Nesting inertial manifolds for reaction and diffusion equations with large diffusivity

A. Rodríguez Bernal\textsuperscript{a}, Robert Willie\textsuperscript{b,c,*}

\textsuperscript{a} University of Madrid, Complutense, Applied Mathematics, 28040 Madrid, Spain
\textsuperscript{b} University of Zimbabwe, Department of Mathematics, Harare, Zimbabwe
\textsuperscript{c} University of KwaZulu-Natal, South Africa

Received 29 April 2006; accepted 31 May 2006

Abstract

We study the asymptotic behaviour in large diffusivity of inertial manifolds governing the long time dynamics of a semilinear evolution system of reaction and diffusion equations. A priori, we review both local and global dynamics of the system in scales of Banach spaces of Hilbert type and we prove the existence of a universal compact attractor for the equations. Extensions yield the existence of a family of nesting inertial manifolds dependent on the diffusion of the system of equations. It is introduced an upper semicontinuity notion in large diffusivity for inertial manifolds. The limit inertial manifold whose dimension is strictly less than those of the infinite dimensional system of semilinear evolution equations is obtained.

\textcopyright 2006 Elsevier Ltd. All rights reserved.

MSC: 35Bxx; 35D40; 35B45; 35K57

Keywords: Semilinear system of reaction and diffusion equations; Well posedness; Universal compact attractor; Inertial manifolds, and limit inertial manifold; Large diffusivity

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be an open bounded regular domain with boundary $\partial \Omega = \Gamma$ and $\varepsilon > 0$ be a given parameter. In this paper, we consider the following weakly coupled system of reaction and
diffusion equations

\[ u_t - \text{Div}(D^\varepsilon \nabla u) + V^\varepsilon u + \lambda u = f(u) \quad \text{in } \Omega \]
\[ D^\varepsilon \frac{\partial u}{\partial n} + B^\varepsilon u = g(u) \quad \text{on } \Gamma, \; t \in (0, T) \]
\[ u(0) = u_0^\varepsilon \quad \text{in } \Omega, \]

with nonlinear boundary conditions, where \( u = (u_1, \ldots, u_m)^\top \) is a vector function, and \( \vec{n} = (n_1, \ldots, n_N)^\top \) is the normal external vector at the boundary \( \Gamma \). The coefficient \( D^\varepsilon = \text{diag} [d_i^\varepsilon] \) for \( i = 1, \ldots, m \) is a diagonal matrix of strictly positive functions \( d_i^\varepsilon \in L^\infty(\Omega) \). Similarly \( V^\varepsilon = \text{diag} [V_i^\varepsilon] \) and \( B^\varepsilon = \text{diag} [b_j^\varepsilon] \) for \( i = 1, \ldots, m \) are diagonal matrices of order \( m \) of functions

\[ V_i^\varepsilon \in L^{p_0}(\Omega) \quad \text{and} \quad b_j^\varepsilon \in L^{q_0}(\Gamma), \quad \forall \varepsilon > 0 \quad (2) \]

where \( p_0, q_0 \)

\[ \begin{cases} \geq 1 & \text{if } N = 1, \\ > 1 & \text{if } N = 2, \\ \geq \frac{N}{2}, & N - 1 \text{ if } N \geq 3 \text{ respectively.} \end{cases} \quad (3) \]

In (2) we further will assume that only the integrated values of the given functions converge as \( \varepsilon \to 0 \). Also given with the system of equations is a real number \( \lambda > 0 \).

Let \( N = f, \) or \( g : \mathbb{R}^m \to \mathbb{R}^m \) then if \( u_0^\varepsilon \in X^\alpha \) with \( \alpha < 3/4 \) we assume that

\[ |N(u) - N(v)| \leq \mu(u, v)|u - v| \quad \text{componentwise, where in } N \geq 3 \]
\[ \mu(u, v) = C(|u|^{\rho_1} + |v|^{\rho_1}) \quad \text{with} \]
\[ \rho_f \leq \frac{N + 4\alpha}{N - 4\alpha}, \quad \rho_g \leq \frac{(N - 1) + 4\alpha}{(N - 1) - 4\alpha} \quad \text{respectively.} \quad (4) \]

If \( N = 2 \) we assume \( \forall \eta > 0, \exists C_\eta \geq 0 \) such that

\[ \mu(u, v) = C_\eta (e^{\eta|u|^2} + e^{\eta|v|^2} + 1). \]

If \( N = 1 \) no growth conditions are posed.

Throughout this paper, we concentrate on Eqs. (1) with nonlinearity given in the first case of (4). The case \( N = 2 \) can easily be obtained with minor changes from the one we study. The case \( N = 1 \) is even much simpler. The main differences are due to Sobolev space embeddings theorems \([1, 5, 15]\). Also important is that the choice of phase space is due to the trace theorem and contribution of the nonlinear boundary conditions. These scales of Banach space are natural diffusion equations

\[ u = (u_1, \ldots, u_m)^\top \]

\[ (\mu(\varepsilon) - \lambda_0) t \quad \text{as } \varepsilon \to 0. \]

Inertial manifolds for dissipative evolution equations and their abstract theory of existence have been studied in the following references [11, 20, 21, 23, 22]. In particular, [20] has studied these manifolds in general scales of Banach spaces. More advanced studies on approximating inertial manifolds for evolution equations of the forms either of Faedo–Galerkin
or Euler–Newton can be found in [21,23]. More precisely, in the first of these references this study is done in Chapter X. The notion of stability of inertial manifolds we wish to investigate can be characterized as follows

\[
\text{If } (X, d) \text{ is a complete metric space and } \mathcal{M}_n, \mathcal{M} \subset X, n \in \mathbb{N} \text{ then }
\lim_{n \to \infty} \sup_{x \in \mathcal{M}_n} \inf_{y \in \mathcal{M}} \text{dist}_X(x, y) = 0.
\]

(6)

This limit question is significantly different from that of approximations to inertial manifolds we have cited above and studied in [21]. However, new concepts we have obtained are consistent with those developed in this cited research monograph.

Concerning the limit problem (5) in semilinear reaction and diffusion equations with nonlinear boundary conditions we have studied in [24–26] the notion (6) in the case of attractors of the dynamical system generated by Eqs. (1). The limit attractor corresponds to the following system of differential equations

\[
\dot{u} + L_0 u = f(u) + \frac{|\Gamma|}{|\Omega|} g(u), \quad u(0) = u_0 \text{ in } \mathbb{R}^m,
\]

(7)

where for \( i = 1, \ldots, m \) the matrix \( L_0 = \text{diag} [\mu_i] : \mathbb{R}^m \to \mathbb{R}^m \) is positive definite. A complete review of early studies of the type problem (1) with standard homogeneous boundary conditions in the case of hypothesis (5) has been provided in [24]. Therein we have discussed the pioneering works of authors in citations [6,9,12,14]. It can be appreciated from these works in view of (7) that the contributions of the nonlinear boundary conditions in (1) to the limit equations are perturbations of the nonlinear vector fields in the domain.

It is worthwhile to note that some of the techniques used by these pioneers are not immediately applicable in proving that the limit equations are in fact (7) as given above. However, a nice extension of results in [14] to our system of equations (1) has been provided in [25]. The results we obtained there roughly speaking that if the system of ordinary differential equations (7) has a local compact attractor then the system of semilinear reaction and diffusion equations (1) also has an attractor provided diffusion in the system is sufficiently large. This attractor for the partial differential equations is in an extended neighbourhood of the attractor for the system (7) in the phase space of the equations. Moreover, there exists an exponential attracting invariant manifold for the system (1) of partial differential equations. This invariant manifold turns out that it contains the above attractor of the partial differential equations. In [26] we have proved the upper semicontinuity of attractors for Eqs. (1) and (7) when large diffusivity (5) is assumed. The convergence of attractors is obtained not in the energy space \( H^1(\Omega) \), but in the strong topology of functions in \( C(\overline{\Omega}) \). The paper [26] yields not only the convergence of attractors, but a complete survey of the asymptotic behaviour of the linear semigroup generated by the homogeneous equations of our system (1). The basis of analysis is a detailed study of the behaviour of the corresponding eigenvalue problem in large diffusivity.

Since inertial manifolds contain global attractors of evolution equations. It is therefore natural to ask about their asymptotic behaviour in the case of dependence on diffusion of the system of equations. The notion to studying this asymptotics is one we have proposed in (6). It is this study that is the central theme of the paper. We now turn to the structure of the paper. It is organized as follows. In Section 2, we briefly review [15,19,26] the generation of a linear analytic semigroup in extended scales of Hilbert spaces by the homogeneous evolution equations associated with the semilinear system of reaction and diffusion equations (1). Then we prove local well posedness of the complete system incorporating the nonlinear boundary conditions. This is a standard
procedure, see for examples in [2–4]. Section 3 is devoted to the study of global existence of the solutions. It also proves the existence of a universal compact attractor for the system of equations (1) in given scales of Banach spaces. In Section 4, we construct the inertial manifold for the system of equations with fixed diffusion coefficients. Section 5 studies the asymptotic behaviour in large diffusivity of the inertial manifolds. In particular, we prove existence of a limit inertial manifold in the sense (6) where $X$ is the scale of spaces $X^\alpha_\varepsilon$, $\alpha < 3/4$. This limit inertial manifold has dimension strictly less than that of the family in diffusion parameter of inertial manifolds corresponding to the system (1) of equations.

The conclusion we have obtained is that as diffusion goes to infinity the inertial manifolds of (1) deform somehow to an inertial manifold of its natural limit of this process the system of ordinary differential equations (7).

2. The local dynamical system

For simplicity of notation, we use the same function space notation of spaces for scalar functions for either vector or matrix functions. This can appreciated from the above section. Also we do not distinguish the scales of spaces $X^\alpha = X^\alpha_1 \times \cdots \times X^\alpha_m$, $\alpha \in \mathbb{R}$ from ones defined by the scalar spatial differential operator. All generic constants will be denoted by $C \geq 0$. The inner product of the space $L^2(\Omega)$ will be denoted as $\langle \cdot, \cdot \rangle$. The norm notations of the Banach space $X$ will be denoted as $\|\cdot\|_X$, while for scales of spaces $X^\alpha_\varepsilon$, $\alpha \in \mathbb{R}$ we use the notation $\|\cdot\|_{\alpha,\varepsilon}$. In what follows we study the generation of a linear semigroup by the spatial differential operator in (1).

2.1. The linear semigroup

Consider the linear elliptic system differential operator of second order in (1) with given but homogeneous boundary conditions. Let $(A^\varepsilon, D(A^\varepsilon))$ be the operator

$$(A^\varepsilon, D(A^\varepsilon)) = (\text{diag}[A^\varepsilon_i], \Pi_{i=1}^m D(A^\varepsilon_i))$$

where for $i = 1, \ldots, m$

$$D(A^\varepsilon_i) = \left\{ u_i \in H^1(\Omega) : -\text{Div}(d^\varepsilon_i(x)\nabla u_i) + (V^\varepsilon_i + \lambda)u_i \in L^2(\Omega) \right\}.$$

Then the operator $A^\varepsilon : D(A^\varepsilon) \subset L^2(\Omega) \rightarrow L^2(\Omega)$, is maximal monotone and self-adjoint. Consequently, the scales of Banach spaces of Hilbert type $D(A^\alpha_\varepsilon) = X^\alpha_\varepsilon$ for $\alpha \in \mathbb{R}$ [15,19] are well defined and endowed with the graph norm

$$\|u\|_{\alpha,\varepsilon}^2 = \sum_{n=1}^{\infty} |\mu_n^{\varepsilon}|^{2\alpha} |u_n^\varepsilon|^2$$

for $u \in X^\alpha_\varepsilon$.

where $u_n = \langle u, \varphi_n^\varepsilon \rangle$, $n \in \mathbb{N}^*$ with inner product that of $L^2(\Omega)$ and $\varphi_n^\varepsilon \in H^1(\Omega)$ are eigenfunctions corresponding to the eigenvalues

$$0 < \mu_1^\varepsilon \leq \cdots \leq \mu_n^\varepsilon \leq \mu_{n+1}^\varepsilon \leq \cdots \nearrow \infty \quad \text{as } n \rightarrow \infty \quad (9)$$
of the operator (8). Here the scales of spaces of negative exponents define the dual spaces of ones in positive exponents.

