Projection of Subvarieties of Grassmannians of Lines

Contents

Introduction

It has been a classical problem, in fact parallel to the study of projective algebraic geometry, what was know as "the geometry of the line". Classical authors like Fano, Castelnuovo, Severi, Marletta or Segre studied the geometry of varieties whose elements, instead of points, were lines. In modern terms, this corresponds to study subvarieties of Grassmannians of lines. In fact, recently modern authors, like Sols, Ran, Goldstein, Verra, Arrondo, Hernández, Alzati, Bertolini or Turrini, have retaken this point of view.

One of the most fruitful approaches of this point of view has been to try to translate known projective results to subvarieties of Grassmannians. As a sample of this translation we have: 1) the classification of varieties of low degree (see for instance [**Fan93**], [**Pap79**], [**HS87**], [**AS92**], [**ABT98**]); 2) the generalization of the theorem of Ellingsrud and Peskine to $G(1,3)$, i.e. the finiteness of the components of the Hilbert scheme of smooth surfaces of $G(1,3)$ which are not of general type; 3) the generalization of Horrocks theorem ([**Ott87**], [**AGn99**]); 4) an extension of the liaison theory ([**AS92**]).

In the present thesis, we will concentrate in the problem of projecting from one Grassmannian into another. This problem was first initiated by Arrondo and Sols in [**AS92**] and later continued by Arrondo in [**Arr99**]. The goal of this thesis is to present a thorough study of all the problems appearing when projecting subvarieties of Grassmannians of lines, extending and sometimes giving more conceptual proofs of the results of Arrondo.

The general problem in the projective space that we want to translate will be the following.

Let $X \subset \mathbb{P}^N$ be an n-dimensional variety, smooth, nondegenerate, i.e. X is not contained in a hyperplane. It is a classical problem to ask which is the smallest integer k such that the projection $\pi_h : \mathbb{P}^N \to \mathbb{P}^k$ (with center $H \cong \mathbb{P}^{N-k-1}$) restricted to X is an isomorphism. To answer this question the fundamental tool is the secant variety SX, i.e. the union of lines of $\mathbb{P}^N$ meeting the variety X in at least two points, maybe infinitely closed to each other. A formal definition can be given in the following way (see [**Zak93**] for instance). Let I be the closure inside $X \times X \times \mathbb{P}^N$ of the set $I_0 = \{(x_1, x_2, z) \in (X \times X \setminus \Delta_X) \times \mathbb{P}^N \mid z \in \langle x_1, x_2 \rangle\}$, and consider the diagram

(1)
$$\begin{array}{ccc} & I & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ X \times X & & \mathbb{P}^N. \end{array}$$

The image $SX = p_2(I)$ of the second projection is called *secant variety* of X.

One can easily prove that given $p \in \mathbb{P}^N$, the projection $\pi_p : \mathbb{P}^N \to \mathbb{P}^{N-1}$ restricted to X is an isomorphism if and only if $p \notin SX$. Moreover, from the diagram (1), the dimension of secant variety cannot be bigger than $2n+1$, which means that every smooth n-dimensional variety $X \subset \mathbb{P}^N$, with $N \geq 2n+2$, can be isomorphically projected to $\mathbb{P}^{2n+1}$.

Indeed, $2n+1$ is the expected dimension for SX, which means that we expect that in general an n-dimensional variety cannot be projected to $\mathbb{P}^k$, with $k \leq 2n$. Then it is an interesting problem to classify varieties that can be projected more than one expects, which is equivalent to find varieties with small secant variety.

In the case $n = 2$, Severi proved that the double Veronese embedding of $\mathbb{P}^2$ is the only smooth surface that can be isomorphically projected from $\mathbb{P}^5$ to $\mathbb{P}^4$ (see [**Sev01**]).

Then, for $n = 3$ there exists a classification of smooth three-folds that can be projected from $\mathbb{P}^7$ to $\mathbb{P}^6$, due to Fujita ([**Fuj82**]).

Finally, for every $n \geq 2$, Zak generalized the result of Severi proving that the only smooth n-dimensional variety that can be projected from $\mathbb{P}^{\frac{n(n+3)}{2}}$ to $\mathbb{P}^{2n}$ is the double Veronese embedding of $\mathbb{P}^n$.

Let us see now what will be the natural translation of this situation to $G(1,N)$, the Grassmannian of lines in $\mathbb{P}^N$. Clearly when we take the projection $\pi_h : \mathbb{P}^N \to \mathbb{P}^k$ and a line $L \subset \mathbb{P}^N$ such that $L \cap H = \emptyset$, then the image of L by the map π_h is still a line $L' \subset \mathbb{P}^k$. So π_h gives rise to a rational map from $G(1,N)$ to $G(1,k)$, defined outside from the family of lines meeting the linear space H. This is what we call a projection in the case of Grassmannians of lines.

