
\frac{d}{dt} \|u(t)\|^2 + \lambda_1(t) \|u(t)\|^2 \leq 0
\]
and by Gronwall’s lemma
\[
\|u(t)\|^2 \leq e^{-\int_s^t \lambda_1(r) \, dr} \|u(s)\|^2.
\]
Exponential stability is therefore guaranteed provided that, for some $R, \beta > 0$ and $t \geq R, s \leq -R$, with $R$ large enough, we have
\[
\int_s^t \lambda_1(r) \, dr \geq 2\beta
\]
which, in turn, is satisfied if
\[
\liminf_{t \to \pm \infty} \lambda_1(t) > 0.
\]
We have thus proved:

**Lemma 3.3.** Let $C \in C^\alpha(\mathbb{R}, L^p(\Omega))$ with $0 < \alpha \leq 1$ and $p > N/2$, $p > 1$. Suppose that
\[
\liminf_{t \to \pm \infty} \lambda_1(t) > 0
\]
where $\lambda_1(t)$ is the first eigenvalue of the problem
\[
\begin{cases}
-\Delta u - C(t, x)u = \lambda(t)u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]
Then $\Delta + C(t, x)$ generates an exponentially stable evolution operator in $L^q(\Omega)$ for all $1 \leq q \leq \infty$. 

\[\text{J.C. Robinson et al. / J. Differential Equations 238 (2007) 289–337 299} \]
3.3. Persistence of exponential stability under perturbation

We now turn our attention to perturbations of the evolution operators $U = U_C$ defined by the solutions of (3.1) in $L^q(\Omega)$, $1 \leq q \leq \infty$. Our goal is to estimate the effects of the perturbation on the exponential type of the resulting evolution operator.

**Proposition 3.4.** Assume that $U = U_C$ is the evolution operator defined by the solutions of (3.1) in $L^q(\Omega)$, $1 \leq q \leq \infty$, as above, and that there exist $M > 0$ and $\beta \in \mathbb{R}$ such that

$$\|U_C(t, s)\|_{L^q(\Omega)} \leq Me^{\beta(t-s)} \text{ for all } t > s. \quad (3.9)$$

Assume that $P \in C^\alpha(\mathbb{R}, L^p(\Omega))$ with $0 < \alpha \leq 1$ and some $p > N/2$, $p > 1$, is a given time-dependent perturbation of $C$, and denote by $P^+$ the positive part of $P$.

(i) If $P^+ \in L^1(\mathbb{R}, L^\infty(\Omega))$ then

$$\|U_{C+P}(t, s)\|_{L^q(\Omega)} \leq K e^{\beta(t-s)} \text{ for all } t > s,$$

for some constant $K$.

(ii) If $P^+ \in L^\sigma(\mathbb{R}, L^p(\Omega))$, with $1 < \sigma < \infty$ and $p > N\sigma/2$, then for every $\varepsilon > 0$ there exists $K_\varepsilon$ such that

$$\|U_{C+P}(t, s)\|_{L^q(\Omega)} \leq K_\varepsilon e^{(\beta+\varepsilon)(t-s)} \text{ for all } t > s.$$

(iii) If $P^+ \in L^\infty(\mathbb{R}, L^p(\Omega))$ then for all $\varepsilon > 0$, there exists $K_\varepsilon$ such that

$$\|U_{C+P}(t, s)\|_{L^q(\Omega)} \leq K_\varepsilon e^{(\beta+\varepsilon)(t-s)} \text{ for all } t > s,$$

for some $\gamma$ which is proportional to $\|P^+\|_{L^\infty(\mathbb{R}, L^p(\Omega))}$ with $\delta = N/2p < 1$.

(iv) If $P^+ \in L^\infty(\mathbb{R}, L^p(\Omega)) \cap L^1(\mathbb{R}, L^p(\Omega))$, $p > N/2$, then for every $\varepsilon > 0$ there exists $K_\varepsilon$ such that

$$\|U_{C+P}(t, s)\|_{L^q(\Omega)} \leq K_\varepsilon e^{(\beta+\varepsilon)(t-s)} \text{ for all } t > s.$$

**Proof.** First we prove that non-positive perturbations do not increase the exponential type of the evolution operator. More precisely, we prove that if $0 \geq P \in C^\alpha(\mathbb{R}, L^p(\Omega))$ with $0 < \alpha \leq 1$ and some $p > N/2$ then

$$\|U_{C+P}(t, s)u_0\| \leq U_C(t, s)\|u_0\|$$

pointwise in $\Omega$ for every $u_0 \in L^q(\Omega)$. To see this note first that if $u_0 \geq 0$ then $U_{C+P}(t, s)u_0 \geq 0$ which implies that $|U_{C+P}(t, s)u_0| \leq U_{C+P}(t, s)\|u_0\|$. Therefore it is enough to prove the claim for non-negative initial data. In such a case, let $u(t, s; u_0) = U_{C+P}(t, s)u_0 \geq 0$ then, since $P \leq 0$ we have, from Theorem 2.13, $0 \leq u(t, s; u_0) \leq U_C(t, s)u_0$ and the claim is proved.

Now let $P$ be as in the statement of the proposition, i.e., $P \in C^\alpha(\mathbb{R}, L^p(\Omega))$ with $0 < \alpha \leq 1$ for some $p > N/2$. Writing $P = P^+ - P^-$ and using the evolution operator $U_{C-P^-}(t, s)$, which
still satisfies (3.9), we have, by the variation of constants formula, that for every \( u_0 \in L^q(\Omega) \) the
solution \( u(t, s; u_0) = U_{C^{-p}}(t, s)u_0 \) satisfies for \( s \leq t_0 \leq t \),
\[
u(t, s; u_0) = U_{C^{-p}}(t - t_0)u(t_0, s; u_0) + \int_{t_0}^{t} U_{C^{-p}}(t, \tau)P^+(\tau)u(\tau, s; u_0) \, d\tau.
\]

**Case (A).** Assume that \( p \geq q' \). Then the term \( P^+(\tau)u(\tau, s; u_0) \) can be estimated, using Hölder’s
inequality, in \( L^r(\Omega) \) with \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \). Hence, using (3.9) and Lemma 3.2, with \( \varepsilon > 0 \), setting \( x = t - s \), \( x_0 = t_0 - s \), \( y = \tau - s \), \( a(y) = \| P^+(y + s) \|_{L^p(\Omega)} \), and denoting \( z(x) = e^{-\beta \varepsilon (t-s)} \| u(t, s; u_0) \|_{L^q(\Omega)} \) we get for every \( 0 \leq x_0 \leq x \)
\[
z(x) \leq M_1 \sigma \int_{x_0}^{x} \frac{M_2}{(x - y)^{\frac{p}{q}}} a(y)z(y) \, dy.
\]
The argument in this case is concluded using the singular Gronwall lemma below, with \( \beta = \frac{N}{2p} < 1 \).

**Case (B).** Assume that \( p < q' \). Now the term \( P^+(\tau)u(\tau, s; u_0) \) can only be estimated (using Hölder’s inequality) in \( L^1(\Omega) \), but since the case \( q = p' \) is included in Case (A) above, we get
\[
\| u(t, s; u_0) \|_{L^q(\Omega)} \leq M e^{\beta (t-s)} \| u_0 \|_{L^q(\Omega)} + \frac{1}{s} \int_{s}^{t} M e^{(\beta + \varepsilon)(t-\tau)} \int_{x_0}^{x} \frac{M_2}{(x - y)^{\frac{p}{q}}} a(y)z(y) \, dy \, d\tau.
\]
and then
\[
\| u(t, s; u_0) \|_{L^q(\Omega)} \leq M_1 e^{\beta (t-s)} \| u_0 \|_{L^q(\Omega)} + M_1 e^{\mu (t-s)} \| u_0 \|_{L^q(\Omega)} \int_{s}^{t} \frac{\| P^+(\tau) \|_{L^p(\Omega)}}{(t - \tau)^{\frac{N}{2p}} (\tau - s)^{\frac{N}{2} \left(\frac{1}{q} - \frac{1}{p} \right)}} \, d\tau,
\]
where \( \mu \) equals \( \beta + 2\varepsilon \) or \( \beta + \varepsilon + \gamma \) according to cases (ii), (iii) or (iv) of the statement. Now the result follows after using Hölder’s inequality and observing that setting \( \tau = s + z(t-s) \) we get
\[
\int_{s}^{t} \frac{d\tau}{(t - \tau)^{\frac{N}{2p}} (\tau - s)^{\frac{N}{2} \left(\frac{1}{q} - \frac{1}{p} \right)}} = (t - s)^{1 - \frac{\sigma'}{q'}} \int_{0}^{1} \frac{dz}{(1 - z)^{\frac{N}{2p}} z^{\frac{N}{2} \left(\frac{1}{q} - \frac{1}{p} \right)}}
\]
with \( \frac{\sigma'}{q'} < 1 \) and \( \frac{\sigma'}{q'} \left(\frac{1}{q} - \frac{1}{p} \right) < 1 \) because \( \frac{\sigma'}{q'} < p < q' \). Therefore,
\[ \|u(t, s; u_0)\|_{L^q(\Omega)} \leq M_1 e^{\mu(t-s)} \|u_0\|_{L^q(\Omega)} (1 + (t-s) \frac{1}{\mu} N^{-\frac{N}{q}} C(\sigma, q, p) \|p^+\|_{L^p(\mathbb{R}_+; L^q(\Omega))}) \]
\[ \leq M_2 e^{(\mu+\varepsilon)(t-s)} \|u_0\|_{L^q(\Omega)}. \]

We now prove the singular Gronwall lemma used above.

**Lemma 3.5 (A singular Gronwall lemma).** Assume that \( a \in L^\infty([0, \infty)) \cap L^\infty_\infty(0, \infty) \) with \( 1 \leq \sigma \leq \infty \) and that \( z(x) \geq 0 \) is a locally bounded function that for every \( 0 \leq x_0 \leq x \) satisfies
\[ z(x) \leq Mz(x_0) + \int_{x_0}^x \frac{a(y)}{(x-y)\beta} z(y) \, dy \] (3.10)
with \( \beta \sigma' < 1 \). Then for \( x \geq 0 \)
\[ 0 \leq z(x) \leq M(\gamma) e^{\gamma x} z(0) \]
where \( \gamma = 0 \) if \( \sigma = 1 \) and \( \beta = 0 \), \( \gamma \) is arbitrarily small if \( 1 < \sigma < \infty \) and \( \beta \sigma' < 1 \), or \( \gamma \) is proportional to \( \|a\|_{L^\infty(0, \infty)}^{\frac{1}{1-\beta}} \) if \( \sigma = \infty \) and \( 0 \leq \beta < 1 \).

In particular, if \( a \in L^\infty([0, \infty)) \cap L^1([0, \infty)) \) and \( 0 \leq \beta < 1 \) then for \( x \geq 0 \)
\[ 0 \leq z(x) \leq M(\gamma) e^{\gamma x} z(0) \]
where \( \gamma \) is arbitrarily small.

**Proof.** Note that the case \( \sigma = 1, \beta = 0 \) reduces to the usual Gronwall lemma and then \( z(x) \leq Mz(0) e^{\int_0^x a(s) \, ds} \) and the result is obvious.

On the other hand the case \( \sigma = \infty \) and \( 0 \leq \beta < 1 \) is a particular case of the singular Gronwall lemma in Henry [12, Lemma 7.1.1, p. 188] which gives \( \gamma = (\|a\|_{L^\infty(0, \infty)} \Gamma(1-\beta))^{\frac{1}{1-\beta}} \).

Therefore, we will consider now the case \( 1 < \sigma < \infty \) and \( \beta \sigma' < 1 \). Note that in this case we can take \( T_0 \) large enough such that \( \|a\|_{L^\sigma(T_0, \infty)} \) is as small as we want. Also, from (3.10) we get that for \( T_0 \leq x_0 \leq x \leq x_0 + T \) we have, denoting \( w(x_0, T) = \sup_{x_0 \leq y \leq x_0 + T} z(y) \) and using Hölder’s inequality
\[ z(x) \leq Mz(x_0) + w(x_0, T) \|a\|_{L^\sigma(x_0, x_0 + T)} \left( \int_{x_0}^x \frac{1}{(x-y)\beta\sigma'} \, dy \right)^{1/\sigma'} \]
\[ \leq Mz(x_0) + w(x_0, T) \delta(T_0, T) \]
where we have set \( \delta(T_0, T) = \|a\|_{L^\sigma(T_0, \infty)} \Gamma(1-\beta) \|T^{1/\sigma'} - T_{\beta, \sigma'} \), for some constant \( C(\beta, \sigma') \).

Now, given \( T_0 \), choose \( T \) such that \( \delta(T_0, T) = \|a\|_{L^\sigma(T_0, \infty)} \Gamma(1-\beta) \|T^{1/\sigma'} - T_{\beta, \sigma'} \| = 1/2 \). Taking the supremum for \( x_0 \leq x \leq x_0 + T \) we get
\[ z(x) \leq w(x_0) \leq 2Mz(x_0) \quad \text{for all } x_0 \leq x \leq x_0 + T. \]
Writing \( x_1 = x_0 + T \) and repeating the process and the estimate above we get a sequence \( x_n = x_0 + nT \) such that
\[
    z(x) \leq (2M)^n z(x_0) \quad \text{for all } x_0 + (n - 1)T \leq x \leq x_0 + nT.
\]
From here it follows that
\[
    z(x) \leq (2M) \frac{z(x_0)}{(2M)^k + 1} \quad \text{for all } x \geq x_0.
\]
Since choosing \( T_0 \) large enough, we can make \( T \) as large as we want, we obtain that for any \( \varepsilon > 0 \)
\[
    z(x) \leq M_1 e^{\varepsilon x} z(x_0) \quad \text{for all } x \geq x_0.
\]
Now, since \( a \in L^\infty_{loc} (0, \infty) \) we can use the estimate for the case \( \sigma = \infty \) on compact sets of \( x \) to get
\[
    z(x_0) \leq M(x_0) e^{k\varepsilon x} z(0)
\]
for some \( k > 0 \). Then
\[
    z(x) \leq C e^{\varepsilon x} z(0) \quad \text{for all } x > 0.
\]
Finally, if \( a \in L^\infty((0, \infty)) \cap L^1((0, \infty)) \) and \( 0 \leq \beta < 1 \) then we can always choose \( \sigma \) such that \( a \in L^\sigma((0, \infty)) \) and \( \beta \sigma' < 1 \) and we are finished. \( \square \)

As a consequence of the above results we get the following corollary which will be of great help below.

**Corollary 3.6.** Under the assumptions of Proposition 3.4, assume furthermore that the evolution operator \( U_C(t, s) \) is exponentially stable in \( L^q(\Omega) \), i.e. that (3.9) is satisfied with \( \beta < 0 \).

(i) If \( P^+ \in L^1(\mathbb{R}, L^\infty(\Omega)) \), or \( P^+ \in L^p(\mathbb{R}, L^p(\Omega)) \) with \( 1 < \sigma < \infty \) and \( p > \frac{N\sigma}{2} \), then the evolution operator \( U = U_{C^+} \) is exponentially stable in \( L^q(\Omega) \).