An equivalent alternative approach to solving the eigenvalue problem of the operator (8) is given as a consequence of Lax–Milgram’s Theorem. This theorem can be found in [5, 17] among others. The basis of its application is proving in Hilbert spaces the existence and uniqueness of solutions to corresponding linear elliptic partial differential problems. Since in our system this part of the equations are weakly coupled, it suffices to analyze (8) in scalar form. Thus the bilinear form

$$a^\varepsilon: H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$$

is continuous, symmetrical and coercive.

These properties of (10) therefore remain valid for the system bilinear form. Consequently, if we maintain the same notation for the system bilinear form as in the left hand side of (10) then Lax–Milgram’s Theorem implies that the operator (8) defines an isomorphism been the spaces

$$H^1(\Omega)$$

and its dual $$H^{-1}(\Omega)$$. This isomorphism incorporate the linear part of the boundary conditions in (1). It is given for any $$u \in H^1(\Omega)$$ by $$L^\varepsilon(u) \in H^{-1}(\Omega)$$ such that

$$a^\varepsilon(u, \varphi) = \langle L^\varepsilon(u), \varphi \rangle \quad \forall \varphi \in H^1(\Omega).$$

Moreover, the restriction $$L^\varepsilon(u) \in L^2(\Omega)$$ is precisely the operator (8) satisfying that it is maximal monotone and self adjoint.

Now if we consider the homogeneous problem of the system (1). Then its solution for initial data in some Sobolev spaces $$H^s(\Omega), s \in \mathbb{R}$$ can be obtained via Hille–Yosida’s Theorem. This theorem can be found in many references among others we cite [5, 17]. The study of solutions of the type homogeneous evolution equation in general scales of Banach spaces is done in the following literature [2, 15, 19]. The homogeneous equations of (1) can be written as

$$\begin{cases} u_t + L^\varepsilon u = 0 \\ u(t_0) = u_0^\varepsilon \in X^\alpha_{e}, \quad \alpha \in \mathbb{R}. \end{cases} \quad (11)$$

Thus relating to the well posedness of (11) we have the following theorem.

**Theorem 2.1** ([15, 19]). Consider the homogeneous equations of the system (1) with initial $$u_0^\varepsilon \in X^\alpha_{e}, \alpha \in \mathbb{R}$$. Then the operator $$(L^\varepsilon, X^\alpha_{e+\frac{1}{2}})$$ is an infinitesimal generator of an analytic semigroup

$$\{S^\varepsilon(t); t \geq 0\} = \{\exp(-L^\varepsilon t), t \geq 0\} \subset X^\alpha_{e}, \quad \alpha \in \mathbb{R}$$

such that the mapping

$$t \mapsto u^\varepsilon(t) = \exp(-L^\varepsilon t)u_0^\varepsilon \in X^\alpha_{e}, \quad \alpha \in \mathbb{R}$$

and the problem (11) has a unique solution of class in $$\alpha = 0$$,

$$u^\varepsilon \in C([0, \infty), X^{-\frac{1}{2}}_{e}) \cap C^\alpha((0, \infty), X^\alpha_{e}), \quad \forall \alpha \in \mathbb{R}.$$ 

In addition, for any $$\alpha_0, \alpha_1 \in \mathbb{R}$$ the mapping $$S^\varepsilon(t): X^{\alpha_0}_{e} \to X^{\alpha_1}_{e}$$ is bounded and there exists $$M > 0$$ such that for any $$0 < \omega \leq \inf\{\mu : \mu \in \sigma(L^\varepsilon)\}$$ we have

$$\|S^\varepsilon(t)\|_{\alpha_0, \alpha_1} \leq \frac{M}{t^{\alpha_1-\alpha_0}}e^{-\omega t}, \quad \text{for all } t > 0 \quad (12)$$
where the spectrum \( \sigma(L^\alpha) = \sigma(A^\alpha) \). The estimate (12) is a smoothening effect of the semigroup.

This theorem says we can solve (13) for any initial data in the scales of spaces \( X^\alpha, \alpha \in \mathbb{R} \). This is not the case of the full system of equations (1). Obviously this is because of the contribution of the nonlinear terms in the equations.

2.2. The local nonlinear semigroup

Now we consider the full system of reaction and diffusion equations (1). Our aim is to prove the existence of a local nonlinear semigroup which is a dynamical system generated by the evolution equations. The system of partial differential equation (1) can be read as

\[
\begin{aligned}
    u_t + L^\alpha u &= h(u) \\
    u(t_0) &= u_0^\alpha \in X^\alpha_\varepsilon, \quad \alpha < \frac{3}{4}
\end{aligned}
\]  

(13)

for almost all \( t \in (0, T) \), where the nonlinearity \( h(u) \in H^{-1}(\Omega) \) is defined by

\[
\langle h(u), \varphi \rangle_{-1,1} \overset{\text{def}}{=} \int_\Omega f(u)\varphi + \int_{\Gamma} g(u)\varphi, \quad \forall \varphi \in H^1(\Omega).
\]

In abbreviation, we write this as

\[
h(u) \overset{\text{def}}{=} f_\Omega(u) + g_\Gamma(u) \in L^{\sigma_1}(\Omega) \oplus X^\beta_\varepsilon \subset H^{-1}(\Omega)
\]

where \( \frac{1}{\sigma_1} + \frac{1}{\beta} = 1 \) and \(-1/4 \geq \beta \geq -1/2 \). In this case, the Nemytskii operators

\[
f_\Omega : X^\alpha_\varepsilon \mapsto L^{\sigma_1}(\Omega) \quad \text{and} \quad g_\Gamma : X^\alpha_\varepsilon \mapsto X^\beta_\varepsilon
\]

(14)

are well defined and the Sobolev type space embeddings of the spaces \( X^\alpha_\varepsilon \subset L^\rho \) are continuous and compact.

**Theorem 2.2.** Consider the system of equations (1) with initial data \( u_0^\alpha \in X^\alpha_\varepsilon, \alpha < 3/4 \). Then there exists at most one locally defined mild solution \( u^\varepsilon \in C([t_0, T), X^\alpha_\varepsilon) \) given by the formula of variations of constants

\[
u^\varepsilon(t, u_0^\varepsilon) = e^{-L^\varepsilon(t-t_0)}u_0^\varepsilon + \int_{t_0}^t e^{-L^\varepsilon(t-s)}h(u(s))\,ds,
\]

(15)

such that the mapping \( (t_0, T) \ni t \mapsto u^\varepsilon(t) \in X^\alpha_\varepsilon, \alpha < 3/4 \) is locally Hölder continuous, and \( u^\varepsilon_\gamma \in C((t_0, t_1), X^\gamma_\varepsilon) \) for any \( \gamma < \beta + 1 \leq 3/4 \).

**Proof.** Technically, we seek solutions to the evolution equation (13) that are continuous functions on \((0, T)\) with values \( u^\varepsilon(t) \in X^\alpha_\varepsilon, \alpha < 3/4 \) and are fixed points of the nonlinear map \( F(u)(t) \) defined in the right hand side of (15) on some subspace \( V \subset C([t_0, t_0 + \delta], X^\alpha_\varepsilon) \) such that \( \|u(t) - u_0\| \leq \delta \) with \( h, \delta > 0 \) sufficiently small. Then use the extension theorems to get a solution on the maximum interval of existence \((0, T)\). This yields existence and uniqueness results in [15,19]. Thus it is sufficient to prove local Lipschitz continuity of the nonlinear terms in (1) to obtain our theorem.

Consider the nonlinear operators (14), denote by \( X \) either \( \Omega \) or \( \Gamma \) and without loss of generality assume \( |X| \geq 1 \). Also recall the following inequality

\[\quad (a + b)^p \leq 2^p(a^p + b^p), \quad a, b \geq 0, \quad p > 0.\]  

(16)
Now from (4) repeatedly using (16) and Hölder’s inequality we obtain that
\[
\int_X |N(u) - N(v)|^{\sigma_i} \leq 2^{2\sigma_i} L_j^{\sigma_i} |X| \left( \int_X |u|^{(\rho-1)\sigma_i} |u - v|^{\sigma_i} + \int_X |v|^{(\rho-1)\sigma_i} |u - v|^{\sigma_i} \right)
\]
\[
\leq 2^{2\sigma_i} L_j^{\sigma_i} \left( \frac{1}{\sigma_i} \left( \int_X |u|^{\rho+1} \right)^{\frac{\sigma_i}{\rho+1}} + \left( \int_X |v|^{\rho+1} \right)^{\frac{\sigma_i}{\rho+1}} + 1 \right)
\]
\[
\times \left( \int_X |u - v|^{\rho+1} \right)^{\frac{\sigma_i}{\rho+1}}
\]
where \( \frac{1}{\sigma_i} + \frac{1}{\rho} = 1 \) for \( i = 1, 2 \) with \( \rho > 1 \) as in (4). Therefore
\[
\| N(u) - N(v) \|_{L^{\sigma_i}(X)} \leq (2^{2\sigma_i+2}) \frac{1}{\sigma_i} L_j |X|^{\sigma_i}
\]
\[
\times \left( \| u \|_{L^{\rho+1}(X)}^{\rho-1} + \| v \|_{L^{\rho+1}(X)}^{\rho-1} + 1 \right) \| u - v \|_{L^{\rho+1}(X)},
\]
which is also valid in the case of \( |X| < 1 \). Moreover, the scales of spaces \( X^{\alpha}_\varepsilon \subset L^{\rho+1} \) where the embeddings are continuous and compact (see [15,18]). Thus the Nemytskii operators in (14) are locally Lipschitz continuous. Now applying results in [15,19] where from Proposition 7.4 and Theorem 7.8 in the last reference, we obtain that the proof to our theorem is complete. □

3. The global semigroup and universal attractor

In this section, we prove the existence of a global nonlinear semigroup and that of a compact attractor for the systems of equations (1). It is important in this sense that the nonlinear terms of the system of equations (1) must satisfy the following assumption.

**DISSIPATIVE CONDITIONS**

\[
\limsup_{|u| \to \infty} \frac{N(u)}{u} < 0, \quad \text{holds for the nonlinear terms in (1).}
\]

(17)

In what follows we shall prove the following theorem.

**Theorem 3.1.** Consider the system of evolution equations (1) and assume in addition to (4) that (D) holds respectively. Then the local dynamical system generated by the equations is bounded dissipative. In other words, there exists a nonnegative constant \( C \geq 0 \) such that if the initial condition of the system is in bounded subsets of the scales of spaces \( X^{\alpha}_\varepsilon, \alpha < \frac{3}{4} \). Then the nonlinear semigroup generated satisfies that

\[
\limsup_{t \to \infty} \| u_\varepsilon(t, u_\varepsilon^0) \|_{\alpha, \varepsilon} \leq C, \quad \forall \alpha < 3/4.
\]

(18)

Moreover, there exists a global compact attractor \( A^\varepsilon \subset X^\alpha, \alpha < 3/4 \) for the semiflow defined by the system of evolution equations.