As in the projective case, it is natural to ask how much a variety $X \subset G(1,N)$ can be isomorphically projected. This is the problem we are going to study throughout the thesis, trying to give a picture of the differences between the projective case and the one of the Grassmannians of lines.

First of all we are going to consider the projection from a point $p \in \mathbb{P}^N$, $\pi_p : G(1,N) \to G(1,N-1)$, restricted to a smooth variety $X \subset G(1,N)$. Clearly $\pi_p|_X$ is not injective if p lies on a plane spanned by two lines of X. Then, defining a *bad plane* as a plane containing at least two lines of X, maybe infinitely near, in Chapter 2 we prove that π_p is an isomorphism if and only if p does not lie on a bad plane for X. Then, since the family of bad planes cannot have dimension bigger than $2n$, we can say that every smooth n-dimensional variety $X \subset G(1,N)$, with $N \geq 2n+3$ can be isomorphically projected to $G(1,2n+2)$.

A first big difference with the projective space is that in the case of the Grassmannians, even if we know that there exist n-dimensional varieties that cannot be projected to a smaller Grassmannian, in general we expect they can. The fact is that, while there exist varieties with a $2n$-dimensional family of bad planes, in general this dimension is expected to be not bigger than n. Then we have that every smooth n-dimensional variety $X \subset G(1,N)$ can be isomorphically projected to $G(1,2n+2)$, but in general we expect to project X until $G(1,n+2)$.

Therefore there are two different kinds of problems when projecting subvarieties of Grassmannians.

On one hand it is interesting to characterize varieties that can be isomorphically projected less than one expects, i.e. n-dimensional varieties $X \subset G(1,k+1)$ that cannot be isomorphically projected to $G(1,k)$, for $n+2 \geq k \geq 2n+1$. In Chapter

2 we study the limit case $k = 2n+1$, proving that an n-dimensional variety $X \subset G(1, 2n+2)$ cannot be projected to $G(1, 2n+1)$ if and only if its points are the ruling lines of a cone on an n-dimensional variety $X' \subset \mathbb{P}^{2n+1}$ that cannot be isomorphically projected to $\mathbb{P}^{2n}$. In Chapter 5 we complete the case $n = 2$.

On the other hand one can ask for the varieties that can be projected more than one expects, i.e. n-dimensional varieties $X \subset G(1, N)$ that can be projected to $G(1, k)$, with $k \leq n+1$.

Therefore, a first natural problem is to study n-dimensional varieties that can be isomorphically projected to $G(1, n+1)$, which is analogous to the problem of projective n-dimensional varieties that can be projected to $\mathbb{P}^{2n}$.

We see in Chapter 2 that, while in the case of a variety $X \subset \mathbb{P}^N$ if we want to know how much X can be isomorphically projected, we just have to compute the dimension of its secant variety, when we take a subvariety X of the Grassmannian $G(1, N)$, we have different secant varieties depending on the Grassmannian to which we want to project X. In particular we define the *k-th secant variety* $S_kX \subset G(k, N)$ and we see that $X \subset G(1, N)$ can be projected to $G(1, N-k-1)$ if and only if S_kX is not the whole $G(k, N)$.

We restrict ourselves to the case $k = 1$ because it is the more natural one, and the first secant variety (which we call simply secant variety) of $X \subset G(1, N)$ is still a subvariety of $G(1, N)$.

Let $\pi_m : G(1, N) \to G(1, N-2)$ be the projection from a line M and $X \subset G(1, N)$ a smooth variety . We define the secant variety $SX \subset G(1, N)$ as the family of lines contained in a $\mathbb{P}^3$ containing at least two lines of X. To be formal, we take the closure J of the set $J_0 = \{(l_1, l_2, l) \in (X \times X \setminus \Delta_X) \times G(1, N) \mid \dim\langle L_1, L_2, L\rangle \leq 3\}$ inside $X \times X \times G(1, N)$, and the diagram

(2)
$$\begin{array}{ccc} & J & \\ {}^{q_1}\swarrow & & \searrow^{q_2} \\ X \times X & & G(1, N). \end{array}$$

We define the secant variety SX as the image of the second projection $q_2(J) \subset G(1, N)$.