(ii) If \( P^+ \in L^\infty(\mathbb{R}, L^p(\Omega)) \) with \( p > \frac{N}{2} \), then the evolution operator \( U = U_{C^+} \) is exponentially stable in \( L^q(\Omega) \) provided that

\[
    \beta + \left( M \| P^+ \|_{L^\infty(\mathbb{R}, L^p(\Omega))} \Gamma(1 - \delta) \right)^{1/(1 - \delta)} < 0,
\]

where \( \delta = \frac{N}{2p} < 1 \).

(iii) If \( P^+ \in L^\infty(\mathbb{R}, L^p(\Omega)) \cap L^1(\mathbb{R}, L^p(\Omega)) \) with \( p > \frac{N}{2} \), then the evolution operator \( U = U_{C^+} \) is exponentially stable in \( L^q(\Omega) \).

A close look at the proof above prompts the following remark which will be used below:

**Remark 3.7.** Notice that Proposition 3.4 and Corollary 3.6 remain true if we only assume that
\[
    P^+ \in L^\sigma([x_0, \infty), L^p(\Omega))
\]
for some $s_0 \in \mathbb{R}$ and $\sigma$ and $p$ as in the statements of these results. In this case we obtain the estimate

$$
\|U_{C^+}^s(t, s)\|_{L^p(\Omega)} \leq M_{s_0} e^{(\beta + \gamma)(t-s)} \quad \text{for all } t > s > s_0
$$

where $\gamma$ is arbitrarily small or depends on $\|P^+\|_{L^p([t_0, \infty), L^p(\Omega))}$ according to the cases above.

In order to obtain a constant $M_{s_0}$ independent of $s_0$ we will then need to have a uniform bound on $\|P^+\|_{L^p([t_0, \infty), L^p(\Omega))}$, which requires $P^+ \in L^p(\mathbb{R}, L^p(\Omega))$.

The next corollary gives a result that will be useful for the study of asymptotically autonomous problems.

**Corollary 3.8.** Let $C \in C^\omega(\mathbb{R}, L^p(\Omega))$ with $0 < \omega \leq 1$ and some $p > N/2$, $p > 1$, such that the evolution operator generated by $\Delta + C(t, x)$ is exponentially stable.

(i) If there exist $C^+ \in L^p(\Omega)$ and $T_0 \in \mathbb{R}$ such that $C - C^+ \in L^\omega([T_0, \infty), L^p(\Omega))$ with either $1 \leq \sigma < \infty$ and $p > \frac{N\sigma}{2}$, or $\sigma = \infty$ and $p > \frac{N}{2}$ and

$$
\lim_{t \to \infty} \|C(t) - C^+\|_{L^p(\Omega)} = 0
$$

then the semigroup generated by $\Delta + C^+$ has exponential decay.

(ii) If there exist $C^- \in L^p(\Omega)$ and $T_0 \in \mathbb{R}$ such that $C - C^- \in L^\omega((-\infty, T_0], L^p(\Omega))$ with either $1 \leq \sigma < \infty$ and $p > \frac{N\sigma}{2}$, or $\sigma = \infty$ and $p > \frac{N}{2}$ and

$$
\lim_{t \to -\infty} \|C(t) - C^-\|_{L^p(\Omega)} = 0
$$

then the semigroup generated by $\Delta + C^-$ has exponential decay.

**Proof.** Since the evolution operator $U_C$ is exponentially stable we have, for some $\beta < 0$,

$$
\|U_C(t, s)\|_{L^p(\Omega)} \leq M e^{\beta(t-s)} \quad \text{for all } t > s.
$$

(i) Set $P(t, x) = C^+(x) - C(t, x)$. Our assumptions imply that for $s_0$ large enough the norm $\|P\|_{L^\omega([t_0, \infty), L^p(\Omega))}$ is as small as we want. Therefore from Proposition 3.4 and Remark 3.7 we know that

$$
\|U_{C^+}^s(t, s)\|_{L^p(\Omega)} \leq M_{s_0} e^{(\beta + \gamma)(t-s)} \quad \text{for all } t > s > s_0,
$$

for arbitrarily small $\gamma$.

Since $C(t, x) + P(t, x) = C^+(x)$ we know that $T_{C^+}(t) = U_{C^+}(t + s_0, s_0)$, $t > 0$, is an autonomous evolution operator, i.e. a semigroup which is actually the semigroup generated by $\Delta + C^+$. Hence, from (3.11), $T_{C^+}(t)$ has exponential decay.

(ii) Set $P(t, x) = C^-(x) - C(t, x)$. Our assumptions now imply that for $t_0$ sufficiently negative the norm $\|P\|_{L^\omega([t_0, \infty), L^p(\Omega))}$ is as small as we want. Therefore from Proposition 3.4 and Remark 3.7 it follows that (3.11) holds for $s_0 \leq s < t \leq t_0$ with arbitrarily small $\gamma$. 

As before, since \( C(t, x) + P(t, x) = C_-(x) \) the semigroup generated by \( \Delta + C_- \) satisfies \( T_{C_-}(t) = U_{C_-P}(t_0, t) \). Hence, from (3.11), we can find \( t \) such that

\[
\| T_{C_-}(t) \|_{\mathcal{L}(L^q(\Omega))} < 1
\]

and once more obtain exponential decay. \( \square \)

4. Complete trajectories for the linear problem

We consider now linear non-autonomous problems.

4.1. The homogeneous case

We begin by studying the homogeneous case. For this, we will consider the following problem

\[
\begin{cases}
  w_t - \Delta w = C(t, x)w & \text{in } \Omega, \; t > s, \\
  w = 0 & \text{on } \partial \Omega, \; t > s, \\
  w(s) = w_0
\end{cases}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) and \( C \in C^\alpha(\mathbb{R}, L^p(\Omega)) \) with \( 0 < \alpha \leq 1 \) and some \( p > N/2 \), \( p > 1 \).

Let \( U_C(t, s) \) be the evolution operator associated with the above problem in \( X = C(\mathbb{T}) \) or in \( X = L^q(\Omega) \) with \( 1 \leq q < \infty \), i.e. \( U_C(t, s)w_0 = w(t, s; w_0) \).

Given \( \beta > 0 \) we define \( \mathcal{D}_\beta = \mathcal{D}_\beta(\mathbb{R}, X) \), the “basin of attraction,” consisting of families of bounded sets that grow slower than \( e^{-\beta t} \) as \( t \to -\infty \), i.e. families of bounded sets of the form \( \{B(t)\} \), such that for some \( \gamma < \beta \) we have

\[
e^{\gamma t} \| B(t) \|_X \to 0 \quad \text{as } t \to -\infty,
\]

where

\[
\| B \|_X := \sup_{b \in B} \| b \|_X.
\]

As remarked before, in a slight abuse of notation we can also include single-valued functions in \( \mathcal{D}_\beta \) via the identity \( \psi \leftrightarrow \{\psi(t)\}_{t \in \mathbb{R}} \). Similarly, we can regard \( \mathcal{D}_\beta \) as containing all fixed bounded sets \( B \), via the identity \( B \leftrightarrow \{B(t)\}_{t \in \mathbb{R}} \), where \( B(t) = B \) for all \( t \in \mathbb{R} \).

Outstandingly we note that although the class \( \mathcal{D}_\beta \) imposes some bound on the growth as \( t \to -\infty \), it imposes no restrictions at all as \( t \to \infty \).

**Theorem 4.1.** Let \( X = C(\mathbb{T}) \) or \( L^q(\Omega) \) with \( 1 \leq q < \infty \). Suppose that the evolution operator \( U_C(t, s) \) for (4.1) is exponentially stable in \( X \), i.e. for some \( \beta > 0 \)

\[
\| U_C(t, s) \|_{\mathcal{L}(X)} \leq M e^{-\beta(t-s)} \quad \text{for all } t > s.
\]

Then the unique complete trajectory for (4.1) in \( \mathcal{D}_\beta \) is the trivial solution. Indeed, \( \mathcal{A} = \{\mathcal{A}(t)\} \), with \( \mathcal{A}(t) = 0 \) is the pullback attractor with respect to \( \mathcal{D}_\beta \).

Moreover, the trivial solution also attracts bounded sets in \( X \) forwards in time.
Proof. It is clear that \( \{0\} \in \mathcal{D}_\beta \) is a complete trajectory for (4.1) so we only have to prove the uniqueness. Let \( \psi \in \mathcal{D}_\beta \) be a complete trajectory. Then

\[
\psi(t) = U_C(t, s)\psi(s) \quad \text{for all } t \geq s
\]

and if we take norms in the above expression, for some \( \gamma < \beta \) we have

\[
\|\psi(t)\|_X \leq \|U_C(t, s)\|_{L(X)}\|\psi(s)\|_X \leq M e^{-\beta(t-s)}M_1(t) e^{{\gamma}s}.
\]

Letting \( s \) tend to \(-\infty\) shows that

\[
\|\psi(t)\|_X = 0 \quad \text{for all } t \in \mathbb{R}
\]

since \( \gamma < \beta \). So the unique bounded complete trajectory in \( \mathcal{D}_\beta \) is 0.

Now let \( \{B(s)\}_s \in \mathcal{D}_\beta \). Then, for all \( w_s \in B(s) \),

\[
\|U_C(t, s)w_s\|_X \leq M e^{-\beta(t-s)}\|B(s)\|_X \\
\leq M e^{-\beta(t-s)}M_1(t) e^{{\gamma}s} = M_2(t) e^{(\beta - \gamma)s} \quad (4.3)
\]

for some \( \gamma < \beta \). Thus, taking limits as \( s \to -\infty \) we have

\[
U_C(t, s)w_s \to 0 \quad \text{as } s \to -\infty
\]

for all \( t \in \mathbb{R} \).

The attraction forwards in time follows immediately from the asymptotic stability condition (4.2). \( \square \)

Remark 4.2. To prove the previous theorem in the case of attraction of bounded sets (i.e. with the basin of attraction consisting of families of bounded sets not depending on time) it is not necessary to assume the exponential stability of \( U \). It is enough to suppose that the evolution semigroup decays to zero as \( s \) tends to \(-\infty\), i.e. that

\[
\|U(t, s)\|_{L(X)} \to 0 \quad \text{as } s \to -\infty.
\]

4.2. The inhomogeneous problem

We now consider the following linear inhomogeneous problem

\[
\begin{cases}
  v_t - \Delta v = C(t, x)v + D(t, x) & \text{in } \Omega, \ t > s, \\
  v = 0 & \text{on } \partial \Omega, \ t > s, \\
  v(s) = v_s
\end{cases} \quad (4.4)
\]

posed in either \( X = C(\overline{\Omega}) \) or \( X = L^q(\Omega) \) with \( 1 \leq q < \infty \).

Assume that \( C \in C^\alpha(\mathbb{R}, L^p(\Omega)) \) with \( 0 < \alpha \leq 1 \) and \( p > N/2 \), \( p > 1 \), and \( D \in L^1_{\text{loc}}(\mathbb{R}, L^r(\Omega)) \), for some \( \frac{qN}{N+2q} < r \leq \infty \) if \( X = L^q(\Omega) \) or \( r > N/2 \) if \( X = C(\overline{\Omega}) \), respectively.
Then there exists a unique mild solution of (4.4) which is given by the variation of constants formula, i.e.

\[ v(t, s; v_0) = U_C(t, s)v_0 + \int_s^t U_C(t, \tau)D(\tau) \, d\tau. \]  

(4.5)

We will also assume the exponential stability of the evolution operator \( U_C \) associated with \( \Delta + C(t, x) \) as in Section 4.1.

The following result establishes the existence of a unique complete trajectory for (4.4) under two different types of conditions on the behaviour of \( D \) as \( t \to -\infty \). Note that regularity properties of the mild solutions above have been established in [19, Theorem 1.2 and Section 3].

**Theorem 4.3.** Let \( X = C(\bar{\Omega}) \) or \( X = L^q(\Omega) \) with \( 1 \leq q < \infty \). Suppose that the evolution operator \( U_C(t, s) \) is exponentially stable in \( X \), i.e.

\[ \left\| U_C(t, s) \right\|_{\mathcal{L}(X)} \leq Me^{-\beta(t-s)} \quad \text{with} \quad \beta > 0 \quad \text{and} \quad M \geq 1. \]

(i) Assume that

\[ D \in \mathcal{D}_\beta(\mathbb{R}, L^r(\Omega)) \]

with \( \frac{Nq}{N+2q} < r \leq \infty \) if \( X = L^q(\Omega) \), \( 1 \leq q < \infty \), or \( N/2 < r \leq \infty \) if \( X = C(\bar{\Omega}) \).

Then there exists a unique complete trajectory \( \phi \in \mathcal{D}_\beta = \mathcal{D}_\beta(\mathbb{R}, X) \) for (4.4).

(ii) Assume now that for some \( \sigma \) with \( 1 \leq \sigma \leq \infty \)

\[ D \in L^\sigma \left( (-\infty, T), L^r(\Omega) \right) \quad \text{for each} \quad T < \infty \]

or that

\[ D \in L^\sigma \left( \mathbb{R}, L^r(\Omega) \right) \]

(which corresponds to \( T = \infty \) above), for some \( r \) with \( \frac{Nq}{N+4q} < r \leq \infty \) if \( X = L^q(\Omega) \) or with \( N/2 < r \leq \infty \) if \( X = C(\bar{\Omega}) \).

Then there exists a complete trajectory for (4.4), \( \phi \in L^\sigma \left( (-\infty, T), X \right) \cap C(\mathbb{R}, X) \), for each \( T < \infty \) (or \( \phi \in L^\sigma (\mathbb{R}, X) \cap C(\mathbb{R}, X) \) if \( T = \infty \)).

Assume in addition either that \( 1 < \sigma \leq \infty \) and \( \frac{N\sigma q}{N\sigma + 2\sigma q} < r \leq \infty \), if \( X = L^q(\Omega) \), or \( N\sigma q/2 < r \leq \infty \), if \( X = C(\bar{\Omega}) \); or that \( \sigma = 1 \) and \( q \leq r \leq \infty \), if \( X = L^q(\Omega) \), or \( r = \infty \), if \( X = C(\bar{\Omega}) \); then \( \phi \in C_b((-\infty, T), X) \subset \mathcal{D}_\beta \) (or \( \phi \in C_b(\mathbb{R}, X) \subset \mathcal{D}_\beta \) if \( T = \infty \)) and is the unique complete trajectory within this class.

In either one of the cases above in which the complete trajectory \( \phi \in \mathcal{D}_\beta \), the family \( A = \{ A(t) \} = \{ \phi(t) \} \) is the pullback attractor for (4.4) with respect to \( \mathcal{D}_\beta \).