**Proof.** To begin, let us define for functions in \( X^{\frac{1}{4}}_\varepsilon \equiv H^1(\Omega) \) the functional

\[
J_\varepsilon(\varphi) = \frac{1}{2} \| \varphi \|_{1/2, \varepsilon}^2 + \frac{\gamma}{2} \int_\Omega |\varphi|^2 - \int_\Omega F(\varphi) - \int_\Gamma G(\varphi)
\]

(19)
where $\gamma \geq 0$,

$$F(\varphi) = \int_0^\varphi f(s) ds \quad \text{and} \quad G(\varphi) = \int_0^\varphi g(s) ds$$

are anti-derivatives. Let $\gamma = 1$ in (19) then it is easy to see that there exists a $c > 0$ and that (D) imply also there is a $\beta > 1$ such that

$$-C + c\|\varphi\|_{1/2, \varepsilon}^2 \leq J_\varepsilon(\varphi) \leq \beta\|\varphi\|_{1/2, \varepsilon}^2 + C. \tag{20}$$

Therefore with $\varphi = u^\varepsilon$ in (19) we obtain that

$$\frac{d J_\varepsilon(u^\varepsilon(t))}{dt} \leq -\sigma J_\varepsilon(u^\varepsilon(t)) + C$$

$$\Rightarrow J_\varepsilon(u^\varepsilon(t)) \leq e^{-\sigma t} J_\varepsilon(u^\varepsilon(0)) + C(1 - e^{-\sigma t}) \tag{21}$$

for all $t \geq 0$ where $\sigma = c^{-1}$. Consequently from the implied part in (21) using (20) we conclude that the estimate (18) holds for $\alpha = 1/2$, provided $u^\varepsilon_0 \in X^\alpha_{\varepsilon}$ is in bounded subsets. Thus by Sobolev type embedding theorems [15,18] we conclude (18) is satisfied for all $\alpha \in \mathbb{R}^+$, $\alpha \leq 1/2$.

Now let $\alpha \in \mathbb{R}^+$ be as in the theorem such that $\alpha > \frac{1}{2}$. Since from the above $h(u) = f_\varepsilon(u) + g_\varepsilon(u) \in L^{01}(\Omega) \oplus X^\beta$ with $-1/4 \geq \beta \geq -1/2$ is bounded for all $t \geq 0$ in $H^{-1}(\Omega)$. Then the variations of constants formula (15) and the smoothening effect of the semigroup (12) for any $t > 0$ imply that

$$\|u^\varepsilon(t, u^\varepsilon_0)\|_{\alpha, \varepsilon} \leq M t^{-\alpha} e^{-\delta t} \|u^\varepsilon_0\|_{L^2(\Omega)} + \int_0^t \|e^{-L^\varepsilon(t-s)}\|_{\alpha, \beta} \|h(u^\varepsilon(s))\|_{\beta} ds$$

$$\leq M t^{-\alpha} e^{-\delta t} \|u^\varepsilon_0\|_{L^2(\Omega)} + M \sup_{t > 0} \|h(u^\varepsilon)\|_{\beta} \int_0^\infty e^{-\delta t} t^{-(\alpha-\beta)} dt, \tag{22}$$

where $0 < \alpha - \beta < 1$. But $u^\varepsilon_0 \in X^\alpha_{\varepsilon}$, $\frac{3}{4} > \alpha > \frac{1}{2}$ is in bounded subsets, hence (18) is concluded on taking the lim sup$_{t \to \infty}$ on both sides of the inequality (22).

The existence of a universal compact attractor $\mathcal{A}^\varepsilon \subset X^\alpha_{\varepsilon}$, $\alpha < 3/4$ for the semilinear system of equations (1) follows from the estimate (18). In fact the linear semigroup $\{S^\varepsilon(t) : t > 0\}$ is analytic and the mapping $u^\varepsilon \in X^\alpha_{\varepsilon} \to h(u^\varepsilon) \in H^{-1}(\Omega)$ is compact. This implies that the set $\{u^\varepsilon(t, u^\varepsilon_0) : t > 0\}$ is precompact. Hence, if $D = \{w \in X^\alpha_{\varepsilon} : \alpha < 3/4 : \|w\|_{\alpha, \varepsilon} \leq 2C\}$ then the $\omega$-limit set

$$\omega(u^\varepsilon_0) = \bigcap_{\tau \in \mathbb{R}^+} \bigcup_{t \geq \tau} u^\varepsilon(t, u^\varepsilon_0) \subset D$$

is the smallest nonempty subset, that is compact, connected, invariant and

$$\lim_{t \to \infty} \inf_{w \in \omega(u^\varepsilon_0)} \|u^\varepsilon(t, u^\varepsilon_0) - w\|_{\alpha, \varepsilon} = 0.$$

Therefore the universal compact attractor $\mathcal{A}^\varepsilon \cong \omega(D) \subset X^\alpha_{\varepsilon}$, $\alpha < 3/4$ for the system of semilinear equations (1) exists. Alternatively one obtains this global compact attractor $\mathcal{A}^\varepsilon \subset H^1(\Omega) \subset L^2(\Omega) \cong X^0_{\varepsilon}$ using Theorem 2.2 Sect.3.2 of [13]. Then using Theorem 1 in [7] and the invariance of the attractors concludes that the global compact attractor $\mathcal{A}^\varepsilon \subset X^\alpha_{\varepsilon}$ for $\alpha < 3/4$. \(\square\)
4. Construction of inertial manifold

In this section, we prove the existence of a globally Lipschitz center-unstable invariant manifold for the system of equations \((1)\). This inertial manifold, contains the global attractor in Theorem 3.1. We notice that, although the abstract theory is standard \([11,15,16,20,22]\), the novel aspect in this section is its concrete application to a system of equations with nonlinear boundary conditions.

Consider the evolution equation \((13)\). Let \(\rho > 0\) be sufficiently large and \(B_\rho\) be an absorbing set for the semiflow generated by the evolution equations of the system \((1)\) of semilinear reaction and diffusion equations. Assume that the global compact attractor in Theorem 3.1 is such that it is contained in this absorbing set, i.e. \(A_\rho \subset B_\rho\).

Let \(\Theta : \mathbb{R}^+ \mapsto [0,1]\) be a smooth function of at least class \(C^1\). Then, define \(\Theta_\rho(s) = \Theta(s/\rho^2)\) for \(s \geq 0\) such that

\[
\Theta_\rho(s) = \begin{cases} 
1 & \text{if } 0 \leq s \leq 2\rho^2, \\
0 & \text{if } s \geq 4\rho^2, \\
\end{cases}
\]

Also let \(h_\rho(u) = \Theta_\rho(\|u\|_{\alpha,\varepsilon})h(u)\) and consider the following evolution system of equations

\[
u_t + L^\varepsilon u = h_\rho(u) \quad \text{on } (0, T) \text{ a.e.,} \quad u(0) = u_0^\varepsilon \in X_\varepsilon^\alpha, \quad \alpha < 3/4. \tag{24}\]

We comment that Eqs. \((24)\) have the same long time dynamics as the system of equations \((1)\) in the neighbourhood of the absorbing set \(B_\rho\).

It is easy to see that the truncated nonlinearity \(h_\rho\) is of at least class \(C^1\) \((X_\varepsilon^\alpha, H^{-1}(\Omega))\), also that its support \(\text{supp } h_\rho \subset \{ u \in X_\varepsilon^\alpha : \|u\|_{\alpha,\varepsilon} \leq 2\rho \}\). In addition, we can verify the following properties are satisfied.

**Lemma 4.1.** Let \(h_\rho(u) \in X_\varepsilon^\beta\) for any \(-1/4 \geq \beta \geq -1/2\). Then there exists universal constants \(C \geq 0\) such that for any \(u, w, v \in X_\varepsilon^\alpha, \alpha < 3/4\) it holds that

\[
\begin{align*}
\|h_\rho(u)\|_{\beta,\varepsilon} & \leq C, \\
\|h_\rho(u) - h_\rho(w)\|_{\beta,\varepsilon} & \leq C\|u - w\|_{\alpha,\varepsilon}, \\
\|Dh_\rho(u)v\|_{\beta,\varepsilon} & \leq C\|v\|_{\alpha,\varepsilon}.
\end{align*}
\]

Moreover, the unique mild solution \(u_\varepsilon^\alpha \in C([0, \infty), X_\varepsilon^\alpha) \cap C^1((0, \infty), X_\varepsilon^\alpha) \cap C^1([0, \infty), X_\varepsilon^0)\) of the equation \((24)\) exists and has the property that it coincides with the solution of the system of equations \((13)\) on \(B_{2\rho}\).

Now we recall Theorem 2.2 in \([26]\).

**Theorem 4.2.** Consider the eigenvalue problem of the operator \((8)\) and denote the eigen pair solutions by \((\mu^\varepsilon_n, \varphi^\varepsilon_n) \subset \mathbb{R}^+ \times H^1(\Omega), n \in \mathbb{N}\). Let \(Z_\varepsilon = \text{span} [\varphi^\varepsilon_1, \ldots, \varphi^\varepsilon_m]\) Then,

\[
L_\varepsilon |_{Z_\varepsilon} \to L_0 = \text{diag}[\mu_n] : \mathbb{R}^m \to \mathbb{R}^m
\]

with associated eigenvectors \(\varphi_n = |\Omega|^{-\frac{1}{2}}\varepsilon_n\) for \(n = 1, \ldots, m\) and \(\{\varepsilon_n : 1 \leq n \leq m\}\) is the canonic basis of \(\mathbb{R}^m\) being strong limits in \(H^1(\Omega)\) as \(\varepsilon \to 0\) given \((5)\) is satisfied.

Furthermore,

\[
\lim_{\varepsilon \to 0} \int_\Omega d_\varepsilon^\alpha(x)|\nabla \varphi_n^\varepsilon|^2 = 0 \quad \text{and} \quad \liminf_{\varepsilon \to 0} \frac{\mu^\varepsilon_{m+1}}{\sigma_1(\varepsilon)} \geq \lambda_2^N \tag{26}\]
where $\lambda_2^N > 0$ is the principal eigenvalue of the $-\Delta$ Neumann eigenvalue problem in homogeneous boundary conditions.

**Proof.** Since we are assuming weak coupledness in the system of Eqs. (1) the system spatial differential operator (8) is not coupled. Thus we produce our argument in the scalar case. Consider the bilinear form (10). Since (8) is a maximal monotone and self-adjoint operator in $L^2(\Omega)$ we can solve the eigenvalue problem

$$
\begin{cases}
-\text{Div}(d^ε(x)\nabla u) + (V^ε + \lambda) u = \mu u & \text{in } \Omega \\
d^ε(x) \frac{\partial u}{\partial n} + b^ε u = 0 & \text{on } \Gamma.
\end{cases}
$$

Thus there exists a sequence of eigenvalues satisfying (9) and associated eigenfunctions $d^ε(x)\nabla u + (V^ε + \lambda) u = \mu u$ in $\Omega$.

Since we are assuming weak coupledness in the system of Eqs. (1) the system spatial boundedness of the first eigenvalue in (29). On the other hand, the bilinear form (10) is strongly in $H^1(\Omega)$, $n \in \mathbb{N}^*$ such that $\|\varphi_n^ε\|_{L^2(\Omega)} = 1$. These eigenvalues are characterized, see in [8], by the Rellich quotient as given by

$$
\mu_n^ε = \inf_{w \in H^1(\Omega) \setminus \{0\}} \|J_ε(w)\| = \sup_{w \in \mathbb{R}^n, w \neq 0} \inf_{w \in \mathbb{R}^n, w \neq 0} J_ε(w) = \inf_{w \in H^1(\Omega)} \sup_{w \in \mathbb{R}^n, w \neq 0} J_ε(w)
$$

where $n \geq 2$, $J_ε(w) = \frac{\varphi(w,w)}{\|w\|^2}$ and the first eigenvalue given by

$$
\mu_1^ε \overset{\text{def}}{=} \inf_{w \in H^1(\Omega) \setminus \{0\}} J_ε(w) = J_ε(\varphi_1^ε)
$$

is strictly positive. It is also simple, since for any two different eigenfunctions $\varphi$, $\psi \in H^1(\Omega) \setminus \{0\}$ associated to it we have that

$$
0 = J_ε(\varphi) - J_ε(\psi) \geq \beta \int_\Omega (\varphi - \psi)^2 \Rightarrow \int_\Omega \varphi \psi = \frac{1}{2} \left( \int_\Omega \varphi^2 + \int_\Omega \psi^2 \right) > 0
$$

for some $\beta > 0$. Thus the eigenfunctions can not be orthogonal in $L^2(\Omega)$ and the simplicity of the first eigenvalue (29) is proved.