With this definition we prove that the projection $\pi_m|_X$ is an isomorphism if and only if $m \notin SX$. We see that there are two distinct "pieces" in SX, i.e. the irreducible component $S'X$ of lines contained in a $\mathbb{P}^3$ spanned by two skew lines of X, and the piece (that can be reducible) $S''X$, i.e. the lines meeting a bad plane. While we can still say few things about $S''X$, the component $S'X$ seems to be more natural and analogous to the secant variety in the projective space.

With the help of this secant variety, in Chapter 3 we study smooth varieties of dimension n that can be isomorphically projected from $G(1, n+3)$ to $G(1, n+1)$. We prove a structure theorem which says that if X is uncompressed (i.e. the union of the lines of X, say $Y \subset \mathbb{P}^N$ has the expected dimension $n+1$) and it can be isomorphically projected from $G(1, n+3)$ to $G(1, n+1)$, then the union $Y \subset \mathbb{P}^{n+3}$ of the lines of X is a scroll of $\mathbb{P}^n$'s. Moreover, either the scroll is rational, or there exists a linear space $K \cong \mathbb{P}^k$ contained in all the $\mathbb{P}^n$'s of the family and such that all the lines of X meet K.

We are then going to consider the next case, i.e. n-dimensional varieties $X \subset G(1,N)$ that can be projected to $G(1,n)$. If this is the case, then there exists a linear space $H \cong \mathbb{P}^{N-n-1}$ which does not intersect the union of lines of X, say $Y \subset \mathbb{P}^N$. In particular we have that $\dim Y$ cannot be bigger than n, and then X is compressed. In Chapter 4 we study this kind of problem in the first interesting case, i.e. 4-dimensional varieties that can be isomorphically projected to $G(1,4)$ (the cases $n = 2,3$ are trivial). First of all we construct a four-fold $X \subset G(1,4)$ which is an extension of the surface $S \subset G(1,3)$ consisting of the lines tangent to a quadratic cone and meeting a fixed line M, and that is projected from $G(1,5)$.

Then we prove that if $X \subset G(1,4)$ is projected from $G(1,5)$, then X must be an extension of the surface $S \subset G(1,3)$ above.

Finally we prove that a four-fold $X \subset G(1,6)$ can never be isomorphically projected to $G(1,4)$.

In Chapter 5 we summarize the notions developed in the thesis to give a complete picture of the projection of surfaces in the Grassmannian of lines.

First of all we consider the problem of surfaces that can be projected less than one expects, giving a classification of surfaces $X \subset G(1,k+1)$ that cannot be isomorphically projected to $G(1,k)$, for $k = 4,5$. In the case $k = 5$ we have that a surface $X \subset G(1,6)$ cannot be projected to $G(1,5)$ if and only if its points correspond to the ruling lines of a cone on a surface $X' \subset \mathbb{P}^5$, such that X' is not the double Veronese embedding of $\mathbb{P}^2$.

In the case $k = 4$ we see that the list of surfaces that cannot be isomorphically projected from $G(1,5)$ to $G(1,4)$ is much bigger.

On the other hand, when considering surfaces that we can project more than the expected, Arrondo and Sols gave a classification of surfaces of $G(1,3)$ projected from $G(1,4)$ (see [**AS92**]). In particular they proved that the only smooth surface that can be projected from $G(1,5)$ to $G(1,3)$ is the Veronese surface.

Finally, considering also singular varieties and using an analogous of the structure theorem, we prove that the only surfaces $X \subset G(1,5)$ with small secant variety (i.e. such that $\dim SX < 8$) are the Veronese surface and the cones on a rational curve. In particular as a corollary we find again that the Veronese surface is the only smooth surface that can be projected from $G(1,5)$ to $G(1,3)$.

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CHAPTER 1

Preliminary Results

1. Definitions and Notations

In this first section we recall the basic definitions and results that we are going to use in the rest of the thesis. Further details can be found in [**Arr96**], [**KL72**] and [**GH94**]. We also illustrate many examples of varieties that will appear in our main results of the next chapters.

1.1. $G(k,N)$ **and the Plücker Embedding.** Given an $(N+1)$-dimensional vector space V over the field of complex numbers $\mathbb{C}$, let us denote by $\mathbb{P}(V) = \mathbb{P}^N$ the projective space of all hyperplanes of V (this is the same as taking the space of all lines in the dual V^*). We denote by $G(k,N)$ the Grassmannian of k-dimensional linear spaces in $\mathbb{P}^N$. This is naturally identified with the set of $(k+1)$-dimensional linear subspaces of V^*, or $(k+1)$-dimensional quotients of V.

General Notation. Let us take an element of the Grassmannian $G(k,N)$; throughout the thesis we will use the following notations:
i) a small letter l, if we refer to it as a point of $G(k,N)$;
ii) a capital letter L, if we consider it as a subspace of $\mathbb{P}^N$;
iii) a calligraphic letter $\mathcal{L}$ if we consider the corresponding vector subspace of V^*.