\( \{ \phi(t) \} \) also attracts bounded sets of \( X \) forwards in time. More precisely, for every bounded set \( B \subset X \) we have

\[ \left\| v(t, s; v_0) - \phi(t) \right\|_X \leq Ke^{-\beta(t-s)}, \quad t > s, \]

(4.6)

for all \( v_0 \in B \), where \( K = K(B) \).
**Proof.** First, we prove the existence of a complete trajectory for (4.4). We set

$$\phi(t) = \int_{-\infty}^{t} U_C(t, \tau) D(\tau) \, d\tau.$$  

(4.7)

If \( \phi(t) \) is well defined then it is a complete trajectory for (4.4) since, given \( t \geq s \),

$$\phi(t) - U_C(t, s) \phi(s) = \int_{-\infty}^{t} U_C(t, \tau) D(\tau) \, d\tau - \int_{-\infty}^{t} U_C(s, \tau) D(\tau) \, d\tau$$

$$= \int_{s}^{t} U_C(t, \tau) D(\tau) \, d\tau$$  \hspace{1cm} (4.8)

and moreover in such a case we will automatically have \( \phi \in C(\mathbb{R}, X) \). We will show below that \( \phi(t) \) is well defined and belongs to \( \mathcal{D}_\beta = \mathcal{D}_\beta(\mathbb{R}, X) \). For now we assume that this has been proved.

Let \( B = \{B(s)\}_{s} \in \mathcal{D}_\beta \) and fix \( \{v_{t}\}_{t} \in B \). Then the solution of (4.4) is given by the variation of constants formula (4.5). Let \( w(t, s; v_{t}) = v(t, s; v_{t}) - \phi(t) \). Then \( w \) solves the homogeneous problem

$$\begin{cases}
\phi_{t} - \Delta w = C(t, x)w & \text{in } \Omega, \ t > s, \\
w(s) = v_{s} - \phi(s), \\
w = 0 & \text{on } \partial \Omega.
\end{cases}$$  \hspace{1cm} (4.9)

So, since \( \{B(s) - \phi(s)\}_{s} \in \mathcal{D}_\beta \), from Theorem 4.1 we have

$$w(t, s; v_{t}) \to 0 \quad \text{as } s \to -\infty$$

uniformly for \( v_{t} \in B(s) \), where \( \{B(s)\}_{s} \in \mathcal{D}_\beta \). Thus, for all \( t \in \mathbb{R} \)

$$v(t, s; v_{t}) \to \phi(t) \quad \text{as } s \to -\infty.$$  

So we have proved that \( \mathcal{A} = \{\phi(t)\} \) is the pullback attractor.

Notice that if we fix \( s \in \mathbb{R} \) and a bounded set \( B \subset X \) we have

$$\left\| v(t, s; v_{0}) - \phi(t) \right\|_{X} \leq Ke^{-\beta(t-s)} \left\| v_{0} - \phi(s) \right\|_{X} \leq K_{1}e^{-\beta(t-s)} \to 0$$  \hspace{1cm} (4.10)

as \( t \to +\infty \), for all \( v_{0} \in B \), where \( K_{1} \) depends on the bounded set \( B \). Hence \( \phi(t) \) also attracts bounded sets of \( X \) forwards in time.

We now prove that \( \phi(t) \) is well defined.

(i) Since \( \frac{Nq}{N+2q} < r \leq \infty \) if \( X = L^{q}(\Omega) \), \( 1 \leq q \leq \infty \), or \( N/2 < r \leq \infty \), if \( X = C(\overline{\Omega}) \) and \( D \in \mathcal{D}_{\beta}(\mathbb{R}, L^{r}(\Omega)) \), then, in (4.7) we get, for each \( t \leq t_{0} \), and for some \( \gamma < \beta \),

$$e^{\gamma t} \phi(t) = \int_{-\infty}^{t} e^{\gamma (t-\tau)} U_C(t, \tau) e^{\gamma \tau} D(\tau) \, d\tau.$$
Using (3.2) we get

$$e^{\gamma t} \left\| \phi(t) \right\|_X \leq M \sup_{\tau \leq t} e^{\gamma \tau} \left\| D(\tau) \right\|_{L^p(\Omega)} \int_{-\infty}^{t} (t - \tau)^{-\frac{N}{2}} \left( \frac{1}{r} - \frac{1}{q} \right) e^{-(\beta - \gamma)(t - \tau)} \, d\tau$$

with $1 < q \leq \infty$.

Now, since $r > \frac{qN}{N + 2q}$, if $1 \leq q < \infty$, or $N/2 < r \leq \infty$ if $q = \infty$, we have that $\frac{N}{2} \left( \frac{1}{r} - \frac{1}{q} \right) < 1$ and, therefore, the integral term above is bounded independently of $t$.

Hence $\phi \in D_\beta$ and, by Theorem 4.1, $\phi$ is the unique complete trajectory in $D_\beta$.

(ii) We assume now $D \in L^\sigma((-\infty, T), L^p(\Omega))$ for each $T < \infty$ (or even $T = \infty$ if $D \in L^\sigma(\mathbb{R}, L^p(\Omega))$) and distinguish below several cases.

**Case (a).** Suppose that $D \in L^\sigma((-\infty, T), L^p(\Omega))$, $1 \leq \sigma \leq \infty$ and $q \leq r \leq \infty$, if $X = L^q(\Omega)$, $1 \leq q < \infty$, or $1 \leq \sigma \leq \infty$ and $r = \infty$ if $X = C(\Omega)$, respectively.

We start with the case $\sigma = \infty$. Then, from (4.7), for $t < T$,

$$\left\| \phi(t) \right\|_X \leq \limsup_{s \to -\infty} \int_{s}^{t} \left\| U_C(t, \tau) D(\tau) \right\|_X \, d\tau$$

$$\leq \limsup_{s \to -\infty} \frac{M}{\beta} (1 - e^{-\beta(\tau - t)}) \sup_{\tau \leq t} \left\| D(\tau) \right\|_X$$

$$\leq \frac{M}{\beta} \left\| D \right\|_{L^\infty((-\infty, T), X)}.$$  \hfill (4.11)

Thus $\phi \in L^\infty((-\infty, T), X)$.

Now we prove the result in the case $\sigma = 1$, i.e. $D \in L^1((-\infty, T), X)$. From (4.7), we get

$$\left\| \phi \right\|_{L^1((-\infty, T), X)} = \int_{-\infty}^{T} \left\| \phi(t) \right\|_X \, dt \leq \int_{-\infty}^{T} \int_{-\infty}^{t} \left\| U_C(t, \tau) D(\tau) \right\|_X \, d\tau \, dt$$

$$\leq \int_{-\infty}^{T} \int_{-\infty}^{t} M e^{-\beta(\tau - t)} \left\| D(\tau) \right\|_X \, d\tau \, dt$$

$$\leq \int_{-\infty}^{T} \int_{-\infty}^{T} M e^{-\beta(\tau - t)} \left\| D(\tau) \right\|_X \, dt \, d\tau$$

$$\leq C \int_{-\infty}^{T} \left\| D(\tau) \right\|_X \, d\tau = C \left\| D \right\|_{L^1((-\infty, T), X)}$$

where we have used Fubini’s Theorem and the boundedness of $\int_{-\infty}^{T} M e^{-\beta(t - \tau)} \, d\tau$ independent of $\tau$ and $T \leq \infty$. Thus, $\phi \in L^1((-\infty, T), X)$. 

Now from the interpolation theorem for $L^p$-spaces (see Theorem 5.2.3, p. 111, in Bergh and Lofstrom [6]) we have that if $D \in L^\sigma((-\infty, T), X)$, $1 \leq \sigma \leq \infty$, then $\phi \in L^\sigma((-\infty, T), X)$ and

$$\|\phi\|_{L^\sigma((-\infty, T), X)} \leq C \|D\|_{L^\sigma((-\infty, T), X)}.$$ 

Finally, we prove that if we take $D \in L^\sigma((-\infty, T), X)$, $1 \leq \sigma < \infty$, then $\phi \in L^\infty((-\infty, T), X)$. Indeed, let $1 < \sigma < \infty$ then, from expression (4.7), using Holder’s inequality, we get for $t < T$,

$$\|\phi(t)\|_X \leq \limsup_{s \to -\infty} \int_s^t \|U_C(t, \tau)D(\tau)\|_X \, d\tau$$

$$\leq M \limsup_{s \to -\infty} \left( \frac{1}{\beta\sigma'} \left[ 1 - e^{-\beta\sigma'(t-\tau)} \right] \right)^{1/\sigma'} \left( \int_{-\infty}^t \|D(\tau)\|_X^\sigma \, d\tau \right)^{1/\sigma}$$

$$\leq \frac{M}{(\beta\sigma')^{1/\sigma'}} \|D\|_{L^\sigma((-\infty, T), X)}. \tag{4.12}$$

Hence $\phi \in L^\infty((-\infty, T), X)$. The case $\sigma = 1$ is proved in an analogous way.

**Case (b).** Let $X = L^q(\Omega)$, $1 \leq q < \infty$. Suppose that $D \in L^\sigma((-\infty, T), L^r(\Omega))$, $1 \leq \sigma \leq \infty$, $\frac{N}{\sigma + 2q} < r < q$. In this case, we need to use $L^p$-$L^q$ smoothing estimates for the evolution operator, see (3.2).

We start with the case $\sigma = \infty$. Suppose that $D \in L^\infty((-\infty, T), L^r(\Omega))$ with $\frac{N}{\sigma + 2q} < r < q$. Then, for $t < T$,

$$\|\phi(t)\|_{L^r(\Omega)} \leq \int_{-\infty}^t \|U_C(t, \tau)D(\tau)\|_{L^r(\Omega)} \, d\tau$$

$$\leq M \sup_{\tau \leq t} \|D(\tau)\|_{L^r(\Omega)} \int_{-\infty}^t (t - \tau)^{-\frac{N}{2} \left( \frac{1}{r} - \frac{1}{q} \right)} e^{-\beta(\tau-t)} \, d\tau \tag{13.14}$$

where we have used (3.2). Now, since $r > \frac{qN}{N + 2q}$ we have $\frac{N}{2} \left( \frac{1}{r} - \frac{1}{q} \right) < 1$ and, therefore, the integral term above is bounded independently of $t$. Thus, from (13.14),

$$\|\phi\|_{L^\infty((-\infty, T), L^r(\Omega))} \leq C \|D\|_{L^\infty((-\infty, T), L^r(\Omega))}.$$ 

Now assume that $\sigma = 1$, i.e. that $D \in L^1((-\infty, T), L^r(\Omega))$. Then, using (3.2) as in (4.13), we have

$$\|\phi\|_{L^1((-\infty, T), L^r(\Omega))} \leq \int_{-\infty}^T \int_{-\infty}^t (t - \tau)^{-\frac{N}{2} \left( \frac{1}{r} - \frac{1}{q} \right)} Me^{-\beta(\tau-t)} \|D(\tau)\|_{L^r(\Omega)} \, d\tau \, dt.$$
\[
\begin{align*}
&= \int_T^{-\infty} \int_0^{T-\tau} M s^{-\frac{\sigma^N (1 - \frac{1}{2})}{r} e^{-\beta s} \mathrm{d}t} \| D(\tau) \|_{L^r(\Omega)} \mathrm{d}\tau \\
&\leq C \int_{-\infty}^T \| D(\tau) \|_{L^r(\Omega)} \mathrm{d}\tau = C \| D \|_{L^1((-\infty, T), L^r(\Omega))}
\end{align*}
\]

where we have used Fubini’s Theorem and the boundedness of \( \int_0^\infty s^{-\frac{\sigma^N (1 - \frac{1}{2})}{r} e^{-\beta s} \mathrm{d}s} \), since \( r > \frac{N\sigma}{N\sigma + 2q} \).

Again, the result in the case \( D \in L^r((-\infty, T), L^r(\Omega)) \), \( 1 < \sigma < \infty \), follows from the interpolation theorem for \( L^p \) spaces as in Case (a) above.

Finally, we show that if \( D \in L^r((-\infty, T), L^r(\Omega)) \), \( 1 < \sigma < \infty \), \( \frac{N\sigma q}{N\sigma + 2q} < r < q \), then \( \phi \in L^\infty((-\infty, T), L^q(\Omega)) \). Indeed, let \( 1 < \sigma < \infty \) then, as in (4.13), using the Hölder inequality, we get for \( t < T \),

\[
\| \phi(t) \|_{L^q(\Omega)} \leq \limsup_{s \to -\infty} \int_s^t \| U_C(t, \tau) D(\tau) \|_{L^r(\Omega)} \mathrm{d}\tau \\
&\leq \left( \int_0^\infty s^{-\frac{\sigma^N (1 - \frac{1}{2})}{r} e^{-\beta s} \mathrm{d}s} \right)^{1/\sigma} \left( \int_{-\infty}^t \| D(\tau) \|_{L^r(\Omega)}^{\sigma} \mathrm{d}\tau \right)^{1/\sigma}
\]

\[
\leq C \| D \|_{L^r((-\infty, T), L^r(\Omega))}
\]

where we have used that \( \frac{\sigma^N (1 - \frac{1}{2})}{r} < 1 \) since \( r > \frac{N\sigma q}{N\sigma + 2q} \) Thus, \( \phi \in L^\infty((-\infty, T), L^q(\Omega)) \).

**Case (c).** Let \( X = C(\Omega) \). Assume \( D \in L^p((-\infty, T), L^r(\Omega)) \) with \( 1 \leq \sigma \leq \infty \), \( r > N/2 \). This case follows as in Case (b) with \( q = \infty \) and, of course, \( r = \infty \) if \( \sigma = 1 \) or \( r > N\sigma/2 \) if \( 1 < \sigma \leq \infty \). \( \square \)

**Remark 4.4.** In the first part of case (ii) of the theorem we prove the existence of a complete trajectory \( \phi \in L^q((-\infty, T), X) \cap C(\mathbb{R}, X) \). In particular, this complete trajectory can be unbounded in time. Furthermore, \( \phi(t) \) can grow very fast as \( t \to -\infty \) and may be even not belong to \( D\beta \). For this reason we cannot prove the uniqueness of such a \( \phi(t) \). Nevertheless, the complete trajectory \( \phi(t) \) is unique in the class \( \phi + D\beta \).

**Remark 4.5.** Observe that with similar arguments as in the proof of the theorem above, if we define \( E_{\alpha} = \{ f \in C(\mathbb{R}, X): e^{-\alpha t} f \in C_b(\mathbb{R}, X) \} \), with \( \beta > \alpha > 0 \), one can show that there exists a unique complete trajectory in \( E_{\alpha} \) provided that \( D \in E_{\alpha} \). In such a case we have to restrict the basin of attraction \( D \) to families of bounded sets in \( D\beta-\alpha \).