Now from (28) we get that

$$
\mu_2^ε = \inf_{w \in H^1(\Omega)} \left\{ \frac{\int_\Omega |\nabla w|^2}{\int_\Omega |w|^2} \right\} + \beta_0 \to \infty
$$

where $\beta_0 > 0$, $\beta_1^ε \to \infty$ as $\varepsilon \to 0$ given (5). Thus the sequence $\{\varphi^ε_1 : \varepsilon > 0\} \subset H^1(\Omega)$ of first eigenfunctions is uniformly bounded in $\varepsilon > 0$. By compactness there exists a function $\varphi \in H^1(\Omega)$ as a weak limit to the above sequence of first eigenfunctions. Furthermore,

$$
\int_\Omega |\nabla \varphi|^2 \leq \liminf_{\varepsilon \to 0} \int_\Omega |\nabla \varphi^ε_1|^2 = 0.
$$

Consequently, $\varphi \notin W = \{w \in H^1(\Omega) : \int_\Omega w = 0\}$. But the Sobolev space embeddings of $H^1(\Omega) \subset L^2(\Omega)$ are compact; this implies that the convergence of the first eigenfunctions is strongly in $L^2(\Omega)$ and normalization of the eigenfunctions in $L^2(\Omega)$ gives that $\varphi = |\Omega|^{-\frac{1}{2}}$. Next, in view of (31) we obtain that the convergence of the first eigenfunctions is strongly in $H^1(\Omega)$, as $\varepsilon \to 0$ given that (5) is satisfied.

The first assertion in (26) is evident. Otherwise, we obtain a contradiction with the boundedness of the first eigenvalue in (29). On the other hand, the bilinear form (10) is
continuous. Hence passing to the limit as \( \varepsilon \to 0 \) in (29) and using hypothesis on the data we get that
\[
\mu_1 = \int_\Omega V + \frac{|\Gamma|}{|\Omega|} \int_\Gamma b + \lambda > 0.
\]

Suppose a similar estimate (30) also holds for the second eigenvalue using the third eigenvalue of the problem (27). This is impossible as we find a contradiction with the orthogonality of the associated second eigenfunction in the limit \( \varepsilon \to 0 \) given (5) holds. Thus in (30) if we define
\[
\lambda_{2, \varepsilon}^N \equiv \inf_{w \in H^1(\Omega) \setminus \{0\}, \int_\Omega w = 0} \left\{ \frac{\int_\Omega |\nabla w|^2}{\int_\Omega |w|^2} \right\}
\]
since \( w \perp \text{span}[v^\varepsilon, |\Omega|^{-\frac{1}{2}}] \) where \( v^\varepsilon \in W \) we get that \( \lambda_{2, \varepsilon}^N \geq \lambda_2^N \). Hence dividing throughout by \( \sigma_1(\varepsilon) \) and noting that \( \lim_{\varepsilon \to 0} \frac{\beta_\varepsilon}{\sigma_1(\varepsilon)} = 1 \), leads to
\[
\lim inf_{\varepsilon \to 0} \frac{\mu_2^\varepsilon}{\sigma_1(\varepsilon)} \geq \lambda_2^N.
\]
Thus the second eigenvalue of the scalar problem of (27) is unbounded as \( \varepsilon \to 0 \) given (5) is verified. Consequently, generalizing this proof to the noncoupled system eigenvalue problem associated with the operator (8) we obtain the conclusion of the theorem. \( \square \)

We comment that the last statements of Theorem 4.2, conclude the existence of an early realization of large eigenvalues, which usually take place as \( n \to \infty \), but in the case of (5) is taking place at the \( m + 1 \) eigenvalue of the eigenvalue problem associated with the differential operator in (8).

Following from Theorem 4.2, let
\[
X_{\varepsilon}^\alpha = Z^\varepsilon \oplus Z_{\perp, \varepsilon}^\alpha, \quad \alpha < 3/4 \quad \text{where} \ Z^\varepsilon = \text{span}[\varphi_1^\varepsilon, \ldots, \varphi_m^\varepsilon]
\]
and
\[
Z_{\perp, \varepsilon}^\alpha = \left\{ w \in X_{\varepsilon}^\alpha, \alpha < 3/4 : \langle w, \varphi \rangle = 0, \forall \varphi \in Z^\varepsilon \right\}.
\]
Next project the evolution system of equations (24) on the space \( X_{\varepsilon}^\beta, -1/4 \geq \beta \geq -1/2 \) to obtain the following system of equations
\[
\begin{align*}
L_0^\varepsilon p &= H_p(u), \quad p(0) = p_0^\varepsilon \in Z^\varepsilon \\
L^\varepsilon q &= H_q(u), \quad q(0) = q_0^\varepsilon \in Z_{\perp, \varepsilon}^\alpha
\end{align*}
\]
where \( p = P^\varepsilon u, q = Q^\varepsilon u \) and \( u = p + q \in Z^\varepsilon \oplus Z_{\perp, \varepsilon}^\alpha = X_{\varepsilon}^\alpha, \alpha < 3/4, L_0^\varepsilon = \text{diag}[\mu_n^\varepsilon] \) for \( n = 1, \ldots, m, H_p = P^\varepsilon h, H_q = Q^\varepsilon h \) and \( P^\varepsilon, Q^\varepsilon \) are orthogonal projection operators in \( L^2(\Omega) \).

Next we define some generic constants. To begin, let
\[
\sigma(\varepsilon) = \sigma_1(\varepsilon)\lambda_{2, \varepsilon}^N, \quad \sigma_D = D\lambda_{2, \varepsilon}^N
\]
then we shall denote by
\[
C(\varepsilon) \approx (\sigma(\varepsilon))^{-(\vartheta+1)} \Gamma(\vartheta) C, \quad C_{\sigma_D} = (\sigma_D)^{-\alpha} C
\]
respectively, where \( C \geq 0 \) is an arbitrary generic constant, \( \Gamma(\cdot) > 0 \) denotes the gamma function, \( 0 < \alpha \leq 1, 0 < \vartheta = (\alpha - \beta) < 1 \) such that for any fixed \( 0 < \varepsilon \leq \varepsilon_0 \) satisfying (5) we have the constants dependent on the diffusion of the system (1), \( C(\varepsilon), C_{\sigma_D} \ll 1 \) are sufficiently small.
4.1. The inertial manifold

Let us recall from [11,21,23] the following definition.

**Definition 4.3.** Let $E$ be a Banach space admitting topological subspaces $E_1, E_2$ such that $E = E_1 \oplus E_2$, where $\dim E_1 < \infty$. We shall denote by $(\mathcal{F}_b, \| \cdot \|_\infty)$ the union of all classes of bounded strongly continuous mappings $\Phi : E_1 \rightarrowrightarrow E_2$ with an upper bound some constant $b > 0$ and such that

$$
\text{supp} \Phi \subset \{ w \in E_1 : \| w \|_E \leq \sigma \}
$$

for some $\sigma \geq 2\rho$. Also these mappings are required to satisfy a uniform Lipschitz condition of a unit constant. The union of all classes of the above mappings is endowed with the supremum operator norm.

In the remainder of the paper, we consider a less general family of maps $\mathcal{F}_b$ that we obtain from the decomposition (32) to our phase space. We now prove the well posedness of the problem (24) $\iff$ (33).

**Theorem 4.4.** Assume $\Phi \in \mathcal{F}_b$. Then there exists at most one solution $(p^\varepsilon, q^\varepsilon) \in C(\mathbb{R}, Z^\varepsilon) \cap C^1(\mathbb{R}, X^\varepsilon_0)$ of Eqs. (33) and has the property that the last component is given by

$$
q^\varepsilon(t) = \int_{-\infty}^{t} S^\varepsilon(t-s)Hq(p^\varepsilon(s) + \Phi(p^\varepsilon(s)))ds, \quad \text{for } t \in \mathbb{R}.
$$

(37)

Moreover, the pair $(p^\varepsilon(t), q^\varepsilon(t))$ solve uniquely Eq. (24) for all $t \in \mathbb{R}$.

**Proof.** The proof is easy to obtain using Lemma 4.1 and Definition 4.3. In fact $H_p$ is globally Lipschitz continuous. Thus the standard finite dimension theory of differential equations yields the solution component $p^\varepsilon(t)$ exists for all $t \in \mathbb{R}$.

At the same time, this implies $H_q \in C(\mathbb{R}, H^{-1}(\Omega))$ is uniformly Lipschitz continuous, a result with which [18] pp. 50 and pp. 52 imply the regularity (36), see also [15]. In addition, Theorem 4.4.7 in [16] gives the representation (37).

The final statement is evident, because the systems of equations (24) and (33) are equivalent. Moreover the decomposition $u^\varepsilon(t) = p^\varepsilon(t) + \Phi(p^\varepsilon(t))$ with $q^\varepsilon(t) = \Phi(p^\varepsilon(t))$ of the solution to the evolution equation (24) for any fixed $t \in \mathbb{R}$ is unique. □

Now we make precise the concept of an inertial manifold [11,20,22].

**Definition 4.5.** Let $S^\varepsilon(t), t \geq 0$ denote the nonlinear semigroup generated by the system of Eqs. (1). An inertial manifold for this system of equations is defined as a subset $\mathcal{M}^\varepsilon \subset X^\varepsilon_{\alpha}, \alpha < 3/4$ satisfying that it is

1. A uniformly Lipschitz finite dimensional manifold.
2. A positively invariant manifold, i.e. $u^\varepsilon_0 \in \mathcal{M}^\varepsilon \Rightarrow S^\varepsilon(t)u^\varepsilon_0 \in \mathcal{M}^\varepsilon$ for all $t \geq 0$.
3. An exponential attractor, i.e. $\text{dist}_{X^\varepsilon_{\alpha}}(S^\varepsilon(t)u^\varepsilon_0, \mathcal{M}^\varepsilon) \rightarrow 0$ exponentially as $t \rightarrow \infty$, for all $u^\varepsilon_0 \in X^\varepsilon_{\alpha}, \alpha < 3/4$. 

A. Rodríguez Bernal, R. Willie / Nonlinear Analysis 67 (2007) 70–93
To prove the existence of an inertial manifold for the system of equations (1) we need be in the framework of Theorem 4.4. Then define

$$\mathcal{J}^\varepsilon \Phi(p_0^\varepsilon) = \int_{-\infty}^{0} S^\varepsilon(-s) H_q(u^\varepsilon(s)) ds, \quad \forall p_0^\varepsilon \in Z^\varepsilon$$

(38)

where $u^\varepsilon(s) = p^\varepsilon(s) + \Phi(p^\varepsilon(s))$ for $s \in \mathbb{R}^-$ is a solution to Eqs. (24). We now state the following theorem.

**Theorem 4.6.** The operator (38) maps the class $\mathcal{F}_b$ onto itself. It is a strict contraction mapping and the graph

$$\text{Graph} \, \Phi^\varepsilon = \cup_{\forall p^\varepsilon \in Z^\varepsilon} [p^\varepsilon, \Phi^\varepsilon(p^\varepsilon)]$$

(39)

where the pair $[p^\varepsilon(t), \Phi^\varepsilon(p^\varepsilon(t))]$, $t \in \mathbb{R}$ solves uniquely the system of equations (33), satisfy (1) and (2) of Definition 4.5.