If we fix coordinates $(x_0 : \ldots : x_N)$ for $\mathbb{P}^N$, or equivalently a basis $\{e_0, \ldots, e_N\}$ of V^*, we can represent an element $l \in G(k,N)$ by a $(k+1)\times(N+1)$ matrix of rank $k+1$, where the rows are the coordinates of a basis of $\mathcal{L}$. This matrix is not unique but depends on the basis of $\mathcal{L}$ we choose. Since the matrix has rank $k+1$ we can always find one of the minors of order $k+1$ which is not zero. Let us suppose that the minor corresponding to the first $k+1$ columns is not equal to zero. Then, after multiplying by a suitable matrix we can represent l in a unique way by the matrix

$$\begin{pmatrix} 1 & \ldots & 0 & a_{0,k+1} & \ldots & a_{0,N} \\ & \ddots & & \vdots & & \vdots \\ 0 & \ldots & 1 & a_{k,k+1} & \ldots & a_{k,N} \end{pmatrix} \tag{3}$$

Geometrically, we are assuming the space L not to meet the linear space $H = \{x_0 = \ldots = x_k = 0\}$, of codimension $k+1$. The rows of the matrix are the intersection points of L with a basis of the space of codimension-k linear spaces containing H.

The set of k-planes like in (3) is an affine open subset of dimension $(N-k)(k+1)$ (in which we can take the $a_{i,j}$'s as coordinates). We can cover $G(k,N)$ by open affine subsets of this kind, which means that the Grassmannian $G(k,N)$ can be viewed as an abstract manifold of dimension $(N-k)(k+1)$.

One can prove that $G(k,N)$ is a projective variety, considering the so called *Plücker embedding.* To describe it, let us take a point $l \in G(k,N)$. It corresponds to a linear $(k+1)$-dimensional subspace $\mathcal{L} = \langle v_0, v_1, \dots, v_k \rangle \subset V^*$. Then the Plücker embedding is given by the following map

$$\begin{aligned} \phi_{k,N} : & G(k,N) \to \mathbb{P}(\bigwedge^{k+1} V) \\ & l \mapsto [v_0 \wedge v_1 \wedge \dots \wedge v_k], \end{aligned}$$

where $[v_0 \wedge v_1 \wedge \dots \wedge v_k]$ is the point of $\mathbb{P}(\bigwedge^{k+1} V)$ corresponding to the vector $v_0 \wedge v_1 \wedge \dots \wedge v_k$. If we change the basis of the space $\mathcal{L}$, the image is the same, up to a constant (the determinant of the matrix defining the basis change), so that the map $\phi_{k,N}$ is well defined.

Definition. The homogeneous coordinates induced in $\mathbb{P}(\bigwedge^{k+1} V)$ by a choice of coordinates in $\mathbb{P}^N$ are called *Plücker coordinates* and they are denoted by $p_{i_0,\dots i_k}$.

Then with this notation, the set of k-planes as in (3) corresponds to the affine subset $\{p_{0,1,\dots,k} \neq 0\}$, or equivalently the equation $p_{0,1,\dots,k} = 0$ represents the set of k-planes meeting $\{x_0 = \dots = x_k = 0\}$.

The Grassmannian of k-planes in $\mathbb{P}^N$ is naturally identified with the Grassmannian of $(N-k-1)$-planes in the dual space $\mathbb{P}^{N^*}$. We say that there exists a duality among the Grassmannians $G(k,N)$ and $G(N-k-1,N)$. From the geometric point of view, if we fix a k-plane in $\mathbb{P}^N$, the space of hyperplanes containing it forms an $(N-k-1)$-plane in $\mathbb{P}^{N^*}$.

Then, for instance, the dual of the Grassmannian $G(1,3)$ is again a $G(1,3)$, because the dual of a line in $\mathbb{P}^3$ is a line.

1.2. Schubert Calculus. We just recall some definitions and basic facts about Schubert calculus, the tool for studying intersections in Grassmannians.