Considering only pullback attraction it is enough to work in

\[
E^-_{\alpha} = \{ f \in C(\mathbb{R}, X): e^{\alpha t} f \in C_{b}((-\infty, \tau), X) \text{ for some } \tau \in \mathbb{R} \},
\]
and analogously if we consider forward attraction we can use

\[ E^+_t = \{ f \in C(\mathbb{R}, X) : e^{-at} f \in C_b((t, \infty), X) \text{ for some } t \in \mathbb{R} \}. \]

4.3. Asymptotic behaviour as \( t \to \pm \infty \)

Given the above theorem it is natural to consider the asymptotic behaviour of the complete trajectory as \( t \to \pm \infty \). In fact with a closer look at the above proof we can show that, in the cases that \( \phi(t) \) remains bounded and integrable as \( t \to \pm \infty \), it actually converges to zero:

**Corollary 4.6.** Let \( X = C(\Omega) \) or \( L^q(\Omega) \) with \( 1 \leq q < \infty \). Suppose that the evolution operator associated with \( \Delta + C(t, x) \) is exponentially stable.

(i) Assume that \( D \in L^q((-\infty, T), L^r(\Omega)) \), for \( T < \infty \), with either \( 1 < \sigma < \infty \) \( \frac{N \sigma' q}{N^2 + 2q} < r \leq \infty \), or \( \sigma = 1 \) and \( q \leq r \leq \infty \), or \( \sigma = \infty \) and \( \frac{N q}{N^2 + 2q} < r \leq \infty \). In the case \( \sigma = \infty \) assume in addition that

\[ \lim_{t \to -\infty} \| D(t) \|_{L^r(\Omega)} = 0. \]

Then \( \phi(t) \to 0 \) in \( X \) as \( t \to -\infty \).

(ii) Assume that \( D \) satisfies the assumptions in Theorem 4.3 and also that \( D \in L^q((T, \infty), L^r(\Omega)) \), for \( T > -\infty \), with \( \sigma \) and \( r \) as in case (i) above. Assume in addition that, if \( \sigma = \infty \),

\[ \lim_{t \to \infty} \| D(t) \|_{L^r(\Omega)} = 0. \]

Then \( \phi(t) \to 0 \) in \( X \) as \( t \to \infty \). Hence, for every bounded set \( B \subset X \), we have

\[ \psi(t, s; \psi_0) \to 0 \quad \text{in } X \text{ as } t \to \infty \]

uniformly for \( \psi_0 \in B \).

**Proof.** Case (i) is a direct consequence of inequalities (4.11)–(4.14).

Therefore it remains to prove that \( \phi(t) \to 0 \) in \( X \) as \( t \to \infty \). In such a case, the rest of the result is a consequence of (4.6). Hence, note that for any solution of (4.4) we have (see (4.5))

\[ \psi(t, s; \psi_0) = \psi(t, s) = U_C(t, s) \psi_0 + \int_s^t U_C(t, \tau) D(\tau) \, d\tau \]

and since the linear evolution operator is exponentially stable, the first term tends to zero as \( t \to \infty \). For the integral term, let \( t > T > s \) to be fixed later. Then

\[ \left\| \int_s^t U_C(t, \tau) D(\tau) \, d\tau \right\|_X \leq \int_s^T \| U_C(t, \tau) D(\tau) \|_X \, d\tau + \int_T^t \| U_C(t, \tau) D(\tau) \|_X \, d\tau. \]
Case (a). Suppose that either $X = L^q(\Omega)$, $1 \leq q < \infty$, and $1 \leq \sigma \leq \infty$, $r \geq q$; or $X = C(\overline{\Omega})$, $1 \leq \sigma \leq \infty$ and $r = \infty$.

Suppose $\sigma = \infty$. Given $\epsilon > 0$, on the one hand we have

\[
\int_0^T \left\| U_C(t, \tau) D(\tau) \right\|_X d\tau \leq \int_0^T M e^{-\beta(t-\tau)} \left\| D(\tau) \right\|_X d\tau \leq \frac{M}{\beta} \left( 1 - e^{-\beta(t-T)} \right) \text{ess sup}_{\tau \geq T} \left\| D(\tau) \right\|_X < \frac{\epsilon}{2} \tag{4.15}
\]

choosing $T$ large enough. On the other hand,

\[
\int_0^T \left\| U_C(t, \tau) D(\tau) \right\|_X d\tau \leq \int_0^T M e^{-\beta(t-\tau)} \left\| D(\tau) \right\|_X d\tau = \frac{M}{\beta} e^{-\beta(t-T)} \left( 1 - e^{-\beta(T-t)} \right) \left\| D \right\|_{L^\infty((t, \infty), X)} < \frac{\epsilon}{2} \tag{4.16}
\]

choosing $t$ large enough. Thus, for all $t$ large enough we have

\[
\left\| \phi(t) \right\|_X < \epsilon,
\]

i.e. $\phi(t) \to 0$ in $X$ as $t \to \infty$.

In the case $D \in L^\sigma((T, \infty), X)$, $1 < \sigma < \infty$, arguing as in (4.12) in the proof of Theorem 4.3, we have

\[
\int_0^T \left\| U_C(t, \tau) D(\tau) \right\|_X d\tau \leq M \left( \frac{1 - e^{-\beta \sigma'(t-T)}}{\beta \sigma'} \right)^{1/\sigma'} \left\| D \right\|_{L^\sigma((t, \infty), X)} < \frac{\epsilon}{2}
\]

and

\[
\int_0^T \left\| U_C(t, \tau) D(\tau) \right\|_X d\tau \leq M e^{-\beta(t-T)} \left( \frac{1 - e^{-\beta \sigma'(T-t)}}{\beta \sigma'} \right)^{1/\sigma'} \left\| D \right\|_{L^\sigma((t, \infty), X)} < \frac{\epsilon}{2}
\]

for $T$ and $t$ large enough.

The case $\sigma = 1$ follows in an analogous way.

Case (b). Suppose that $X = L^q(\Omega)$, $1 \leq q < \infty$, and $1 \leq \sigma \leq \infty$, $\frac{Nq}{N+2q} < r < q$.

Suppose that $\sigma = \infty$. In such a case, $r > \frac{Nq}{N+2q}$. Arguing as in (4.13) in the proof of Theorem 4.3, we have that for $T$ large and $t \to \infty$
\[
\int t \int \| U_C(t, \tau) D(\tau) \|_{L^s(\Omega)} \, d\tau \\
\leq \text{ess sup}_{t \geq T} \| D(\tau) \|_{L^t(\Omega)} \int t \int \frac{e^{-\beta(t-\tau)}}{(t-\tau)^\frac{3}{2}(-\frac{1}{q})} \, d\tau < \epsilon \\
\]

and

\[
\int s \int \| U_C(t, \tau) D(\tau) \|_{L^s(\Omega)} \, d\tau \\
\leq M(t-T)^{-\frac{N}{2}(-\frac{1}{2}(-\frac{1}{q})\sigma')} \int s \int e^{-\beta(t-\tau)} \| D(\tau) \|_{L^t(\Omega)} \, d\tau \\
\leq \frac{K}{\beta} e^{-\beta(t-T)} \left(1 - e^{-\beta(T-t)}\right) \| D \|_{L_\infty((s, \infty), L^t(\Omega))} < \epsilon
\]

for some constant \( K > 0 \).

Suppose now that \( 1 < \sigma < \infty \). Again, arguing as in (4.14) in the proof of Theorem 4.3, we have that for \( T \) and \( t \) large enough

\[
\int t \int \| U_C(t, \tau) D(\tau) \|_{L^s(\Omega)} \, d\tau \\
\leq M \left( \int t \int \frac{e^{-\beta\sigma'(t-\tau)}}{(t-\tau)^{\frac{N}{2}(-\frac{1}{2}(-\frac{1}{q})\sigma')}} \, d\tau \right)^{1/\sigma'} \| D \|_{L^{\sigma'((T,\infty), L^t(\Omega))}} < \epsilon
\]

and

\[
\int s \int \| U_C(t, \tau) D(\tau) \|_{L^s(\Omega)} \, d\tau \\
\leq M(t-T)^{-\frac{N}{2}(-\frac{1}{2}(-\frac{1}{q})\sigma')} \left( \int s \int e^{-\beta\sigma'(t-\tau)} \, d\tau \right)^{1/\sigma'} \| D \|_{L^{\sigma'((s, \infty), L^t(\Omega))}} \\
\leq Ke^{-\beta(t-T)} \left(1 - e^{-\beta\sigma'(T-t)}\right) \frac{1}{\beta\sigma'} \| D \|_{L_\infty((s, \infty), L^t(\Omega))} < \epsilon
\]

for some constant \( K > 0 \).

**Case (c).** Let \( X = C(\mathbb{T}) \) and assume that \( D \in L^\sigma((T, \infty), L^t(\Omega)) \) with \( 1 \leq \sigma \leq \infty \), \( r > N/2 \). This case follows as in Case (b) with \( q = \infty \) and, of course, \( r = \infty \). \( \square \)
Now, by combining the arguments from the proofs of Theorem 4.3 and Corollary 4.6, we obtain the following result.

**Corollary 4.7.** Let $X = C(\Omega)$ or $X = L^q(\Omega)$ with $1 \leq q < \infty$. Suppose that the evolution operator associated with $\Delta + C(t, x)$ is exponentially stable.

Assume $D_0$ is such that Theorem 4.3 applies and denote by $\phi_0$ the corresponding complete trajectory. Assume in addition that $D = D_0$ is such that Corollary 4.6 also applies.

Then (4.4) has a complete trajectory, $\phi$, and $\phi - \phi_0 \to 0$ as $t$ tends either to $+\infty$ or to $-\infty$, according to the cases in Corollary 4.6.

Note that this corollary can be applied for example if $D_0 \in D_\beta(\mathbb{R}, L^p(\Omega))$ while $D - D_0 \in L^q((-\infty, T), L^r(\Omega))$. In such a case $D$ might not satisfy the assumptions in Theorem 4.3.

### 4.4. The periodic problem

Finally we consider the $T$-periodic problem associated with (4.4), i.e. we suppose that $C(t, x)$ and $D(t, x)$ are $T$-periodic functions. In this case the unique complete trajectory given by Theorem 4.3 is $T$-periodic.

**Corollary 4.8.** Let $X = L^q(\Omega)$, $1 \leq q < \infty$, or $X = C(\Omega)$. Assume $C \in C^\alpha(\mathbb{R}, L^p(\Omega))$ with $0 < \alpha \leq 1$ and some $p > N/2$, the evolution operator associated with $\Delta + C(t, x)$ is exponentially stable and

$$D \in L^\infty(\mathbb{R}, L^r(\Omega))$$

for some $\frac{Nq}{N+r} < r \leq \infty$ if $X = L^q(\Omega)$ or $N/2 < r \leq \infty$ if $X = C(\Omega)$.

If $C(t, x)$ and $D(t, x)$ are $T$-periodic functions then the unique complete trajectory $\phi \in D_\beta$ for (4.4) is $T$-periodic.

**Proof.** Note that the hypotheses in Theorem 4.3 hold since $D \in D_\beta(\mathbb{R}, L^r(\Omega))$. Let $\phi \in D_\beta$ be the unique complete trajectory given by Theorem 4.3. Then,

$$\phi_t(t) - \Delta \phi(t) = C(t, x)\phi(t) + D(t, x)$$

and, by the periodicity of $C$ and $D$ we have

$$\phi_t(t) - \Delta \phi(t) = C(t + T, x)\phi(t) + D(t + T, x)$$

which after a change of variables gives

$$\phi_t(t - T) - \Delta \phi(t - T) = C(t, x)\phi(t - T) + D(t, x).$$

So, $w(t) = \phi(t - T)$ is a complete trajectory of the problem (4.4). But, from the Theorem 4.3 this complete trajectory is unique, and so we have $\phi(t) = w(t)$ for all $t \in \mathbb{R}$, that is, $\phi(t) = \phi(t - T)$ for all $t \in \mathbb{R}$. In other words, $\phi$ is $T$-periodic. \qed
5. Asymptotically autonomous and asymptotically periodic linear problems

In this section we study the linear evolution problem

\[
\begin{cases}
    v_t - \Delta v = C(t, x)v + D(t, x) & \text{in } \Omega, \quad t > s, \\
    v(s) = v_0, & \text{on } \partial\Omega \\
    v = 0 & \text{on } \partial\Omega
\end{cases}
\]  \tag{5.1}

where \( C(t, x) \) and \( D(t, x) \) converge in some sense as \( t \to \pm \infty \).

When \( C \) and \( D \) converge to time-independent functions we will show below that under suitable conditions the pullback and forwards asymptotic behaviour of the solutions of (5.1) is described in terms of suitable functions \( \phi^\pm(x) \) which can be characterised as solutions of some elliptic problems.

An analogous result will be proved for the case where \( C \) and \( D \) converge to periodic functions.

**Theorem 5.1.** Let \( X = L^2(\Omega) \), \( 1 \leq q < \infty \), or \( X = C(\overline{\Omega}) \). Suppose also that the evolution operator associated with \( \Delta + C(t, x) \) is exponentially stable.

(i) Assume that there exists \( C^- \in L^p(\Omega) \) for some \( p > N/2, \quad p > 1 \), such that, for every \( T < \infty \),

\[
C - C^- \in L^\sigma((-\infty, T), L^p(\Omega))
\]

with \( p > \frac{N\sigma'}{N - 2q} \), and

\[
\lim_{t \to -\infty} \|C(t) - C^-\|_{L^p(\Omega)} = 0
\]

if \( \sigma = \infty \).

Also, assume that there exists \( D^- \in L^r(\Omega) \) such that, for every \( T < \infty \),

\[
D - D^- \in L^\sigma((-\infty, T), L^r(\Omega))
\]

with either \( 1 < \sigma < \infty, \frac{N\sigma'}{N - 2q} < r \leq \sigma, \sigma = 1 \) and \( q \leq r < \infty \) or \( \sigma = \infty, \frac{Nq}{N + 2q} < r \leq \infty \), and

\[
\lim_{t \to -\infty} \|D(t) - D^-\|_{L^r(\Omega)} = 0.
\]

Then there exists a unique solution \( \phi^- \) of

\[
\begin{cases}
-\Delta \phi^- = C^-(x)\phi^- + D^-(x), \\
\phi^-|_{\partial\Omega} = 0
\end{cases}
\]  \tag{5.2}

and (5.1) has a pullback attractor by the unique complete trajectory for (5.1) \( \mathcal{A} = \{\mathcal{A}(t)\}_{t} = \{\phi(t)\}_{t} \) which satisfies \( \phi(t) \to \phi^- \) in \( X \) as \( t \to -\infty \).

(ii) Assume that \( C \) and \( D \) satisfy the assumptions in Theorem 4.3. In addition assume that there exists \( C^+ \in L^p(\Omega) \) for some \( p > N/2, \quad p > 1 \), such that for \( -\infty < T \),

\[
C \in D_p(\mathbb{R}, L^p(\Omega)) \quad \text{and} \quad C - C^+ \in L^\sigma((T, \infty), L^p(\Omega))
\]
with $p > \frac{N\sigma'}{2}$, and
\[
\lim_{t \to \infty} \| C(t) - C^+ \|_{L^p(\Omega)} = 0
\]
if $\sigma = \infty$.