**Proof.** To begin we fix the constant $b > 0$ of the class $\mathcal{F}_b$ in Definition 4.3. With this in mind, note that the last estimate (26) of Theorem 4.2 and the smoothing effect (12) of the linear semigroup imply

$$\| \mathcal{J}^\varepsilon \Phi(p_0^\varepsilon) \|_{\alpha, \varepsilon} \leq \int_{-\infty}^{0} \| S^\varepsilon(-s) \|_{\alpha, \beta} \| h_\rho(u^\varepsilon(s)) \|_{\beta, \varepsilon} ds$$

$$\leq C \int_{-\infty}^{0} e^{\sigma(\varepsilon)s} (s) ds \leq b,$$

since for any $0 < \varepsilon \leq \varepsilon_0$ such that (5) is satisfied it follows (26) imply that $\sigma(\varepsilon) > \sigma_D.$ Hence using (35) we can choose $b = (\sigma_D)^{-(\delta+1)} C \Gamma(\delta).$

Next we use the truncation properties of the nonlinearity (23) to obtain $\mathcal{J}^\varepsilon \Phi = 0$ if and only if $H_q(u(s)) = 0.$ But $H_q(u(s)) = 0$ if $\|u^\varepsilon(s)\|_{\alpha, \varepsilon} \geq 2 \rho.$ This implies $H_q(u(s)) \neq 0$ if $\|u^\varepsilon(s)\|_{\alpha, \varepsilon} < 2 \rho.$ Since $\|u^\varepsilon\|_{\alpha, \varepsilon} \geq \|p^\varepsilon\| - \|\Phi(p^\varepsilon)\|_{\alpha, \varepsilon} \geq \|p^\varepsilon\| - b$ we get that $H_q(u(s)) \neq 0$ on $\|p^\varepsilon\| < 2 \rho + b = \sigma.$ Clearly the closure $\overline{\{p^\varepsilon \in Z^\varepsilon : \|p^\varepsilon\| < \sigma\}}$ contains supp$\mathcal{J}^\varepsilon \Phi.$

Let $p_i^\varepsilon(\varepsilon) \in Z^\varepsilon, \, i = 1, 2$ be arbitrary. Then the second line in (25) yields

$$\| \mathcal{J}^\varepsilon \Phi(p_0^\varepsilon) - \mathcal{J}^\varepsilon \Phi(p_0^\varepsilon) \|_{\alpha, \varepsilon} \leq \int_{-\infty}^{0} \| S^\varepsilon(-s) \|_{\alpha, \beta} \| h_\rho(u_1^\varepsilon(s)) - h_\rho(u_2^\varepsilon(s)) \|_{\beta, \varepsilon} ds$$

$$\leq C \int_{-\infty}^{0} e^{\sigma(\varepsilon)s} (s) ds \| p_1^\varepsilon(s) - p_2^\varepsilon(s) \| ds.$$

(40)

Now following the results of [22]-Lemma 2.9 we have in general for any solutions $(p_1^\varepsilon, \Phi_1(p_1^\varepsilon)), \, i = 1, 2$ to the projected evolution equations (33) that

$$\| p_1^\varepsilon(s) - p_2^\varepsilon(s) \| \leq \left( \| p_1^\varepsilon(t) - p_2^\varepsilon(t) \| + C \sigma_0 \| \Phi_1 - \Phi_2 \|_{\infty} \right) e^{\sigma_D(t-s)}$$

(41)

for any $s \leq t.$ This implies with $t = 0$ in (40) that

$$\| \mathcal{J}^\varepsilon \Phi(p_0^\varepsilon) - \mathcal{J}^\varepsilon \Phi(p_0^\varepsilon) \|_{\alpha, \varepsilon} \leq C(\varepsilon) \| p_1^\varepsilon - p_2^\varepsilon \|,$$

(42)

which give uniform Lipschitzness of unit constant of the mapping (38). Consequently all the above conclude that (38) as a mapping $\mathcal{J}^\varepsilon : \mathcal{F}_b \rightarrow \mathcal{F}_b$ is well defined.
In the following, we notice that if \((p^0_i, \Phi_i(p^0_i))\) \(\in Z^k \times F_b\) for \(i = 1, 2\) solves the problem (33) with the same initial data \(p^0_i(0) = p^0_i\) with \(i = 1, 2\) in the first solutions components. Then

\[
|\mathcal{J}^e \Phi_1(p^0_0) - \mathcal{J}^e \Phi_2(p^0_0)| \leq \int_{-\infty}^{0} \|S^e_t(s)\|_{\alpha, \beta} \|h_\rho(u^e_1(s)) - h_\rho(u^e_2(s))\|_{\beta} ds \\
\leq 2C \int_{-\infty}^{0} \frac{e^{\sigma(s)\alpha}}{(s)^{\alpha-\beta}} \|p^e_1(s) - p^e_2(s)\| ds + C(\varepsilon),
\]

since \(\Phi \in F_b\) and the Definition 4.3 imply \(\|\nabla \rho \|_{L(Z^e, Z^u_{\perp, \perp})} \leq 1\). Hence using (41) with \(t = 0\) we obtain

\[
|\mathcal{J}^e \Phi_1(p^0_0) - \mathcal{J}^e \Phi_2(p^0_0)| \leq C(\varepsilon)\|\Phi_1 - \Phi_2\|_{\infty},
\]

which is possible always if \(C(\varepsilon) \ll 1\) is relatively small. Thus the inertial form (38) onto \(F_b\) is uniformly Lipschitz. It is also a strict contraction mapping. Thus by Banach Fixed Theorem there exists a unique fixed point \(\Phi^e \in F_b\) such that \(\Phi^e = \mathcal{J}^e \Phi\).

It only remains to show that \(\text{Graph} \Phi^e = \mathcal{M}^e\) is finite dimensional. But this is evident from the characterization (39), since the subspace \(Z^e\) is finite dimensional. To show that it is positively invariant, first notice that \(\mathcal{J}^e \Phi : Z^e \mapsto Z^u_{\perp, \perp}\) is a continuous function of \(p^e(t), t \in \mathbb{R}\) solution to the first equation in (33). Indeed,

\[
\mathcal{J}^e \Phi(p^e(t)) = \int_{-\infty}^{t} S^e_t(t-s)H_q(u^e(s))ds \quad \text{for any } t \in \mathbb{R}. \tag{44}
\]

Thus for any two solutions \(p^e_i \in Z^e\) for \(i = 1, 2\) to the first equation of (33) we have

\[
\|\mathcal{J}^e \Phi(p^e_1(t)) - \mathcal{J}^e \Phi(p^e_2(t))\|_{\alpha, \varepsilon} \leq C \int_{-\infty}^{t} \frac{e^{-\sigma(e)(t-s)}}{(t-s)^{\alpha-\beta}} \|p^e_1(s) - p^e_2(s)\| ds.
\]

Consequently (41) yields

\[
\|\mathcal{J}^e \Phi(p^e_1(t)) - \mathcal{J}^e \Phi(p^e_2(t))\|_{\alpha, \varepsilon} \leq C(\varepsilon)\|p^e_1(t) - p^e_2(t)\| \quad \text{for any } t \in \mathbb{R}
\]

which prove the assertion. Moreover, the solution component \(p^e(t)\) of the evolution equations (33) is defined for all \(t \in \mathbb{R}\). Thus \(\mathcal{J}^e \Phi \in C(\mathbb{R}, Z^u_{\perp, \perp})\).

To complete the proof of the theorem, it suffices to observe that \(\mathcal{J}^e \Phi = \Phi^e\) is a fixed point. Hence \((p^e(t), \Phi^e(p^e(t)))\) solves the projected evolution equations (33) for all \(t \in \mathbb{R}\). In particular, the Graph \(\Phi^e = \mathcal{M}^e\) is positively invariant. In fact it is easy to see that

\[
\mathcal{J}^e \Phi(p^e(t)) = \mathcal{J}^e \Phi(p^e(s); p^e(t)) = \mathcal{J}^e \Phi(p^e(s + t)) = \int_{-\infty}^{t} S^e_t(t)H_q(u^e(s))ds = \Phi(p^e(t)) = q^e(t), \quad \forall t \in \mathbb{R}
\]

where we have used uniqueness of solutions \(p^e(s; p^e(t)) = p^e(t + s; p^0_0), s \in \mathbb{R}\) as well as the representation (37). Thus \((p^e(t), q^e(t))\) for all \(t \in \mathbb{R}\) solves the problem (33). In particular we obtain the graph of the fixed point to \(\mathcal{J}^e \Phi\) is positively invariant. 

Up to now we have constructed a uniformly Lipschitz invariant manifold \(\mathcal{M}^e \subset X^u_{\varepsilon}, \alpha < 3/4\). To obtain an inertial manifold it only remains to show assertion (3) in Definition 4.5 holds. The proof we give below of this property runs exactly as in [22] Theorem 3.6-7. This is so because in our opinion it is much simpler. Moreover it is in line with the general technique in application. This method is one of appropriate use of the variation of constants formula. However there is an
alternative proof to the fact that this invariant manifold is an exponential attractor for the system of equations (1) \iff (24) see [11]. This approach uses the squeezing property which we state as follows.

(S1) SQUEEZING PROPERTY. For every \( r > 0 \) there is a \( K_1 \) depending on \( r \) and also on the operator (8) such that if the initial data \( u^\xi_0, w^\xi_0 \in X^\alpha_\varepsilon, \alpha < 3/4 \) with \( \|u^\xi_0\|_{\alpha,\varepsilon}, \|w^\xi_0\|_{\alpha,\varepsilon} \leq r. \) Then the nonlinear semigroup \( \{S^t(\cdot), t \geq 0\} \) defined by the system of equations (1) satisfy

\[
\|S^t(t)u^\xi_0 - S^t(t)w^\xi_0\|_{\alpha,\varepsilon} \leq \exp(K_1t)\|u^\xi_0 - w^\xi_0\|_{\alpha,\varepsilon}, \quad \forall t \geq 0.
\]

The proof given in [11] is not trivial. It uses equivalent formulations of the property (S1). These are proved elsewhere in references therein cited. These formulations are the following.

(S2) EQUIVALENT FORMULATION OF THE SQUEEZING PROPERTY. For every \( T, \gamma, r > 0 \) there exists constants \( K_i = K(T, \gamma, r) > 0 \) with \( i = 2, 3 \) such that for every \( 0 \leq t \leq T \) and for any bounded initial data \( u^\xi_0, w^\xi_0 \in B_r \subset X_\alpha^\varepsilon, \alpha < 3/4 \) it holds that

\[
\|Q^\xi(S^t(t)u^\xi_0 - S^t(t)w^\xi_0)\|_{\alpha,\varepsilon} \leq \gamma\|P^\xi(S^t(t)u^\xi_0 - S^t(t)w^\xi_0)\|
\]

\[
\|S^t(t)u^\xi_0 - S^t(t)w^\xi_0\|_{\alpha,\varepsilon} \leq K_2 \exp(-K_3\varepsilon^{1+i} t)\|u^\xi_0 - w^\xi_0\|_{\alpha,\varepsilon}
\]

(45)

where \( \delta > 0 \) is a small parameter.

It is correct to say the inequalities (45) state either the solution component of the infinite dimensional part of the system of equations (33) is estimated in terms of the finite dimensional one or the trajectories of (24) for fixed time \( t \geq 0 \) are sufficiently close to each other in norm of the phase space. This is true because \( \mu^{\xi}_{m+1} \gg 1 \) is sufficiently large (cf. (26) of Theorem 4.2), although we won’t use the approach in [11] to obtain condition (3) of Definition 4.5. The verification of the squeezing property (S1) for the evolution equation (24) is relatively easier.

**Proof.** Indeed, if \( u^\xi(t, u^\xi_0) \) and \( w^\xi(t, w^\xi_0) \) are any two solutions to (24) corresponding to the initial data \( u^\xi_0, w^\xi_0 \in B_r \subset X^\alpha_\varepsilon, \alpha < 3/4 \), then the variation of constant formula and the smoothing effect (12) of the semigroup imply that

\[
\|u^\xi(t) - w^\xi(t)\|_{\alpha,\varepsilon} \leq Ce^{-\delta t}\|u^\xi_0 - w^\xi_0\|_{\alpha,\varepsilon} + C \int_0^t e^{-\delta(t-s)}\|h_{\rho}(u^\xi(s)) - h_{\rho}(w^\xi(s))\|_{C_\alpha}\,ds
\]

\[
\leq Ce^{-\delta t}\|u^\xi_0 - w^\xi_0\|_{\alpha,\varepsilon} + C \int_0^t e^{-\delta(t-s)}\|u^\xi(s) - w^\xi(s)\|_{\alpha,\varepsilon}\,ds.
\]

Thus by Gronwall’s Lemma pp. 188 of [15] we get that

\[
e^{\delta t}\|u^\xi(t) - w^\xi(t)\|_{\alpha,\varepsilon} \leq C\|u^\xi_0 - w^\xi_0\|_{\alpha,\varepsilon} E_{\alpha}(\theta t), \quad \text{for all } t > 0
\]

where \( \theta = (C\Gamma(\alpha - \beta))^{1/\beta} > 0. \) Since \( E_{\alpha}(\theta t) \sim (\alpha - \beta + 1)^{-1} e^{\theta t} \) as \( t \to \infty \) it follows that for every \( \eta > 0 \) there is \( C_\eta > 0 \) such that

\[
e^{\delta t}\|u^\xi(t) - w^\xi(t)\|_{\alpha,\varepsilon} \leq C_\eta\|u^\xi_0 - w^\xi_0\|_{\alpha,\varepsilon} e^{(\theta + \eta)t}, \quad \forall t > 0
\]

consequently the squeezing property (S1) is obtained with \( K_1 = (\delta - \theta - \eta). \)

Now we prove assertion (3) of Definition 4.5. But before we do this, let us recall the topological concept of the metric of a point \( \psi \in X \) in a Banach space \( X \) from a subset, say \( N \subset X. \) This is defined as

\[
dist_X(\psi, N) = \inf_{\psi \in X, \psi \in N} \|\psi - \varphi\|_X.
\]

(46)
The main theorem is the following.

**Theorem 4.7.** Let \( u^\varepsilon (t, u^\varepsilon _0) \) for \( t \geq 0 \) denote the semiflow of the system of equations (1) with at \( t = 0 \) the initial condition \( u^\varepsilon _0 \in X^\varepsilon , \alpha < 3/4 \). Then, there exists \( C > 0 \) such that

\[
\text{dist}_{\mathcal{X}^\varepsilon} (u^\varepsilon (t, u^\varepsilon _0), \mathcal{M}^\varepsilon) \leq Ce^{-\sigma_2(\varepsilon)t} \| u^\varepsilon_0 \|_{\alpha,\varepsilon} \quad \forall t \geq 0,
\]

where \( \sigma_2(\varepsilon) = \sigma (\varepsilon) - \theta - \eta > 0 \) for some \( \theta, \eta > 0 \).

In other words, the invariant manifold \( \mathcal{M}^\varepsilon = \text{Graph} \mathcal{F}^\varepsilon \) of the system of equations (1) in Theorem 4.6 is a global exponential attractor. Moreover, by construction the universal compact attractor in Theorem 3.1; \( \mathcal{A}^\varepsilon \subset \mathcal{M}^\varepsilon \) is contained in the inertial manifold.

**Proof.** Consider the system (33) and carry out a transformation of variables given by \( (p, q) \mapsto (p, r) \) where \( r = q - \Phi^\varepsilon (p) \) with \( \Phi^\varepsilon \in \mathcal{F}_b \) being the unique fixed point of the inertial mapping (38).

In other words, define an equivalent system

\[
\begin{cases}
p_t + L^\varepsilon_0 p = H_p(u^\varepsilon), & p(0) = p^\varepsilon_0 \in Z^\varepsilon \\
r_t + L^\varepsilon r = G(u^\varepsilon), & r(0) = r^\varepsilon_0 \in Z^\varepsilon_{\alpha,\varepsilon}
\end{cases}
\]

where \( u^\varepsilon = p + \Phi^\varepsilon (p) + r \) is a solution to the system (24),

\[ G(u^\varepsilon) = H_q(p + \Phi^\varepsilon (p) + r) - \nabla_p \Phi^\varepsilon (p) H_p(p + \Phi^\varepsilon (p) + r). \]

In the following, we use the facts that \( \| \nabla_p \Phi^\varepsilon \|_{\mathcal{L}(Z^\varepsilon, Z^\varepsilon_{\alpha,\varepsilon})} \leq 1 \), \( \mathcal{M}^\varepsilon \) is invariant imply that \( G(p + \Phi^\varepsilon (p)) = 0 \) for all \( p \in Z^\varepsilon \), hence we can write

\[ G(u^\varepsilon) = H_q(p + \Phi^\varepsilon (p) + r) - H_q(p + \Phi^\varepsilon (p)) - \nabla_p \Phi^\varepsilon (p) [H_p(p + \Phi^\varepsilon (p) + r) - H_p(p + \Phi^\varepsilon (p))]. \]

Next we observe that if we compute (46) with solution coordinates \( (p^\varepsilon, \Phi^\varepsilon (p^\varepsilon) + r^\varepsilon) \in X^\varepsilon_{\alpha} \) of Eqs. (48), \( (p^\varepsilon, \Phi^\varepsilon (p^\varepsilon)) \in \mathcal{M}^\varepsilon \) of the system of equations (33), then we only need to find an estimate for the solution component \( r^\varepsilon \in Z^\varepsilon_{\alpha,\varepsilon} \) to the new Eqs. (48). Therefore, the variations of constants formula (15) and the smoothening properties (12) of the semigroup [15,18], the estimate (26) in Theorem 4.2 leads to that

\[
\| r^\varepsilon (t) \|_{\alpha,\varepsilon} \leq Ce^{-\sigma(\varepsilon)t} \| r^\varepsilon_0 \|_{\alpha,\varepsilon} + C \int_0^t \frac{e^{-\sigma(\varepsilon)(t-s)}}{(t-s)^{\alpha-\beta}} \| G(u^\varepsilon(s)) \|_{\beta,\varepsilon} ds
\]

\[
\leq Ce^{-\sigma(\varepsilon)t} \| r^\varepsilon_0 \|_{\alpha,\varepsilon} + C \int_0^t \frac{e^{-\sigma(\varepsilon)(t-s)}}{(t-s)^{\alpha-\beta}} \| r^\varepsilon (s) \|_{\alpha,\varepsilon} ds
\]

where we have taken into account (49), the first and second properties of the truncated nonlinearity in (25).

As a result from (50) using Gronwall’s Lemma [15] pp. 188, arguing as in the last part of the proof to show the squeezing property (S1), we get

\[ \| r^\varepsilon (t) \|_{\alpha,\varepsilon} \leq C\eta e^{-\sigma_2(\varepsilon)t} \| r^\varepsilon_0 \|_{\alpha,\varepsilon}, \quad \forall t > 0 \]

where \( \sigma_2(\varepsilon) = (\sigma (\varepsilon) - \theta - \eta) > 0 \) for any \( \varepsilon > 0 \) sufficiently small. This implies (47) since from the metric (46) we find

\[ \text{dist}_{\mathcal{X}^\varepsilon} ((p^\varepsilon, \Phi^\varepsilon (p^\varepsilon) + r^\varepsilon), \mathcal{M}^\varepsilon) \leq \| r^\varepsilon (t) \|_{\alpha,\varepsilon}, \]

hence our uniformly Lipschitz finite dimensional invariant manifold $\mathcal{M}^\varepsilon$ verifies condition (3) of Definition 4.5. Consequently it is an inertial manifold for the system of equations (1) (as far as the long time dynamics are concerned) and by construction contains the universal attractor $\mathcal{A}^\varepsilon$ in Theorem 3.1 of the solutions to the system (1) of equations. This completes the proof of the theorem. □

5. Asymptotics of inertial manifold

In this subsection we provide information on the asymptotic behavior of the inertial manifolds of the system of equations (1) in the limit case $\varepsilon = 0$ of large diffusivity. The following theorem is preliminary.

Lemma 5.1. Consider the semigroup $T(t) = e^{-L_0 t}; t > 0$ where $L_0 = \text{diag}[\lambda_n]$ for $n = 1, \ldots, m$ is defined as in Theorem 4.2. Then

$$
\lim_{\varepsilon \to 0} \sup_{t > 0} e^{\beta} \| S^\varepsilon (t) - T(t) \|_{\alpha_1, \alpha_0} = 0, \quad \forall \alpha_0, \alpha_1 \in \mathbb{R}
$$

(51)

where $S^\varepsilon(t), t \geq 0,$ is the semigroup generated by the homogeneous equations of (1) in Theorem 2.1 and $\inf\{\mu_n; 1 \leq n \leq m\} > \delta.$

Proof. This proof is for completeness. Let $w \in X_\varepsilon^{\alpha_0}, \alpha_0 \in \mathbb{R}$ then write a realization of the finite dimensional semigroup as

$$
T(t)w = \sum_{n=1}^{m} e^{-\mu_n t} (\mu_n^\varepsilon)^{\alpha_1} w_n \phi_n \in X_\varepsilon^{\alpha_1}, \quad t \geq 0
$$

(52)

where $\phi_n = |\Omega|^{-1/2} \tilde{e}_n, \{\tilde{e}_n; 1 \leq n \leq m\}$ is the canonical basis of $\mathbb{R}^m$, $w_n = \langle w, \phi_n \rangle, n = 1, \ldots, m.$ Since the spectral representation of the semigroup to (13) is given by $S^\varepsilon(t)w = \sum_{n=1}^{\infty} e^{-\mu_n t} |\mu_n^\varepsilon|^{\alpha} w_n \phi_n \in X_\varepsilon^\alpha, t \geq 0$ for any $w \in X_\varepsilon^\alpha$, where $(\mu_n^\varepsilon, \phi_n^\varepsilon), n \in \mathbb{N}$ are solutions to the system eigenvalue problem in the operator (8) and $w_n = \langle w, \phi_n^\varepsilon \rangle.$ Then

$$
e^{2\beta} \| (S^\varepsilon(t) - T(t))w \|^2_{\alpha_1, \varepsilon} \leq \sum_{n=m+1}^{\infty} e^{-2\mu_n t} |\mu_n^\varepsilon|^{2(\alpha_1 - \alpha_0)} |\mu_n^\varepsilon|^{2\alpha_0} |w_n^\varepsilon|^2 + \sum_{n=1}^{m} \left( e^{-2\mu_n t} - e^{-\mu_n t} \right)^2 |\mu_n^\varepsilon|^{2(\alpha_1 - \alpha_0)} |\mu_n^\varepsilon|^{2\alpha_0} |w_n^\varepsilon|^2 + 2e^{2\mu_n t} |\mu_n^\varepsilon|^{2(\alpha_1 - \alpha_0)} |\mu_n^\varepsilon|^{2\alpha_0} |w_n^\varepsilon|^2 \| \phi_n^\varepsilon - \phi_n \|^2_0.
$$

This enables us to deduce the following

$$
e^{2\beta} \| (S^\varepsilon(t) - T(t))w \|^2_{\alpha_1, \varepsilon} \leq \left( \sup_{n \geq m+1} \left\{ \frac{|\mu_n^\varepsilon|^{\alpha_1 - \alpha_0}}{e^{\mu_n t_0}} \right\}^2 \right) + \left( \sup_{1 \leq n \leq m} \left\{ \sup_{t \geq t_0} \left\{ e^{-\mu_n t} - e^{-\mu_n t} \right\}^2 \right\} \right) + 2e^{2\mu_n t_0} \left( \sup_{1 \leq n \leq m} \left\{ |\mu_n^\varepsilon|^{\alpha_1 - \alpha_0} \| \phi_n^\varepsilon - \phi_n \|^2_0 \right\}^2 \right) \| w \|^2_{\alpha_0}.$$

for any $0 < t_0 \leq t$. Thus implying the conclusion on taking into account the asymptotic behaviour in (5) of eigenvalues and eigenfunctions of the eigenvalue problem associated with the operator (8) in Theorem 4.2. □

5.1. Upper semicontinuity of attractors

This subsection is complementary. This is because its results can be obtained from those in our previous works [25,27]. Its concern is the asymptotic dynamics of the attractors for the system (1) of equations when (5) is assumed. The theorem to be proved is the following.