Also, assume that there exists a $D^+ \in L^r(\Omega)$ such that, for $-\infty < T$,
\[
D \in D_\beta([\mathbb{R}, L^r(\Omega)]) \quad \text{and} \quad D - D^+ \in L^s((-T, \infty), L^r(\Omega))
\]
with either $1 < \sigma < \infty$, $\frac{N\sigma'q}{N\sigma' + 2q} < r \leq \infty$, $\sigma = 1$ and $q \leq r \leq \infty$ or $\sigma = \infty$, $\frac{Nq}{N + 2q} < r \leq \infty$, and
\[
\lim_{t \to \infty} \| D(t) - D^+ \|_{L^r(\Omega)} = 0.
\]

Then there exists a unique solution $\phi^+$ of
\[
\begin{cases}
-\Delta \phi^+ = C^+(x)\phi^+ + D^+(x), \\
\phi^+ |_{\partial \Omega} = 0
\end{cases}
\]
and for every bounded set $B \subset X$ we have $v(t, s, u_0) \to \phi^+$ in $X$ as $t \to \infty$, uniformly for $u_0 \in B$. Moreover, there exists a pullback attractor $\mathcal{A}$ given by $\mathcal{A}(t) = \phi(t)$ for all $t \in \mathbb{R}$ where $\phi(t)$ is the unique complete trajectory for (5.1) which satisfies $\phi(t) \to \phi^+$ as $t \to \infty$.

\textbf{Proof.} From Corollary 3.8 we have that $C^\pm$ are such that the semigroups generated by $\Delta + C^\pm(x)$ have exponential decay. Thus, problems (5.2) and (5.3) have unique solutions $\phi^\pm$. Take then any $v_0 \in X$ and let $v(t, s, v_0)$ be the unique solution of (5.1). Then $w = v(t, s, v_0) - \phi^\pm$ satisfies
\[
\begin{cases}
w_t - (\Delta + C(t, x))w = D(t, x) + (\Delta + C(t, x))\phi^\pm = \tilde{D}^\pm(t, x), \\
w(s) = v_0 - \phi^\pm, \\
w|_{\partial \Omega} = 0
\end{cases}
\]
where
\[
\tilde{D}^\pm(t, x) = D(t, x) + (\Delta + C(t, x))\phi^\pm = (D(t, x) - D^\pm(x)) + (C(t, x) - C^\pm(x))\phi^\pm.
\]

Note that, by elliptic regularity, $D^\pm \in L^r(\Omega)$ implies that $\phi^\pm \in L^r(\Omega)$ for all $s$ such that $\frac{1}{r} - \frac{\sigma}{\sigma'} < \frac{1}{r}$ and then, for each $t$, $(C(t) - C^\pm)\phi^\pm \in L^m(\Omega)$ with $\frac{1}{m} = \frac{1}{r} + \frac{1}{p} > \frac{1}{r} - \frac{\sigma}{\sigma'} + \frac{1}{p}$ and since $p > N/2$ we can take $m > r$. Therefore, for each $t$, $\tilde{D}^\pm(t) \in L^r(\Omega)$.

Hence in case (i), note that we have $\tilde{D}^-(t, x) = (D(t, x) - D^-(x)) + (C(t, x) - C^-(x))\phi^-$ and for all $T \in \mathbb{R}$, $\tilde{D}^- \in L^\sigma((-\infty, T), L^r(\Omega))$ with either $1 < \sigma < \infty$, $\frac{N\sigma'q}{N\sigma' + 2q} < r \leq \infty$, $\sigma = 1$ and $q \leq r \leq \infty$ or $\sigma = \infty$, $\frac{Nq}{N + 2q} < r \leq \infty$, and
\[
\lim_{t \to -\infty} \| D^-(t) \|_{L^r(\Omega)} = 0.
\]
Then by part (i) of Corollary 4.6 we get the result.

For case (ii) we have \( \hat{D}^+(t, x) = (D(t, x) - D^+(x)) + (C(t, x) - C^+(x))\phi^+ \) which satisfies \( \hat{D}^+ \in D_p(\mathbb{R}, L^r(\Omega)) \) and for all \( T \in \mathbb{R}, \hat{D}^+ \in L^q((T, \infty), L^r(\Omega)) \) with either \( 1 < \sigma < \infty, \frac{N\sigma q}{N\sigma + 2q} < r \leq \infty, \sigma = 1 \) and \( q \leq r \leq \infty \) or \( \sigma = \infty, \frac{Nq}{N + 2q} < r \leq \infty \), and

\[
\lim_{t \to -\infty} \left\| \hat{D}^+(t) \right\|_{L^r(\Omega)} = 0.
\]

Hence part (ii) of Corollary 4.6 gives the result. \( \Box \)

Analogously, for the case of asymptotically periodic problems, we have the following result.

**Theorem 5.2.** Let \( X = C(\mathbb{R}) \) or \( X = L^q(\Omega) \) with \( 1 \leq q < \infty \).

Suppose that the evolution operators associated with \( \Delta + C(t, x) \) and \( \Delta + C^+(t, x) \) are exponentially stable, where \( C^\pm \in C^\alpha(\mathbb{R}, L^p(\Omega)) \), with \( 0 < \alpha \leq 1 \) and some \( p > N/2, p > 1 \), are \( T \)-periodic functions.

In addition assume that \( D^\pm \in L^\infty(\mathbb{R}, L^r(\Omega)) \), for some \( \frac{Nq}{N + 2q} < r \leq \infty \) if \( X = L^q(\Omega) \) or \( N/2 < r \leq \infty \) if \( X = C(\mathbb{R}) \), are \( T \)-periodic functions.

Define \( \phi^\pm(t) \) as the unique complete trajectories of the periodic problems

\[
\begin{aligned}
&c^\pm_t - \Delta c^\pm = C^\pm(t, x)c^\pm + D^\pm(t, x), \\
&\text{such that } c^\pm|_{x=\partial \Omega} = 0, \quad (5.5)
\end{aligned}
\]

which are \( T \)-periodic by Corollary 4.8.

(i) Assume that for every \( T_0 < \infty \),

\[
C - C^- \in L^p((-\infty, T_0), L^p(\Omega))
\]

with \( p > \frac{N\sigma q}{N + 2q} \), and

\[
\lim_{t \to -\infty} \left\| C(t) - C^-(t) \right\|_{L_p(\Omega)} = 0
\]

if \( \sigma = \infty \).

Also, assume that, for every \( T_0 < \infty \),

\[
D - D^- \in L^\alpha((-\infty, T_0), L^r(\Omega))
\]

with either \( 1 < \sigma < \infty, \frac{N\sigma q}{N\sigma + 2q} < r \leq \infty, \sigma = 1 \) and \( q \leq r \leq \infty \) or \( \sigma = \infty, \frac{Nq}{N + 2q} < r \leq \infty \), and

\[
\lim_{t \to -\infty} \left\| D(t) - D^-(t) \right\|_{L^r(\Omega)} = 0.
\]

Then, (5.1) has a pullback attractor given by a complete trajectory for (5.1) \( \hat{A} = \{A(t)\}_t = \{\phi(t)\} \) which satisfies \( \phi(t) - \phi^-(t) \to 0 \) in \( X \), as \( t \to -\infty \).

(ii) Assume that for \( -\infty < T_0 \),

\[
C \in D_p(\mathbb{R}, L^p(\Omega)) \quad \text{and} \quad C - C^+ \in L^\sigma((T_0, \infty), L^p(\Omega))
\]
with \( p > \frac{Nq'}{r} \), and
\[
\lim_{t \to \infty} \left\| C(t) - C^+(t) \right\|_{L^p(\Omega)} = 0
\]
if \( \sigma = \infty \).

Also, assume that, for \( -\infty < T_0 \),
\[
D \in D_{\beta}(\mathbb{R}, L'(\Omega)) \quad \text{and} \quad D - D^+ \in L^\sigma ((T_0, \infty), L'(\Omega))
\]
with either \( 1 < \sigma < \infty \), \( \frac{Nq}{N+q} < r \leq \infty \), \( \sigma = 1 \) and \( q \leq r \leq \infty \) or \( \sigma = \infty \), \( \frac{Nq}{N+2q} < r \leq \infty \), and
\[
\lim_{t \to \infty} \left\| D(t) - D^+(t) \right\|_{L^r(\Omega)} = 0.
\]

Then, for every bounded set \( B \subset X \), we have \( v(t, s, u_0) - \phi^+(t) \to 0 \) in \( X \), as \( t \to \infty \), uniformly for \( t_0 \in B \). Moreover, there exists a pullback attractor \( A \) given by \( A(t) = \{ \phi(t) \} \) for all \( t \in \mathbb{R} \) where \( \phi(t) \) is the unique complete trajectory for (5.1) which satisfies \( \phi(t) - \phi^+(t) \to 0 \) as \( t \to \infty \).

**Proof.** Since the evolution operators associated with \( \Delta + C^\pm(t, x) \) are exponentially stable we know, from Corollary 4.8, that problems (5.5±) have unique complete trajectories \( \phi^\pm(t) \) which are \( T \)-periodic. Take any \( v_0 \in X \) and let \( v(t, s, v_0) \) be the unique solution of (5.1). Then \( w = v(t, s, v_0) - \phi^\pm(t) \) satisfies
\[
\begin{cases}
   w_t - (\Delta + C(t, x)) w = \tilde{D}^\pm(t, x), \\
   w(s) = v_0 - \phi^\pm(s), \\
   w|_{\partial \Omega} = 0
\end{cases}
\]
(5.6)

where
\[
\tilde{D}^\pm(t, x) = (D(t, x) - D^\pm(t, x)) + (C(t, x) - C^\pm(t, x))\phi^\pm(t).
\]

Note that, by parabolic regularity, for each \( t, D^\pm(t) \in L'(\Omega) \) implies that \( \phi^\pm(t) \in L^r(\Omega) \) for all \( s \) such that \( \frac{1}{r} - \frac{2}{N} < \frac{1}{2} \) and then, for each \( t, (C(t) - C^\pm(t))(\phi^\pm(t) \in L^m(\Omega) \) with \( \frac{1}{m} = \frac{1}{2} + \frac{1}{p} > \frac{1}{r} - \frac{2}{N} + \frac{1}{p} \) and since \( p > N/2 \) we can take \( m > r \). Therefore, for each \( t, \tilde{D}^\pm(t) \in L^r(\Omega) \).

Hence in case (i), note that we have
\[
\tilde{D}^-(t, x) = (D(t, x) - D^-(t, x)) + (C(t, x) - C^-(t, x))\phi^-(t)
\]
and for all \( T_0 \in \mathbb{R}, \tilde{D}^- \in L^\sigma ((-\infty, T_0], L'(\Omega)) \) with either \( 1 < \sigma < \infty \), \( \frac{Nq}{N+q} < r \leq \infty \), \( \sigma = 1 \) and \( q \leq r \leq \infty \) or \( \sigma = \infty \), \( \frac{Nq}{N+2q} < r \leq \infty \), and
\[
\lim_{t \to -\infty} \left\| \tilde{D}^-(t) \right\|_{L^r(\Omega)} = 0.
\]

Then from part (i) of Corollary 4.6 we get the result.
For the case (ii) we have

\[ \tilde{D}^+(t, x) = \left( D(t, x) - D^+(t, x) \right) + \left( C(t, x) - C^+(t, x) \right) \phi^+(t) \]

which satisfies \( \tilde{D}^+ \in D_\beta (\mathbb{R}, L'(\Omega)) \) and for \( T_0 \in \mathbb{R} \), \( \tilde{D}^+ \in L^\sigma ([T_0, \infty), L'(\Omega)) \) with either \( 1 < \sigma < \infty, \frac{N\sigma q}{N\sigma + 2q} < r \leq \infty \), \( \sigma = 1 \) and \( q \leq r \leq \infty \) or \( \sigma = \infty, \frac{Nq}{N+2q} < r \leq \infty \), and

\[ \lim_{t \to \infty} \| \tilde{D}^+(t) \|_{L'(\Omega)} = 0. \]

Hence part (ii) of Corollary 4.6 gives the result. \( \square \)

6. The nonlinear problem

We will now consider the nonlinear non-autonomous problem

\[
\begin{aligned}
\begin{cases}
    u_t - \Delta u = f(t, x, u) & \text{in } \Omega, \ t > s, \\
    u(s) = u_0 \in X, \\
    u = 0 & \text{on } \partial \Omega
\end{cases}
\end{aligned}
\]  

(6.1)

where \( X = C(\overline{\Omega}) \) and \( f(t, x, u) \) as in Section 2; see (2.2), (2.3). Hence, from the results quoted in Section 2, see Theorem 2.12, (6.1) has a unique locally defined smooth solution for every \( u_0 \in X \).

Suppose that \( f \) satisfies the dissipativity condition

\[ uf(t, x, u) \leq C(t, x)u^2 + D(t, x)|u| \]  

(6.2)

with \( C \in C^\alpha (\mathbb{R}, L^p (\Omega)) \), for some \( \alpha \) with \( 0 < \alpha \leq 1 \) and some \( p > N/2 \), \( p > 1 \), and that \( D \geq 0 \) with values in \( L^r (\Omega) \). Our key assumption is that the evolution operator associated with \( \Delta + C(t, x) \), which we continue to denote by \( U_C(t, s) \), is exponentially stable.

To ensure that the solutions of (6.1) are globally defined forward in time we only need to prove that the solutions of (6.1) are bounded for all \( t \geq s \), which will follow from the dissipativity property of \( f \) (6.2) (see (6.6) in the proof of Lemma 6.2 below).

Then the solutions of the problem (6.1) define an evolution operator given by

\[ U(t, s)u_0 = u(t, s; u_0), \quad t \geq s, \]

and this operator is order-preserving by Theorem 2.13 (see also [5]).

The next result guarantees the existence of two extremal complete trajectories for (6.1) which are ‘attracting’ in a certain sense. A related result can be found in Langa and Suárez [15] for abstract evolution operators given the assumption either of the existence of a pair of sub- and super-trajectories, or the existence of a pullback attractor for the system embedded in an order interval. In the first case, the authors prove the existence of extremal complete trajectories between the sub- and the super-trajectory (see Remark 6.5 below for more details).

**Theorem 6.1.** Suppose that \( X = C(\overline{\Omega}) \) and that \( f \) satisfying (2.2), (2.3), is as in Theorem 2.12, and satisfies (6.2) with \( C \in C^\alpha (\mathbb{R}, L^p (\Omega)) \) for some \( \alpha \) with \( 0 < \alpha \leq 1 \) and some \( p > N/2 \), \( p > 1 \).
Assume in addition that the evolution operator associated with $\Delta + C(t, x)$ is exponentially stable with exponent $\beta$ and that $D(t, x)$ is such that the linear problem (4.1) has a pullback attractor in the class $D_\beta(\mathbb{R}, X)$, given by a complete trajectory $\{\phi(t)\}_t$, e.g. as in Theorems 4.3, 5.1 or 5.2.

Then the solutions of (6.1) are global and we can define $U(t, s)$, the evolution operator defined by the solutions of (6.1), for all $t \geq s$.

Moreover, there exist two extremal complete trajectories that are elements of $D_\beta$, $\varphi_M$ and $\varphi_m$, maximal and minimal, respectively, in the sense that any other complete trajectory for $U$ in $D_\beta$, $\psi$, satisfies $\varphi_m(t) \leq \psi(t) \leq \varphi_M(t)$ for all $t \in \mathbb{R}$.