**Theorem 5.2.** Consider the system of equations (1) with initial data in $X^\alpha_\varepsilon$, $\alpha < 3/4$. Also consider the ordinary differential system of equations (7) and assume the condition (D) holds.

Then the system of equations (7) has a universal compact attractor $\mathcal{A} \subset \mathbb{R}^m$. Moreover, the family of attractors $\{A^{\varepsilon} \cup \mathcal{A}; 0 \leq \varepsilon \leq \varepsilon_0\}$ satisfy

$$\lim_{\varepsilon \to 0} \sup_{u^{\varepsilon} \in A^{\varepsilon}} \inf_{u_0 \in \mathcal{A}} \|u^{\varepsilon}(t; u_0^{\varepsilon}) - u(t; u_0)\|_{\alpha, \varepsilon} = 0, \quad \forall \alpha < 3/4.$$  \hspace{1cm} (53)

Thus the attractors are upper semicontinuous in $\varepsilon = 0$ in norm of the spaces $X^\alpha_\varepsilon$, $\alpha < 3/4$.

**Proof.** By assumption (D) solutions to the ordinary differential system of equations (7) are bounded dissipative. Moreover, this system of equations is finite dimensional. Thus the existence of a universal compact attractor $\mathcal{A} \subset \mathbb{R}^m$ is evident.

Next consider (18) with $\alpha = 1/2$ and write the system of equations (24) as one elliptic by passing the term in the time partial derivative to the right hand side of the equations. Then results in [26] conclude the system of ordinary differential equations (7) is a limit of the infinite dimensional equations (1) when (5) is assumed. Now the variations of constants formula (15) implies the solution component in $Z_{\alpha, \varepsilon}^\perp$ converges strongly in norm of $X^\alpha_\varepsilon$, $\alpha < 3/4$ to the constant function zero given (5) satisfied. This is because of the last statement in (26). The rest follows by the first part of Theorem 4.2, since the solution component of (1) in $Z^\varepsilon$ verifies a finite dimensional system of equations.

Thus solutions of (1) converge strongly to those of (7) in norm of $X^\alpha_\varepsilon$, $\alpha < 3/4$ given (5) is satisfied. Consequently, (53) is concluded and the proof of the theorem is complete. □

5.2. Nesting inertial manifolds

Finally we study the behaviour of the inertial manifolds of the system of equations (1) in the limit (5) of large diffusion. The not trivial question is that of the existence of a limit inertial manifold. This question in analogy has been answered affirmatively in [10,11,21,22] but is not of the same type of asymptotic behaviour we wish to study.

To render a solution we need make the following assumption.

**(H)** Consider the operator (8) for fixed $1 < i < \frac{m}{2} + 1 : \forall i + 1, \ldots, m \Rightarrow \lambda = D\lambda_2^N$

where $D \gg 1$ is sufficiently large.

Now Theorem 4.2 implies the following decomposition of the eigenvalues associated with the operator (8),

$$\sigma(A^\varepsilon) = \sigma_1(A^\varepsilon) \cup \sigma_2(A^\varepsilon) \cup \sigma_3(A^\varepsilon)$$
where the spectral sets $\sigma_i(A^\varepsilon)$, $i = 1, 2$ are bounded, while the set $\sigma_3(A^\varepsilon)$ is unbounded given (5) is verified. Moreover, the projection operators $P^\varepsilon = P_1^\varepsilon + P_2^\varepsilon$, $Q^\varepsilon$ in $L^2(\Omega)$ are well defined in the scales of spaces $X^\varepsilon_M$, $\alpha \in \mathbb{R}$. In particular, if $\alpha < 3/4$ we have

$$X^\varepsilon_M = Z_1^\varepsilon \oplus Z_2^\varepsilon \oplus Z_{\perp,\varepsilon}$$

where the subspaces $Z_i^\varepsilon$, $i = 1, 2$ are of a finite dimension and $\dim Z_1^\varepsilon + \dim Z_2^\varepsilon = m$.

Consider the following system of evolution equations

$$\begin{cases}
\dot{p}_i + L_0^\varepsilon p_i = h_i(u), & p_i(0) = p_i^0(\varepsilon) \in Z_i^\varepsilon, \quad i = 1, 2,
\dot{q}_t + L^\varepsilon q = h_3(u), & q(0) = q_0^\varepsilon \in Z_{\perp,\varepsilon},
\end{cases}$$

(54)

where $p_i = P_i^\varepsilon u$, $i = 1, 2$, $q = Q^\varepsilon u$ so that $u = p_1 + p_2 + q$, $h_i = P_i^\varepsilon h_\rho$ for $i = 1, 2$, $h_3 = Q^\varepsilon h_\rho$. Our main theorem is the following.

**Theorem 5.3.** Consider the system of semilinear equations (1) and assume condition (H) is satisfied. Then there exists a finite dimensional inertial manifold $N^\varepsilon = N^\varepsilon_1 \cup M^\varepsilon \subset X^\varepsilon_M$ with $N_1^\varepsilon \subset Z_1^\varepsilon \oplus Z_2^\varepsilon$ for this system of equations, that embeds the inertial manifold component $M^\varepsilon$ of Theorems 4.6 and 4.7. Moreover we have that

$$\lim_{\varepsilon \to 0} \sup_{\varepsilon^* \in N^\varepsilon} \inf_{\varepsilon \in N^\varepsilon} \| \varphi^\varepsilon - \varphi \|_{\alpha, \varepsilon} = 0 \quad \text{for any } \alpha < 3/4,$$

(55)

where $N$ is an inertial manifold for the ordinary differential system of equations (7). In other words, the family of inertial manifolds $\{N^\varepsilon \cup N, 0 < \varepsilon \leq \varepsilon_0\}$ is upper-semicontinuous in $\varepsilon = 0$.

Note that the first part of the theorem is proved if we show the existence of the inertial manifold component $N_1^\varepsilon \subset Z_1^\varepsilon \oplus Z_2^\varepsilon$ for the finite dimensional equations in the vector system (54). It is readily seen that the proof of the referred statements can be obtained from Theorems 4.6 and 4.7. However we will produce it for completeness in our presentation of the results and in order to achieve a comprehensible proof we break this into two parts as comprising the theorem.

**Proof.** Part 1. Consider the vector system (54) of equations. Note that if $\Phi^\varepsilon \in \mathcal{F}_b$ then Theorem 4.4 imply that we can solve the system of equations (24) for all time $t \in \mathbb{R}$. Thus in equivalence we are able to solve its projection (54) in $Z_1^\varepsilon \oplus Z_2^\varepsilon \oplus Z_{\perp,\varepsilon}, \alpha < 3/4$ for all time $t \in \mathbb{R}$.

Next define the inertial form

$$\mathcal{J}_1^\varepsilon \Phi(p^\varepsilon(t)) = \int_{-\infty}^t T^\varepsilon(t-s)h_2(u^\varepsilon(s))ds, \quad \text{for any } t \in \mathbb{R}^{-}, p^\varepsilon \in Z_1^\varepsilon$$

(56)

where the semigroup $T^\varepsilon(t)$, $t \geq 0$ is the restriction of the semigroup $S^\varepsilon(t)$, $t \geq 0$ on the subspace $Z^\varepsilon$, $u^\varepsilon(s) = p^\varepsilon(s) + \Phi(p^\varepsilon(s)) + q^\varepsilon(s)$ is a solution to the semilinear evolution equations (24).

It is easy to see that Definition 4.3 holds for the inertial form in (56). Indeed, starting with the unit uniform Lipschitz condition. It follows from Lemma 4.1 and the estimates (41) that

$$\| \mathcal{J}_1^\varepsilon \Phi(p^\varepsilon(t)) - \mathcal{J}_1^\varepsilon \Phi(r^\varepsilon(t)) \| \leq \int_{-\infty}^t \| T^\varepsilon(t-s)\|_{\alpha, \beta} \| h_2(u^\varepsilon_1(s)) - h_2(u^\varepsilon_2(s)) \|_\beta ds \leq C_{\sigma_D} \| p^\varepsilon(t) - r^\varepsilon(t) \| \leq \| p^\varepsilon(t) - r^\varepsilon(t) \|$$

for any $p^\varepsilon, r^\varepsilon \in Z_1^\varepsilon, t \in \mathbb{R}^{-}$ since $C_{\sigma_D} \ll 1$ is small. Furthermore, if we choose $b = 2C_{\sigma_D}$ then
\[ \|J^e_1 \Phi(p^e(t))\| \leq \int_{-\infty}^{t'} \|T^e(t-s)\|_{\alpha, \beta} h_2(u^e_s(s)) \|_\beta ds \]
\[ \leq Ce^{-2\sigma_D t} \int_{-\infty}^{t'} \frac{e^{2\sigma_D s}}{(t-s)^{\alpha-\beta}} ds \leq b, \]
which yields the desired boundedness in Definition 4.3.

It remains only to show the second condition of the same definition holds. But this follows a similar argument as that in the second step of proving Theorem 4.6. Thus (56) maps elements of the class \( \mathcal{F}_b \) onto elements of the same class. In particular, when \( t = 0 \) we have the inertial form is well defined.

In the following, for any \( \Phi_1, \Phi_2 \in \mathcal{F}_b \) it holds that
\[ \|J^e_1 \Phi_1(p^e(t)) - J^e_1 \Phi_2(p^e(t))\| \leq \int_{-\infty}^{t'} \|T^e(t-s)\|_{\alpha, \beta} h_2(u^e_1(s)) - h_2(u^e_2(s)) \|_\beta ds \]
\[ \leq C \int_{-\infty}^{t'} \frac{e^{\sigma_D (t-s)}}{(t-s)^{\alpha-\beta}} \|\Phi_1(p^e(s)) - \Phi_2(p^e(s))\| ds \]
\[ \leq 2C \int_{-\infty}^{t'} \frac{e^{\sigma_D (t-s)}}{(t-s)^{\alpha-\beta}} \|p(s) - r(s)\| ds + 2Cb \int_{-\infty}^{t'} \frac{e^{2\sigma_D (t-s)}}{(t-s)^{\alpha-\beta}} ds. \]
Therefore on using (41) with \( p^e(t) = r^e(t) \) for \( s \leq t \) we find that
\[ \|J^e_1 \Phi_1(p^e(t)) - J^e_1 \Phi_2(p^e(t))\| \leq C_{\sigma_D} \|\Phi_1 - \Phi_2\|_\infty + C_{\sigma_D} \|\Phi_1 - \Phi_2\|_\infty. \quad (57) \]
Thus \( J^e_1 \) satisfy the uniform Lipschitz of unit constant condition. In addition, it is a strict contraction mapping on a subset of \( \mathcal{F}_b \). Consequently, the Banach Fixed Point Theorem yields the existences of a unique \( \phi^e \in \mathcal{F}_b \) such that \( J^e_1 \phi = \phi^e \in \mathcal{F}_b \). In particular, the Graph \( \phi^e = N^e_1 \) is positively invariant and its finite dimension is \( i - 1 \).