The order interval $I(t) = [\varphi_m(t), \varphi_M(t)]$ is forward invariant and attracts the dynamics of the system uniformly in the pullback sense, i.e. for all $t \in \mathbb{R}$ we have

$$\varphi_m(t, x) \leq \liminf_{s \to -\infty} u(t, s, x; v_s) \leq \limsup_{s \to -\infty} u(t, s, x; v_s) \leq \varphi_M(t, x)$$

uniformly in $x \in \overline{\Omega}$ for all $v_s$ with $v_s \in B(s)$, where $\{B(s)\}_s \in D_\beta$. Moreover, $\varphi_M(t)$ is globally asymptotically stable from above in the pullback sense, i.e. for all $v \in D_\beta(\mathbb{R}, X)$, $v \geq \varphi_M$ we have

$$\lim_{s \to -\infty} u(t, s; v_s) = \varphi_M(t).$$

Similarly, $\varphi_m(t)$ is globally asymptotically stable from below in the pullback sense.

As a consequence, there exists a pullback attractor for $U$ with respect to $D_\beta$, denoted by $A = [A(t)]_t$, and

$$A(t) \subset [\varphi_m(t), \varphi_M(t)] \quad \text{for all } t \in \mathbb{R}.$$ 

Moreover, $\varphi_m(t), \varphi_M(t) \in A(t)$ for all $t \in \mathbb{R}$.

We prove the theorem in two steps. First, we prove that the solutions of (6.1) are asymptotically bounded by the unique complete trajectory of the linear problem (4.4) (with $C$ and $D$ from (6.2)), which is non-negative and then we prove Theorem 6.1 proper.

**Lemma 6.2.** Under the assumptions of Theorem 6.1, the solutions of (6.1) are global and satisfy

$$\limsup_{s \to -\infty} \|u(t, s, x; v_s)\| \leq \phi(t, x) \quad \text{for all } t \in \mathbb{R}$$

uniformly in $x \in \overline{\Omega}$ for every $v_s$ with $v_s \in B(s)$ where $\{B(s)\}_s \in D_\beta$, where $\phi(t) \geq 0$ is the pullback attractor in the class $D$, given by a complete trajectory $\{\phi(t)\}_t$, for the problem

$$\begin{cases}
 v_t - \Delta v = C(t, x)v + D(t, x) & \text{in } \Omega, \ t > s, \\
 v = 0 & \text{on } \partial \Omega.
\end{cases}$$

Moreover, the order intervals $[-\phi(t), \phi(t)]$ are forward invariant for (6.1).
Remark 6.3. In particular, since the limit in (6.4) is uniform in $x \in \overline{\Omega}$, the order interval $[-\phi(t) - \delta, \phi(t) + \delta]$ is pullback absorbing at time $t$ for the solutions of (6.1). In fact, for any fixed $t \in \mathbb{R}$ and $\delta > 0$, there exists a time $s_0$ such that

$$-\phi(t) - \delta \leq u(t, s; v_s) \leq \phi(t) + \delta$$

for all $s < s_0$.

Proof of Lemma 6.2. We know that there exists a unique bounded complete trajectory for (6.5) which we denote by $\phi(t)$. Furthermore, $A = \{A(t)\} = \{\phi(t)\}$ is the pullback attractor for this problem. Given $u_0 \in X$, let $v(t, s, x; u_0)$ be the solution at time $t$ of the problem (6.5) starting from $u_0$ and $u(t, s, x; u_0)$ the solution at time $t$ of (6.1) with initial data $u_0$. We fix $\{B(s)\}_s \in \mathcal{D}$ and $v_s \in B(s)$. By (6.2) and the comparison principle, see Theorem 2.13, [19, Section 5] and [2],

$$|u(t, s, x; v_s)| \leq v(t, s, x; |v_s|)$$

(6.6)

while both solutions exist. In particular, from here we get bounds on the solution of (6.1) on finite time intervals and hence the solution is defined for all $t > s$.

Now, we have

$$\lim_{s \to -\infty} v(t, s, x; |v_s|) = \phi(t, x)$$

in $C(\overline{\Omega})$. Thus

$$\limsup_{s \to -\infty} u(t, s, x; v_s) \leq \phi(t, x)$$

uniformly in $x \in \overline{\Omega}$ and $v_s \in B(s)$. Arguing with $-v(t, s, x; |v_s|)$ instead of $v(t, s, x; |v_s|)$, we have

$$\limsup_{s \to -\infty} |u(t, s, x; v_s)| \leq \phi(t, x)$$

for all $|v_s|$ in $\{B(s)\}_s \in \mathcal{D}$.

Finally, notice that if $\{u_s\}$ is such that $u_s \leq \phi(s)$ then, by the comparison principle,

$$u(t, s; u_s) \leq u(t, s; \phi(s)) \leq v(t, s; \phi(s)) = \phi(t) \quad \text{for all } t > s.$$ 

Taking now $\{u_s\}$ such that $u_s \geq -\phi(s)$ we have

$$u(t, s; u_s) \geq u(t, s; -\phi(s)) \geq -v(t, s; \phi(s)) = -\phi(t) \quad \text{for all } t > s.$$ 

Thus,

$$U(t, s)[-\phi(s), \phi(s)] \subset [\phi(t), \phi(t)],$$

i.e., $[-\phi(t), \phi(t)]_t$ is forward invariant for $U$. □

Using this lemma we can now prove Theorem 6.1. See Fig. 1 for an illustration of the proof.
Fig. 1. behavioural of a nonlinear non-autonomous evolution operator bounded by asymptotically autonomous linear ones.

**Proof of Theorem 6.1.** Let $U(t, s)$ be the nonlinear evolution operator associated with (6.1). We know that this operator is order-preserving by Theorem 2.13. Moreover, $\phi(t) \geq 0$ is a super-trajectory since the solution of (6.1) starting from $\phi(s)$ satisfies, by (6.6),

$$u(t, s; \phi(s)) \leq v(t, s; \phi(s)) = \phi(t)$$

where $v$ is the solution of the linear problem (6.5). For the last equality we have used the fact that $\phi(t)$ is a complete trajectory for the linear problem.

Next we prove that, since $\phi(t)$ is a super-trajectory of the nonlinear problem, $U(t, s)\phi(s)$ is monotonic as $s \rightarrow -\infty$ and $U(t, s)\phi(s) \rightarrow \phi_M(t)$ as $s \rightarrow -\infty$ uniformly in $x$ for all $t \in \mathbb{R}$. Indeed, for a fixed $t \in \mathbb{R}$ we have, from the definition of a super-trajectory

$$U(t, s)\phi(s) \leq \phi(t) \quad \text{for all } s \leq t,$$

in particular,

$$U(s + \epsilon, s)\phi(s) \leq \phi(s + \epsilon) \quad \text{for all } \epsilon > 0.$$

Thus, by monotonicity,

$$U(t, s)\phi(s) = U(t, s + \epsilon)U(s + \epsilon, s)\phi(s) \leq U(t, s + \epsilon)\phi(s + \epsilon).$$

Therefore, $\{U(t, s)\phi(s)\}_s$ is non-increasing as $s \rightarrow -\infty$. Moreover, it is bounded from below (by $-\phi(t) - \delta$ for some $\delta > 0$, see Remark 6.3). Thus, it converges pointwise to a certain bounded function that we denote by $\psi_M(t) \in L^\infty(\Omega)$.

Notice that we can write

$$U(t, s)\phi(s) = U(t, t - 1)U(t - 1, s)\phi(s),$$
where \( \{U(t-1,s)\phi(s)\}_{s \leq s_0} \) is bounded (for some \( s_0 \)). Thus, by the smoothing property of the evolution operator (see Theorem 2.12 and subsequent remarks) we know that
\[
\left\{ U(t,s)\phi(s) \right\}_{s \leq s_0} = U(t, t-1)[U(t-1,s)\phi(s)]_{s \leq s_0}
\]
is pre-compact. So, \( U(t,s)\phi(s) \to \varphi_M(t) \in C_0(\Omega) \) uniformly in \( \Omega \) as \( s \to -\infty \).

The continuity of \( U(t,s) \) implies that \( \varphi_M(t) \) is a complete trajectory for (6.1). Indeed,
\[
U(t,s)\varphi_M(s) = U(t,s) \lim_{r \to -\infty} U(s,r)\phi(r)
\]
\[
= \lim_{r \to -\infty} U(t,s)U(s,r)\phi(r) = \lim_{r \to -\infty} U(t,r)\phi(r)
\]
\[
= \varphi_M(t). \tag{6.7}
\]

We now prove that, asymptotically in the pullback sense, all trajectories of Eq. (6.1) lie below \( \varphi_M \), uniformly in \( x \). Fix \( \{B(s)\}_{s \in \mathcal{D}} \) and \( v_s \in B(s) \). From (6.6) we have
\[
u(t,s;v_s) \leq v(t,s;|v_s|) \quad \text{for all } t \geq s.
\]
Letting the evolution operator act on both sides, we have by monotonicity
\[
U(r,t)U(t,s;v_s) = u(r,s;v_s) \leq U(r,t)v(t,s;|v_s|) \quad \text{for all } r \geq t \geq s.
\]
Taking limits as \( s \) goes to \( -\infty \) we have, for each \( x \in \Omega \),
\[
\limsup_{s \to -\infty} u(r,s,x;v_s) \leq U(r,t)\varphi_M(t,x) \tag{6.8}
\]
for all \( t \leq r \), where we have used the continuity of \( U(t,s) \). Letting \( t \) tend to \( -\infty \) we have
\[
\limsup_{s \to -\infty} u(r,s,x;v_s) \leq \varphi_M(r,x),
\]
as claimed. The maximality of \( \varphi_M(t) \) follows from this inequality.

From inequality (6.8) we obtain the global asymptotic stability from above in the pullback sense for the maximal complete trajectory. Indeed, let \( r \in \mathbb{R} \) be fixed and assume \( v_r \geq \varphi_M(s) \) for all \( s \). Then, by monotonicity, for \( x \in \Omega \),
\[
\varphi_M(r,x) = u(r,s;\varphi_M(s)) \leq u(r,s,x;v_s)
\]
for all \( s \leq r \). Now, taking limits as \( s \to -\infty \) and using (6.8) we have
\[
\varphi_M(r,x) \leq \liminf_{s \to -\infty} u(r,s,x;v_s) \leq \limsup_{s \to -\infty} u(r,s,x;v_s) \leq U(r,t)\varphi_M(t,x)
\]
for all \( t \leq r \). Taking now limits as \( t \to -\infty \) we obtain
\[
\varphi_M(r,x) \leq \liminf_{t \to -\infty} u(r,s,x;v_s) \leq \limsup_{s \to -\infty} u(r,s,x;v_s) \leq \varphi_M(r,x).
\]
Therefore, \( u(r,s;v_s) \to \varphi_M(r) \) as \( s \to -\infty \) which proves the asymptotic stability from above.
The result for the minimal complete trajectory is proved in an analogous way.
To prove the forward invariance of $I(t)$, take $[u_t]$, such that

$$
\varphi_m(s) \leq u_s \leq \varphi_M(s)
$$

for all $s \in \mathbb{R}$. Then, letting the evolution operator act, we have, by the comparison principle,

$$
\varphi_m(t) = U(t, s)\varphi_m(s) \leq u(t, s; u_s) \leq U(t, s)\varphi_M(s) = \varphi_M(t).
$$

So, $U(t, s)I(s) \subset I(t)$, i.e., $I(t)$ is forward invariant.

We now show the existence of the pullback attractor $\mathcal{A}$. As we pointed out in Remark 6.3 the time-dependent order interval $[-\phi(t) - \delta, \phi(t) + \delta]$ in $C(\mathcal{T})$ is an absorbing set at time $t$ for $U(t, s)$ in the pullback sense. Let

$$
J(t) = U(t, t-1)[-\phi(t-1) - \delta, \phi(t-1) + \delta].
$$

From the smoothing effect of $U(t, s)$ we know that $J(t)$ is compact in $C(\mathcal{T})$. Moreover, $J(t)$ is a pullback absorbing set. Thus, from Theorem 2.7 there exists a pullback attractor $\mathcal{A}$ for $U(t, s)$.

Finally, it is clear that $\mathcal{A}(t) \subset I(t)$ and $\varphi_m(t), \varphi_M(t) \in \mathcal{A}(t)$ for all $t \in \mathbb{R}$. \qed

**Remark 6.4.** Observe that if $C$ and $D$ satisfy the assumptions in Corollary 4.6 then $\phi(t)$ converges to 0 in $X = C(\mathcal{T})$ as $t \to \infty$ or $t \to -\infty$. In particular the same holds true for the solutions of the nonlinear problem (6.1).

On the other hand, if $C$ and $D$ satisfy the assumptions in Theorem 5.1 or Corollary 4.8 or Theorem 5.2 then $\phi(t)$ is asymptotically constant or periodic.

**Remark 6.5.** Notice that we obtain global information about the dynamics of problem (6.1) as well as uniform properties for the asymptotic behaviour of their solutions. Namely, we obtain information about the dynamics of the problem in the whole phase space (in fact, in the basin of attraction $\mathcal{D}_B$) and the uniform convergence of solutions to the order interval defined by the two extreme complete trajectories.

Moreover, from the proof above it is easy to extend the results obtained for the particular case of problem (1.1) to the general framework of order-preserving evolution operators as considered in Langa and Suárez [15]. Their paper gives a related result (their Theorem 3.4) that guarantees the existence of extremal complete trajectories between an ordered couple of sub- and super-trajectories.

We now consider the $T$-periodic problem associated with (1.1), i.e., we suppose that $f(t, x, u)$ is a $T$-periodic function. This kind of problem has been widely studied (see e.g. Daners and Koch Medina [9] or Hess [13]). Moreover, we suppose that $f$ satisfies (6.2) with $T$-periodic functions $C(t, x)$ and $D(t, x)$.

A simple application of our main result, Theorem 6.1, gives the existence of extremal $T$-periodic solutions for the nonlinear problem.

**Corollary 6.6.** In the $T$-periodic equation case, the extremal solutions of (1.1) given in Theorem 6.1 are $T$-periodic. In particular, there exist two $T$-periodic extremal solutions of (1.1).
Proof. From Corollary 4.8 we know that the unique complete trajectory of (6.5) in $D_{\beta}$ is $T$-periodic. We only have to check that the maximal complete trajectory from Theorem 6.1 is $T$-periodic. But, we know that

$$U(t, s)\phi(s) \to \varphi_M(t) \quad \text{as} \quad s \to -\infty.$$ 

We can use now that $\phi(s)$ and $f(t, x, u)$ are $T$-periodic functions and then

$$U(T + t, T + s)\phi(T + s) = U(t, s)\phi(s)$$

where the left-hand side of the equality tends to $\varphi_M(T + t)$ as $s \to -\infty$ and the right-hand side tends to $\varphi_M(t)$ as $s \to -\infty$. So, $\varphi_M(t) = \varphi_M(T + t)$ and $\varphi_M$ is $T$-periodic as we wanted to prove. The same argument applies for $\psi_\text{m}(t)$. \qed

Remark 6.7. To study this type of equation it is usual to consider the Poincaré map associated with (1.1): $S = U(t, 0)$, where $U(t, s)$ is the evolution operator given by the solutions of (1.1) (see e.g. Hess [13]).

In this case the evolution operator generated by $\Delta + C(t, x)$ is exponentially stable if and only if the Poincaré map $S_C$ associated with this operator has spectral radius less than one (see Hess [13]).

In such a case, this implies that $1 \in \rho(S_C)$ (the resolvent of $S_C$) and by Proposition 6.9 in Daners and Koch Medina [9] we obtain the existence of a unique periodic solution for the linear problem as stated in Corollary 4.8. However, we have given another proof that follows straightforwardly from the fact that we are dealing with equations with periodic coefficients.

7. Asymptotic behaviour forwards in time

In order to study the asymptotic behaviour forwards in time of non-autonomous equation, natural concepts are those of asymptotically compact evolution operators and uniform attractors as defined by Haraux [11] and by Chepyzhov and Vishik [7]:

Definition 7.1. (i) We say that $U(t, s)$ is asymptotically compact at $\sigma \in \mathbb{R}$ if there exists a compact set $K_{\sigma} \subset X$, which may depend on $\sigma$, that attracts bounded sets of $X$ forwards in time for the one parameter family $U_\sigma(t, 0) = U(t + \sigma, \sigma)$, $t \geq 0$.

We say that $U(t, s)$ is asymptotically compact if it is asymptotically compact for all $s \in \mathbb{R}$.

(ii) We say that $U(t, s)$ is uniformly asymptotically compact if there exist a compact subset $K \subset X$ such that for any bounded set $B \subset X$

$$\lim_{t \to \infty} \sup_{s \in \mathbb{R}} \text{dist}(U(t + s, s)B, K) = 0.$$ 

(iii) We say that a compact set $F_U$ is the uniform attractor for $U(t, s)$ if it is the minimal compact set satisfying (ii) above.

As we will show below, in some cases, this notion of attractor could be too strong when studying the asymptotic behaviour forwards in time of a non-autonomous equation. For this reason we now construct another kind of attractor giving information about the forward dynamics.
Consider an asymptotically compact evolution operator \( U(t, s) \). Then for a fixed \( s \in \mathbb{R} \) by definition there exists a compact set \( K_s \) that attracts bounded sets of initial data forwards in time from the initial time \( s \).

Then, in a standard way, given a bounded set \( B \subset X \) we define the \( \omega \)-limit set from time \( s \) as

\[
\omega_s(B) = \left\{ u \in X : \exists n \uparrow \infty, v_n \in B, \text{ s.t. } U(t_n, s)v_n \to u \text{ as } n \to \infty \right\}.
\]

With this, it is clear that \( F_s = \omega_s(K_s) \subset K_s \) is the minimal compact set that attracts bounded sets of \( X \) forwards in time for the one parameter family \( U_s(t, 0) = U(t + s, s), t \geq 0 \).

It is not difficult to show, see Haraux [11], that there exists a monotone relationship between this family of compact sets. Namely,

\[ F_s \subseteq F_t \quad \text{for all } s < t. \]

An interesting situation occurs therefore when the compact set \( K_o \) in Definition 7.1 is independent of \( \sigma \), that is, there exists a compact set \( K \subset X \) such that for all \( s \in \mathbb{R} \) and any bounded set \( B \subset X \),

\[ \lim_{t \to \infty} \text{dist}_X(U(t, s)B, K) = 0. \]

In such a case we get the existence of a forward attractor in the sense of Definition 2.6 for the problem (1.1) that can be characterised as

\[ F = \bigcup_{s \in \mathbb{R}} F_s \subset K \]

where \( F_s = \omega_s(K) \subset K \).

To see how these ideas apply to (6.1) we take \( X = C(\overline{\Omega}) \). Suppose that \( f \) is continuous, locally Hölder in \( t \), locally Lipschitz in \( u \), and satisfies (6.2) with \( C \in C^\alpha(\mathbb{R}, L^p(\Omega)) \) with \( 0 < \alpha \leq 1 \) and some \( p > N/2 \), and \( D \) is such that there exists a unique complete trajectory \( \phi \in L^\infty(\mathbb{R}, X) \) for the linear problem (4.4) satisfying

\[ \|v(t, s; u_x) - \phi(t)\|_X \leq Me^{-\beta(t-s)} \]

for all \( u_x \in L^\infty(\mathbb{R}, X) \) and \( M = M(|u_x|_x) \) (see Theorems 4.3, 5.1 and 5.2 for such conditions on \( C \) and \( D \)). Then, given a bounded set \( B \subset X \) and \( \epsilon > 0 \) there exists a time \( T = T(\epsilon) \) such that for every \( u_0 \in B \),

\[ \|v(t, s; u_0) - \phi(t)\|_X < \epsilon \quad \text{for all } t - s \geq T. \quad (7.1) \]

In particular, for \( R = \|\phi\|_{L^\infty(X)} + 1 \),

\[ \|v(t, s; u_0)\|_X \leq R \quad \text{for all } t - s \geq T. \]

Moreover, we know that \( |u(t, s; u_0)| \leq v(t, s; |u_0|) \) for all \( t > s \). Hence, for all \( t - s \geq T \),

\[ \|u(t, s; u_0)\|_X \leq \|v(t, s; u_0)\|_X \leq R, \quad (7.2) \]
i.e., for all \( s \in \mathbb{R} \), \( B_X(0, R) \) is an absorbing set forward in time for \( U(t, s) \). Furthermore, by the smoothing property of the evolution operator, the solutions of the nonlinear problem enter some ball in a space \( Y \) compactly embedded in \( X \), \( B_Y(0, R_Y) \subset B_X(0, R) \). Thus, \( K = \overline{B_Y(0, R_Y)} \subset \overline{B_X(0, R)} \) (where the closure is taken in \( X \)) is a forward absorbing compact set not depending on \( s \).

It is now clear that in this case the sets \( \mathcal{F}_s \) and \( \mathcal{F} \) as defined above can also be described as

\[
\mathcal{F}_s = \bigcup_{B \subset B(X)} \omega_s(B)
\]

where \( B(X) \) denotes the set of all bounded sets of \( X \) and

\[
\mathcal{F} = \bigcup_{s \in \mathbb{R}} \omega_s(B_X(0, R)).
\]

**Remark 7.2.** Notice that the construction above can be carried out for (6.1) without the boundedness assumption on \( \phi(t) \). Namely, everything above remains true if we allow \( \phi \in \mathcal{D}_\gamma \) for some \( 0 < \gamma < \beta \) since, in that case,

\[
\| v(t, s; u_s) - \phi(t) \|_X \leq M e^{-\beta(t-s)} \| u_s - \phi(s) \| \leq M_1 e^{-\beta(t-s)} e^{-\gamma s}
\]

\[
\leq M_1 e^{-\gamma s} e^{-(\beta-\gamma)(t-s)} = M_1 e^{-\gamma s} e^{-(\beta-\gamma)(t-s)}.
\]

And for \( t > 0 \) we have

\[
\| v(t, s; u_s) - \phi(t) \|_X \leq M_1 e^{-(\beta-\gamma)(t-s)}.
\]

Thus, since \( \beta - \gamma > 0 \), given \( \epsilon > 0 \), there exists \( T = T(\epsilon) > 0 \) such that for all \( t - s > T \), \( t > 0 \),

\[
\| v(t, s; u_s) - \phi(t) \|_X < \epsilon
\]

and now the argument follows as above.

We now state a result about the structure of the forward attractor \( \mathcal{F} \) for (6.1). For this, let \( \{B(t)\}_t \) be a family that is invariant under \( U(t, s) \), i.e. \( U(t, s)B(s) = B(t) \) for all \( t > s \). We denote by \( \omega(B) \) the set

\[
\omega(B) = \{ u \in X : \exists n \uparrow \infty, v_n \in B(l_n), \text{ s.t. } v_n \to u \text{ as } n \to \infty \}.
\]

**Proposition 7.3.** If \( \{B(t)\}_t \) is an invariant set for \( U(t, s) \) then

\[
\omega(B) \subset \mathcal{F},
\]

where \( \mathcal{F} \) is the forwards attractor of \( U(t, s) \) as defined above.

In particular, if \( A \) is a pullback attractor for \( U(t, s) \) then \( \omega(A) \subset \mathcal{F} \). Moreover, if \( \psi(t) \) is a complete trajectory for \( U(t, s) \) then \( \omega(\psi) \subset \mathcal{F} \).
Proof. From (7.2) and the smoothing effect, if we take \( r - s > T \) as above, we have
\[
B(r) = U(r, s)B(s) \subset K
\]
where \( K = \overline{B}_r(0, \overline{R}_r) \subset \overline{B}_X(0, \overline{R}) \). Thus, for all \( t > r \),
\[
B(t) = U(t, r)B(r) \subset U(t, r)K.
\]
Taking limits as \( t \) goes to \(+\infty\) we have that
\[
\omega(B) \subset \omega_r(K) \subset \mathcal{F}_r \subset \mathcal{F}
\]
where \( \omega_r(K) \) is the \( \omega \)-limit set from time \( r \) defined above. \( \square \)

Remark 7.4. As a consequence, the attractor \( \mathcal{F} \) can be defined in cases where the uniform attractor cannot, since boundedness of \( \phi \) is needed in the definition of the uniform attractor. Indeed, it can be shown (see Chepyzhov and Vishik [7]) that
\[
\mathcal{F}_U = \bigcup_{t \in \mathbb{R}} \mathcal{A}(t)
\]
where \( \mathcal{A} \) is the pullback attractor attracting bounded sets.

Let us consider a linear problem whose unique complete trajectory is unbounded backward in time. Then, the only set that satisfies Definition 7.1 is
\[
\mathcal{F}_U = \bigcup_{t \in \mathbb{R}} \phi(t)
\]
which is an unbounded set. Therefore \( \mathcal{F}_U \) is not compact. However, the (non-uniform) forward attractor
\[
\mathcal{F} = \omega(\phi) = \{ u \in X: \exists t_n \uparrow \infty, u_n = \phi(t_n), \text{ s.t. } u_n \to u \text{ as } n \to \infty \}
\]
still exists.

As a particular case we consider now the case of asymptotically autonomous problems. In fact, suppose that \( f \) satisfies the dissipativity condition (6.2) with \( C(t, x) \to C^+(x), D(t, x) \to D^+(x) \) as \( t \to \infty \) as in Theorem 5.1. From the previous results we have, for every \( t \in \mathbb{R} \),
\[
\mathcal{A}(t) \subset [\varphi_m(t), \varphi_M(t)] \subset [-\phi(t), \phi(t)] \subset \overline{B}_X(0, \overline{R})
\]
and all these sets are forwards invariant. Hence denoting \( I = \{ I(t) \}_t = [\varphi_m(t), \varphi_M(t)] \), we have from Theorem 5.1 and Proposition 7.3,
\[
\omega(\mathcal{A}) \subset \omega(I) \subset \mathcal{F} \subset [-\phi^+, \phi^+].
\]
Now observe that since \( \{ \psi_m(t) \}_t, \{ \varphi_M(t) \}_t \) are relatively compact complete trajectories we can consider \( \omega(\varphi_M) \) and \( \omega(\varphi_m) \), which are compact connected sets of \( X \). Thus,

\[
\omega(A) \subset \omega(I) \subset [\psi_m, \psi_M] \cap \mathcal{F} \subset [-\phi^+, \phi^+]
\]

where

\[
\psi_m(x) = \inf_{v \in \omega(\varphi_m)} v(x) \quad \text{and} \quad \psi_M(x) = \sup_{v \in \omega(\varphi_M)} v(x).
\]

Note that a completely analogous analysis can be carried out backwards in time when \( C(t,x) \to C^-(x) \) and \( D(t,x) \to D^-(x) \) as \( t \to -\infty \) by considering the \( \alpha \)-limit set of an invariant set,

\[
\alpha(B) = \{ u \in X : \exists \delta_n \uparrow -\infty, \ v_n \in B(t_n), \text{ s.t. } v_n \to u \text{ as } n \to \infty \}.
\]

Continuing with the forward behaviour, assume in addition that \( f(t,x,u) \to g(x,u) \) as \( t \) goes to \( \infty \), uniformly in \( x \in \mathcal{D} \), for \( u \) in bounded sets of \( X \). Then, it is shown in Mischaikow et al. [17] that the evolution operator associated with Eq. (6.1) is asymptotically autonomous in the sense of Thieme (see Thieme [23]). Thus, the \( \omega \)-limit set of any point \( \omega_s(u_0) \) is invariant under the semiflow \( S(t) \) defined by the solutions of the limit equation

\[
\begin{cases}
    v_t - \Delta v = g(x,v), & t > 0, \\
    v(0) = v_0, \\
    v|_{\partial \Omega} = 0
\end{cases}
\]

(see Theorem 2.5, p. 760 in Thieme [23]). Moreover, if the equilibria of the limit problem are isolated, the existence of a Lyapunov function for the limit problem (7.3) (see Hale [10] or Henry [12]) implies that they are not chained in a cyclic way in the sense of Definition 1.3 in Mischaikow et al. [17].

Then, from Theorem 4.2 and Corollary 4.3, p. 762, in Thieme [23] the \( \omega \)-limit set of each solution of the non-autonomous problem is an equilibrium point for \( S(t) \). So, it follows that

\[
\omega(\varphi_m) = \{ \varphi_m^\infty \}, \quad \omega(\varphi_M) = \{ \varphi_M^\infty \} \subset \mathcal{F},
\]

for some equilibria \( \varphi_m^\infty \leq \varphi_M^\infty \) of the limit autonomous problem. Moreover, we have

\[
\omega(A) \subset \omega(I) \subset [\varphi_m^\infty, \varphi_M^\infty] \subset [-\varphi_m^+, \varphi_M^+] \subset [-\phi^+, \phi^+]
\]

where \( \varphi_m^+, \varphi_M^+ \) are the extremal equilibria of the limit problem, see Rodríguez-Bernal and Vidal-López [20] and Vidal-López [24].

Even more, \( \omega(A) \) is contained in the attractor of the limit problem (see Theorem 3.7.2, p. 45, in Hale [10] or Theorem 4.3.6, p. 96, in Henry [12]).

On the other hand, from the arguments above it is then clear that we also have

\[
\mathcal{F} \subset [\varphi_m^+, \varphi_M^+]
\]

since \( \mathcal{F} \) can be obtained as a union of \( \omega \)-limit sets of fixed bounded sets and, from Theorem 3.7.2, p. 45, in Hale [10] or Theorem 4.3.6, p. 96, in [12], it must be an invariant set for the limit problem.
8. An example: The non-autonomous logistic equation

We now consider the non-autonomous logistic equation

$$\begin{cases}
u_t - \Delta u = f(t, x, u) & \text{in } \Omega, \ t > s, \\ u = 0 & \text{on } \partial \Omega, \\ u(s) = v_0
\end{cases}$$

(8.1)

with the model nonlinearity

$$f(t, x, u) = m(t, x)u - n(t, x)u^3$$

(8.2)

where $m \in C^\alpha(\mathbb{R}, L^p(\Omega))$ with $0 < \alpha \leq 1$ and $p > N/2$, $p > 1$, and $n \geq 0$ is continuous and locally Hölder in $t$.