That the set \( N^e_1 \) is an inertial manifold follows as in the proof of Theorem 4.7 with obvious modifications. This proves that the invariant manifold is an exponential attractor with \( \sigma_2(\epsilon) = \sigma_D \) and the rest is evident. \qed

It only remains to prove the second statement of Theorem 5.3. Before this note that following the vector formulation (54) and uniqueness of the fixed point to the inertial form we have
\[ J^e \Phi = J^e_1 \Phi + J^e_2 \Phi = \phi^e \in \mathcal{F}_b \]
where the representation is unique and the realization of the inertial form component \( J^e_2 \) is exactly identically to that of \( J^e \) defined in (38) while restricted on a smaller finite dimensional subspace \( Z^e_2 \).

Hence it should be understood why we have maintained the same notation \( \mathcal{M}^e \) for the inertial manifold defined by the graph of the fixed point to \( J^e_1 \phi \). More precisely, the inertial manifold \( N^e \) embeds this component, i.e. \( N^e_2 = \mathcal{M}^e \subset N^e \).

It is interesting to note that the result of the first statement in Theorem 5.3 is in agreement with the general theory developed in [11] Section 5.2-3 on vector formulation of the operator \( J^e \) and nesting of inertial manifolds.

Let us pass to the proof of the second half of Theorem 5.3. First observe that there are two equivalent ways to it. These are the following.

(A1) The first approach proves
\[ \phi^e \to \phi \quad \text{strongly in } C_b(Z^e, Z^e_{\epsilon, \perp}) \quad \text{and } \Phi \in \mathcal{F}_b \]
as $\varepsilon \to 0$, where $\text{Graph } \Phi^\varepsilon = \mathcal{N}^\varepsilon$ is the inertial manifold for the system (1). Then

$$\text{Graph } \Phi = \mathcal{N}$$

is an inertial manifold for (7) and (55) is verified.

(A2) The second approach shows that

$$\exists! \mathcal{N} \subset \mathbb{R}^m - \text{ inertial manifold for (7)}$$

and for any pair $(p, \Phi^\varepsilon(p)) \in \mathcal{N}^\varepsilon$, $(r, \Phi(r)) \in \mathcal{N} \Rightarrow (55)$ is satisfied.

We opt for the second alternative. To highlight the initial ideas we observe that by virtue of assumption (H) and Theorem 4.2, there exists a large spectral gap in the spectrum $\sigma(L_0) = \sigma_1(L_0) \cup \sigma_2(L_0)$. In addition, the convergence of the eigenfunctions in Theorem 4.2 imply a strong convergence in operator norm of the space $L^2(\Omega)$ for the projection operators used in (33). Let $V, V_\bot$ be subsets of $\mathbb{R}^m$ associated with the spectral sets $\sigma_i(L_0), i = 1, 2$ such that

$$\mathbb{R}^m = V \oplus V_\bot.$$  \hspace{1cm} (58)

Then consider the vector system of equations

$$\begin{cases}
    \dot{u}_1 + L_{0,1}u_1 = h_1(u), & u_1(0) = u_1^0 \in V \\
    \dot{u}_2 + L_{0,2}u_2 = h_2(u), & u_2(0) = u_2^0 \in V_\bot
\end{cases}$$  \hspace{1cm} (59)

where $Q = I - P$, $P$ are projection operators in $\mathbb{R}^m$ and $u_1 = Pu, u_2 = Qu, h_1 = Ph_p, h_2 = Qh_p$. This is a projected system of the ordinary differential equations (7) in the neighbourhood of the attractor $A \subset \mathbb{R}^m$ in Theorem 5.2.

**Proof.** Part 2. A2.EXISTENCE. Consider the mapping

$$\mathcal{J} \Phi(u_1(t)) = \int_{-\infty}^{t} T(t-s)h_2(u(s))ds, \quad \text{for any } t \in \mathbb{R}^-, u_1 \in V$$  \hspace{1cm} (60)

where $u(s) = u_1(s) + \Phi(u_1(s)), T(t) = e^{-L_0t}, t \in \mathbb{R}$. Next if we set $b = (2\sigma_D)^{-1}C$ it follows that

$$\|\mathcal{J} \Phi(u_1(t))\| \leq \int_{-\infty}^{t} \|T(t-s)\|\|h_2(u(s))\|ds \leq b$$

on using the properties (25).

Now use the truncation properties (23) of the nonlinear term and the definition of the mapping (60). It follow there is $\sigma = 2\rho + b > 0$ such that $\mathcal{J} \Phi \subset \{w \in V : \|w\| \leq \sigma\}$. On the other hand from (41) there exists $C_{\sigma_D} = (\sigma_D)^{-1}C > 0$ such that for any two solutions $(w, \Phi(w)), (v, \Phi(v))$ of the system of equations (59) we have

$$\|\mathcal{J} \Phi(w(t)) - \mathcal{J} \Phi(v(t))\| \leq C_{\sigma_D}\|w(t) - v(t)\| \quad \text{for any } t \in \mathbb{R}^-.$$  \hspace{1cm} (61)

In particular when $t = 0$ in (60) all the above conclude the mapping (60) is well defined on a subset of $\mathcal{F}_b$ onto itself. Therefore on this subset is an inertial form.

Let $\mathcal{V} \subset \mathcal{F}_b$ denote the above subset. It follows from a similar argument as in obtaining the estimate (57) but taking into account that there is no regularizing effect of the semigroup in the estimates, that (60) is uniformly Lipschitz continuous and is a strict contraction mapping on $\mathcal{V} \subset \mathcal{F}_b$. Hence there exists a unique fixed point $\Phi \in \mathcal{V}$ such that $\mathcal{J} \Phi = \Phi$. This also implies the $\text{Graph } \Phi = \mathcal{N}$ is a uniformly Lipschitz invariant manifold for the system (59).
In addition, if we consider the system
\[\begin{align*}
\dot{u}_1 + L_{0,1}u_1 &= h_1(u), \quad u_1(0) = u_0^1 \in V \\
\dot{u}_3 + L_{0,2}u_3 &= F(u), \quad u_3(0) = u_0^3 \in V_\perp
\end{align*}\]
where \(u_3 = u_2 - \Phi(u_1)\) and
\[F(u) = h_2(u) - L_{0,2} \Phi(u_1) - \nabla u_1 \Phi(h_1(u) - L_{0,1}u_1)\]
with \(u = u_1 + \Phi(u_1) + u_3\). Then \(F(u_1 + \Phi(u_1)) = 0\) if \(u_3 = 0\). Thus we can write
\[F(u) = h_2(u_1 + \Phi(u_1) + u_3) - h_2(u_1 + \Phi(u_1)) + \nabla \Phi(h_1(u_1) + \Phi(u_1) - h_1(u_1 + \Phi(u_1) + u_3))\].

It now follows by the variation of constant formula and unit Lipschitz property of the mappings of the class \(V\) that
\[\|u_3(t)\| \leq e^{-2\sigma_Dt} \|u_0^3\| + \int_0^t e^{-2\sigma_D(t-s)} \|F(u(s))\| ds \]
\[\leq e^{-2\sigma_Dt} \|u_0^3\| + 2C \int_0^t e^{-2\sigma_D(t-s)} \|u_3(s)\| ds.\]

Thanks to the standard Gronwall’s Inequality [17] pp. 10 it follows that
\[\|u_3(t)\| \leq e^{-2(\sigma_D-C)t} \|u_0^3\|, \quad \forall t \geq 0,\]
where \(\sigma_D - C > 0\). Thus for any pair of coordinate solutions \((u_1, \Phi(u_1) + u_3) \in \mathbb{R}^m\) of the equation (7), \((u_1, \Phi(u_1)) \in \mathcal{N}\) of the vector system of equations (59) from (46) we conclude that
\[\text{dist}((u_1, \Phi(u_1) + u_3), \mathcal{N}) \leq \|u_3(t)\|\]
Consequently we have proved that the invariant manifold is an exponential attractor. This yields the subset \(\mathcal{N} \subset \mathbb{R}^m\) is an inertial manifold for the system of ordinary differential equations (7).

A2. CONVERGENCE. Let \(\mathcal{N}^{\alpha}\) denote the inertial manifold of the first part of the theorem. Then, for any \((p^\varepsilon, \Phi^\varepsilon (p^\varepsilon)) \in \mathcal{N}^{\alpha} ,(r, \Phi(r)) \in \mathcal{N}\) we find that
\[\| (p^\varepsilon, \Phi^\varepsilon (p^\varepsilon)) - (r, \Phi(r)) \|_{\alpha, \varepsilon} \leq 2 \| p^\varepsilon - r \| + \| \Phi^\varepsilon (r) - \Phi(r) \|_{\alpha, \varepsilon}\]
(61)
where we have added a zero, used the norm inequality and the fact that \(\| \nabla_p \Phi^\varepsilon \| \leq 1\) on \(L(Z^\varepsilon, Z_{\varepsilon, \perp}^\alpha)\). On the other hand, for any \(t \in \mathbb{R}^-\) one has
\[\| \Phi^\varepsilon (r) - \Phi(r) \|_{\alpha, \varepsilon} \leq C \int_{-\infty}^t \| S^\varepsilon (t-s) - T(t-s) \|_{\alpha, \beta} ds + \int_{-\infty}^t \| T(t-s) \|_{\alpha, \beta} \| h(u^\varepsilon(s)) - h(u(s)) \|_{\beta} ds \]
\[\leq C \int_{-\infty}^t \| S^\varepsilon (t-s) - T(t-s) \|_{\alpha, \beta} ds + C_{\sigma_D} \| \Phi^\varepsilon (r) - \Phi(r) \|,\]
where \(u^\varepsilon(s) = r(s) + \Phi^\varepsilon (r(s)), u(s) = r(s) + \Phi(r(s))\), using the same line of argument as in PART 1 leading to the uniform Lipschitz continuity and strict contraction mapping of (56). Therefore, the Dominated convergence theorem of Lebesgue and the convergence in operator norm of semigroups Lemma 5.1 imply
\[ \| \Phi^\varepsilon(r) - \Phi(r) \|_{\alpha, \varepsilon} \leq C(1 - C_{\sigma_D})^{-1} \int_{-\infty}^t \| S^\varepsilon(t-s) - T(t-s) \|_{\alpha, \beta} ds \to 0 \]

for any \( t \in \mathbb{R}^+ \) fixed, as \( \varepsilon \to 0 \).

Now returning to (61) since the family of solutions \( \{ u^\varepsilon(t) = p^\varepsilon(t) + \Phi^\varepsilon(p^\varepsilon(t)) : 0 < \varepsilon \leq \varepsilon_0 \} \) of the evolution equation (24) is bounded independent of \( \varepsilon > 0 \) in \( X^\alpha_{\varepsilon} \), \( \alpha < 3/4 \) for all time \( t \in \mathbb{R} \) the weak compactness and convergence of the eigenfunctions in Theorem 4.2 conclude

\[ p^\varepsilon(t) \to r(t) = u(t) \quad \text{strongly in } Z^\varepsilon \]

as \( \varepsilon \to 0 \), for any fixed time \( t \in \mathbb{R} \). This implies (55) since

\[ \text{dist}(p^\varepsilon, \Phi(p^\varepsilon), \mathcal{N}) \leq \| (p^\varepsilon, \Phi(p^\varepsilon)) - (r, \Phi(r)) \|_{\alpha, \varepsilon} \to 0 \]

for any \( (p^\varepsilon, \Phi(p^\varepsilon) \in \mathcal{N}^\varepsilon) \) as \( \varepsilon \to 0 \). Since the limit is independent of subsequences in \( \varepsilon > 0 \) we have proved PART 2 of Theorem 5.3. \( \square \)

Acknowledgements

This research was carried out while the second author was visiting the University of Madrid, Complutense in 2005. He would like to thank this institute for their hospitality. Both authors were partially supported by grant BFM 2003-03810 of the Ministry of Higher Education, Spain. We would like to sincerely thank the referee for his useful comments and suggestions, mainly the request that the paper be self contained and that we should carry out proof readings of the manuscript for language misprints.

References


Further reading