We will show how our techniques can be applied to this problem, although it will be clear from the analysis that much more general classes of nonlinear terms could be considered.

We start with the case in which the asymptotic dynamics of (8.1) is trivial.

**Theorem 8.1.** Suppose that $n(t, x) \geq 0$ and that $m(t, x)$ is such that the evolution operator associated with $\Delta + m(t, x)$ is exponentially stable. Then $\|u(t, s; u_0)\|_{L^\infty(\Omega)} \to 0$ as $t \to +\infty$ or $s \to -\infty$ uniformly for $u_0$ in bounded sets of $X = C(\overline{\Omega})$.

In particular, the pullback attractor of (8.1) is $\mathcal{A}(t) = \{0\}$ for all $t \in \mathbb{R}$ and the forward attractor is $\{0\}$.

**Proof.** Notice using $n(t, x) \geq 0$ that $f(t, x, u)$ satisfies

$$f(t, x, u)u = m(t, x)u^2 - n(t, x)u^4 \leq m(t, x)u^2$$

(8.3)

for all $t \in \mathbb{R}$.

Now, since $m(t, x)$ is such that the evolution operator associated with $\Delta + m(t, x)$ is exponentially stable, it follows from Theorem 6.1, with $C(t, x) = m(t, x)$ and $D(t, x) = 0$ and (8.3) that there exist two extremal bounded complete trajectories for (8.1). But, in this case, both are the same and equal to the trivial one (see Theorem 4.1). So the pullback and forward attractors are 0. In fact, $\phi(t) \equiv 0$ for all $t \in \mathbb{R}$. \quad \Box

Suppose now that $m(t, x)$ is such that the evolution operator associated with $\Delta + m(t, x)$ is not exponentially stable. In the following result we give conditions to have the existence of a pullback attractor.

**Theorem 8.2.** Let $X = C(\overline{\Omega})$. Suppose that the evolution operator generated by $\Delta + m(t, x)$ is not exponentially stable but there exists a decomposition $m(t, x) = m_1(t, x) + m_2(t, x)$ with $m_2(t, x) \geq 0$ and

$$m_1 \in C^\alpha(\mathbb{R}, L^p(\Omega)) \quad \text{with } 0 < \alpha \leq 1 \text{ and some } p > N/2,$$
such that the evolution operator associated with $\Delta + m_1(t, x)$ is exponentially stable with exponent $\beta$. Let
\[
D = \left(\frac{m^3}{n^2}\right)^{1/2}
\]
and suppose that either

(i) $D \in D_\beta(\mathbb{R}, L^r(\Omega))$ with $N/2 < r < \infty$; or
(ii) for $T < \infty$, $D \in L^r((0, T), L^r(\Omega))$ with $1 < \sigma \leq \infty$ and $N\sigma'/2 < r \leq \infty$ if $1 < \sigma < \infty$, or $N/2 < r \leq \infty$ if $\sigma = \infty$.

Then Theorem 6.1 applies and

(1) There exists a pullback attractor with respect to $D_\beta$, $A(t) \subset [\varphi_m(t), \varphi_M(t)]$ where $\varphi_m(t)$ and $\varphi_M(t)$ are the extremal complete trajectories from Theorem 6.1. In particular, the set $[\varphi_m(t), \varphi_M(t)]$ is a forward invariant set that is pullback attracting at time $t$.

(2) For non-negative solutions there also exists a pullback attractor $A_+(t) \subset [0, \varphi_M(t)]$. In particular the set $[0, \varphi_M(t)]$ is a pullback attracting invariant set for non-negative solutions.

**Remark 8.3.** Assumption (i) implies that $\phi \in D_\beta(\mathbb{R}, C(\bar{\Omega}))$, while assumption (ii) implies that, for each $T < \infty$, $\phi \in L^{\infty}((0, T), C(\bar{\Omega})) \subset D_\beta$.

**Proof.** From Young’s inequality applied to $f$ we have
\[
f(t, x, u) \leq m_1(t, x)u + \left(\frac{8m_3^2(t, x)}{27n(t, x)}\right)^{1/2}
\]
for all $u \geq 0$. A similar expression holds for $u < 0$. Thus,
\[
f(t, x, u)u \leq m_1(t, x)u^2 + \left(\frac{8m_3^2(t, x)}{27n(t, x)}\right)^{1/2}|u|.
\]
Hence, if either (i) or (ii) hold, we can apply Theorem 6.1 with $C(t, x) = m_1(t, x)$ and $D(t, x) = (\frac{8m_3^2(t, x)}{27n(t, x)})^{1/2}$ to deduce the existence of two extremal complete trajectories ($\varphi_m$ and $\varphi_M$) and a pullback attractor $A(t)$ such that
\[
A(t) \subset [\varphi_m(t), \varphi_M(t)].
\]
Moreover, since 0 is a solution of (8.1) and the comparison principle holds, the maximal complete trajectory is non-negative and the minimal one non-positive. So, provided we consider only non-negative solutions, the pullback attractor $A_+(t)$ satisfies
\[
A_+(t) \subset [0, \varphi_M(t)] \quad \text{for all } t \in \mathbb{R}
\]
where $\varphi_M(t)$ is the maximal complete trajectory. \qed
Remark 8.4. Notice that Theorem 4.3, part (ii), gives sufficient conditions on $D$ to conclude that $\phi$, and therefore $\phi_m, \phi_M$ are in $C_b(\mathbb{R}, C(\overline{\Omega}))$. In such a case the arguments in Section 7, regarding the asymptotic behaviour forward in time, apply.

Also, note that Corollary 4.6 gives conditions on $D$ to conclude that

$$\phi_m(t), \phi_M(t) \to 0$$

as $t$ goes to $-\infty$ or $\infty$, in cases not covered by Theorem 8.1.

However, in general solutions of (8.1) may not be bounded as $t \to +\infty$. For example if $n = n(t)$ tends to zero as $t \to +\infty$ and $m(t, x) = \lambda$, a positive constant larger than the first eigenvalue of the Laplace operator in $\Omega$ with Dirichlet boundary conditions (see Langa and Suárez [15], Lemma 4.5). Indeed, assume that on a suitable smooth subdomain $\Omega_0 \subset \Omega$ we have $0 < N(t) = \max_{x \in \Omega_0} n(t, x) \to 0$ as $t \to \infty$ and $m = m(x)$. Then clearly solutions of

$$\begin{cases}
  w_t - \Delta w = m(x)w - N(t)w^3 & \text{in } \Omega_0, \ t > s, \\
  w = 0 & \text{on } \partial \Omega_0, \\
  w(0) = w_0 \geq 0
\end{cases}$$

give lower bounds for the non-negative solutions of (8.1) restricted to $\Omega_0$. Therefore, if the first eigenvalue of $-\Delta - m(x)I$ with Dirichlet boundary conditions in $\Omega_0$ is negative, using the arguments from the proof of Lemma 4.5 in Langa and Suárez [15], we can show that $w(t, x)$ becomes unbounded in $\Omega_0$ and so do the solutions of (8.1).

We now give four examples which show that sometimes the linear bounds appearing in Lemma 6.2 may have desirable properties even though no special behaviour is prescribed for the nonlinear term. For example, $\phi(t)$ can be independent of $t$ or $T$-periodic, while the reaction term $f$ is not. We will assume in (8.2) that $n \geq 0$ and that $m(t, x)$ admits a decomposition of the form

$$m(t, x) = m_1(t, x) + m_2(t, x)$$

such that the evolution operator generated by $\Delta + m_1(t, x)$ is exponentially stable.

Example 1. Suppose that $m_1(t, x)$ is $T$-periodic and that $m_2(t, x) = a(t)g^2(t, x)$, where $g(t, x) \geq 0$ is also $T$-periodic and $a(t) \geq 0$ is arbitrary. Set

$$n(t, x) = a^3(t)h^2(t, x)$$

for some $T$-periodic function $h(t, x) \geq 0$. Then $f$ satisfies (6.2) with

$$C(t, x) = m_1(t, x) \quad \text{and} \quad D(t, x) = \frac{8g^3(t, x)}{27h(t, x)}$$

which are $T$-periodic functions. Hence, $\phi(t)$ is a $T$-periodic solution of the linear problem, that is, we obtain a $T$-periodic bound for the pullback attractor of the nonlinear problem.
Example 2. Assume now that
\[ m_1(t, x) = m_0(x), \quad m_2(t, x) = a(t, x)g_0^2(x), \quad \text{and} \quad n(t, x) = a^3(t, x)h_0(x) \]
where \( h_0 \geq 0, m_0, g_0 \geq 0 \) do not depend on \( t \) and \( a(t, x) \geq 0 \) is arbitrary. Then, \( f \) satisfies (6.2) with
\[ C(t, x) = m_0(x) \quad \text{and} \quad D(t, x) = \frac{8g_0^3(x)}{27h_0(x)} \]
and the linear problem given by (6.5) is an autonomous parabolic equation. So, its (unique) equilibrium gives bounds for the nonlinear problem, that is, we have a time-independent bound for the pullback attractor of the nonlinear problem.

Example 3. Suppose now that \( m_1(t, x) \) is \( T \)-periodic,
\[ m_2(t, x) = a(t, x)b(t, x)g^2(t, x), \quad \text{and} \quad n(t, x) = a^3(t, x)b(t, x)h^2(t, x), \]
where \( g(t, x) \geq 0 \) and \( h(t, x) \geq 0 \) are both \( T \)-periodic and \( a(t, x) \geq 0 \) is arbitrary. Suppose that
\[ 0 \leq b(t, x) \rightarrow b_0(x) \]
uniformly in \( x \) as \( t \rightarrow \infty \). In this case
\[ C(t, x) = m_1(t, x) \quad \text{and} \quad D(t, x) = \frac{8b(t, x)g^3(t, x)}{27h(t, x)}. \]
Notice that if we denote \( D^+(t, x) = 4b_0(x)g^3(t, x)/27h(t, x) \) then \( D(t, x) - D^+(t, x) \rightarrow 0 \) uniformly in \( \Omega \), as \( t \) goes to infinity. Thus \( \phi(t) \) is the unique complete trajectory of the asymptotically \( T \)-periodic problem (6.5) and therefore \( \phi(t) \) is asymptotically \( T \)-periodic.

Example 4. In the previous example, suppose that \( m_1 \geq 0, g \geq 0 \) and \( h \geq 0 \) do not depend on \( t \), i.e. \( m_1(t, x) = m_0(x) \),
\[ m_2(t, x) = a(t, x)b(t, x)g^2(x), \quad \text{and} \quad n(t, x) = a^3(t, x)b(t, x)h(x), \]
with \( a(t, x) \geq 0 \) arbitrary and \( 0 \leq b(t, x) \rightarrow b_0(x) \) uniformly in \( x \) as \( t \rightarrow \infty \). Therefore
\[ C(t, x) = m_1(x) \]
and
\[ D(t, x) = \frac{8b^2(t, x)g_0^3(x)}{27h(x)} \rightarrow D^+(x) = \frac{8b_0^2(x)g_0^3(x)}{27h(x)} \]
uniformly as \( t \rightarrow \infty \).

Thus \( \phi(t) \) satisfies an asymptotically autonomous linear problem. In particular, from the result in Section 7 we have bounds for the asymptotic behaviour of the solutions of the nonlinear problem forward in time.

Note that in all these examples Theorem 8.2 gives conditions for the existence of the pullback attractor.
9. Some other problems

With minor modifications the results of previous sections can be translated to problems other than our model example (1.1).

For example, the results about linear equations in Section 3 remain true for more general equations than (3.1) involving time-dependent operators and boundary conditions.

As a first example, using the results in [9], we can consider operators of the form

\[ A(t, D)u = - \sum_{i,j=1}^{N} a_{ij}(x, t) \partial_i \partial_j u + \sum_{i=1}^{N} a_i(x, t) \partial_i u + a(x, t) u \]

with suitable smooth coefficients and either Dirichlet boundary conditions or time-independent boundary conditions of Robin type

\[ Bu = \frac{\partial u}{\partial n} + b(x) u \]

with no sign conditions on the smooth coefficient \( b(x) \). All these operators satisfy the maximum principle [9, p. 120] and the estimates in (3.2).

Existence results for the corresponding nonlinear problems, along the lines of those given in Theorem 2.12, can be obtained from the results in [9] and [16], assumed \( f(t, x, u) \) is a continuous function, locally Hölder in \( t \) and locally Lipschitz in \( u \).

The analysis of complete trajectories in Section 4 can therefore be carried out without major changes. Of course, the asymptotically autonomous or periodic cases in Section 5 would require a specific although similar treatment.

All the results for the nonlinear equations in Section 6 then follow for this example.

We could also consider the following problem, with non-autonomous nonlinear boundary conditions:

\[
\begin{cases}
  u_t - \Delta u = f(t, x, u) & \text{in } \Omega, \ t > s, \\
  \frac{\partial u}{\partial n} + b(t, x)u = g(t, x, u) & \text{on } \partial \Omega, \\
  u(s) = u_s
\end{cases}
\]  \quad (9.1)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \), \( b(t, x) \) is smooth and \( f(t, x, u) : \mathbb{R} \times \Omega \times \mathbb{R} \to \mathbb{R} \) and \( g(t, x, u) : \mathbb{R} \times \partial \Omega \times \mathbb{R} \to \mathbb{R} \) are continuous, locally Hölder in \( t \), locally Lipschitz in \( u \) and satisfy

\[ f(t, x, u)u \leq C(t, x)|u|^2 + D(t, x)|u| \quad \text{for all } u \in \mathbb{R}, \quad (9.2) \]

\[ g(t, x, u)u \leq B(t, x)|u|^2 + E(t, x)|u| \quad \text{for all } u \in \mathbb{R}, \quad (9.3) \]

for some suitable smooth functions \( C, D, B \) and \( E \). Note that we make no sign assumptions on \( b(t, x) \).

In this case the main assumption would be that the evolution operator defined by
\begin{equation}
\begin{aligned}
\begin{cases}
    v_t - \Delta v = C(t, x)v & \text{in } \Omega, \ t > s, \\
    \frac{\partial v}{\partial n} + b(t, x)v = B(t, x)v & \text{on } \partial \Omega, \\
    v(s) = v_s
\end{cases}
\end{aligned}
\end{equation}

is exponentially stable.

The technical details will be presented elsewhere.

Acknowledgments

A.R.-B. and A.V.-L. were partially supported by Project MTM2006-08262 MEC, Spain. A.V.-L. was also supported by a Marie Curie visiting fellowship. J.C.R. is a Royal Society University Research Fellow.

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