The idea comes from the projective space. Indeed, if we have a subvariety $X \subset \mathbb{P}^N$ of codimension i, its degree is the number of points obtained intersecting X with a sufficiently general linear subspace of dimension i. If we take the *Chow ring* of $\mathbb{P}^N$, $A(\mathbb{P}^N) = \bigoplus_{i=0}^N A^i(\mathbb{P}^N)$ (where the group $A^i(\mathbb{P}^N)$ is the free abelian group of subvarieties of codimension i, modulo rational equivalence), we know that it is generated by the class of a hyperplane H. The multiplicative structure on $A(\mathbb{P}^N)$ is given by the intersection product. In particular, for each $i = 1, \dots, N$, the group $A^i(\mathbb{P}^N)$ is freely generated by the class H^i (the intersection of i general hyperplanes, i.e. the class of a linear space of codimension i). The class of a variety X of codimension i is an element $[X] \in A^i(\mathbb{P}^N)$ and then we can write $[X] = dH^i$, where d is the degree of X (further details about the Chow ring can be found for instance in [**Ful84**]).

In the case of Grassmannians the method is the same. But while in the projective space a variety has only one degree, here we can find more than one.

Definitions. Let us fix a flag of $k+1$ non-empty linear subspaces of $\mathbb{P}^N$, $H_0 \subsetneq H_1 \subsetneq \dots \subsetneq H_k$. We define the *Schubert variety* associated to this flag to be the set

$$\Omega(H_0, \dots, H_k) = \{l \in G(k,N) \mid \dim(L \cap H_i) \geq i, \ i = 0, \dots, k\}.$$

The equivalence class of a Schubert variety under projectivities is called a *Schubert cycle* and it is denoted by $\Omega(\alpha_0, \dots, \alpha_k)$, where $\alpha_i = \dim H_i$. It turns out that $\dim \Omega(\alpha_0, \dots, \alpha_k) = \sum_{j=0}^k (\alpha_j - j)$.

The *special Schubert cycle* σ_a of codimension a is the cycle of k-planes meeting a linear space of dimension $N-k-a$, i.e. $\sigma_a = \Omega(N-k-a, N-k+1, \dots, N)$.

We can state the following well-known result about the Chow ring $A(G) = \bigoplus_{i=0}^{(N-k)(k+1)} A^i(G)$.

THEOREM 1.1. *For any $i = 0, \dots, (k+1)(N-k)$, the Chow group $A^i(G)$ is freely generated by all Schubert cycles $\Omega(\alpha_0, \dots, \alpha_k)$ such that $\sum_{j=0}^{k}(\alpha_j - j) = (N-k)(k+1) - i$.*

Moreover, concerning the multiplicative structure of the Chow ring, let us state the few results we will use in what follows.

THEOREM 1.2. *(Pieri's Formula) The intersection product of a Schubert cycle and a special Schubert cycle is given by*

$$\Omega(\alpha_0, \dots, \alpha_k) \cdot \sigma_a = \sum \Omega(\beta_0, \dots, \beta_k).$$

where the sum is taken over all β_i's such that $\alpha_{i-1} + 1 \leq \beta_i \leq \alpha_i$ and $\sum \beta_i = \sum \alpha_i - a$.

THEOREM 1.3. *If two Schubert cycles have complementary dimension, then their intersection $\Omega(\alpha_0, \dots, \alpha_k) \cdot \Omega(\beta_0, \dots \beta_k)$ is zero unless $\alpha_i + \beta_{k-i} = N$ for all $i = 0, \dots, k$. In this last case $\Omega(\alpha_0, \dots, \alpha_k) \cdot \Omega(\beta_0, \dots, \beta_k) = \Omega(0, 1, \dots, k)$, i.e. the class of a point.*

This last theorem is just telling us that the bases for two complementary dimensions are orthogonal with respect to the intersection product. Now we can extend the notion of degree to the Grassmannians. To do that, we take a subvariety $X \subset G = G(k, N)$ of codimension r. Its class $[X]$ is an element of the Chow group $A^r(G)$.

DEFINITION. We call *multidegree* of X the set of coefficients of the class $[X] \in A^r(G)$ with respect to the basis given by the Schubert cycles of codimension r.

The multidegree of X can also be viewed as the set of the degrees of the intersection numbers of X with all the Schubert cycles of dimension r.

EXAMPLE 1. Let us consider the concrete example of the Grassmannian of lines in $\mathbb{P}^3$, $G = G(1,3)$. This is a 4-dimensional variety, and hence the Chow ring is given by $A(G) = \bigoplus_{i=0}^{4} A^i(G)$, where the generators of the groups $A^i(G)$ are the following

$$\begin{aligned}
A^0(G) &= \langle \Omega(2,3) \rangle \\
A^1(G) &= \langle \Omega(1,3) \rangle \\
A^2(G) &= \langle \Omega(0,3), \Omega(1,2) \rangle \\
A^3(G) &= \langle \Omega(0,2) \rangle \\
A^4(G) &= \langle \Omega(0,1) \rangle.
\end{aligned}$$

Hence $A^0(G)$ is generated by the class of the whole Grassmannian G, $A^1(G)$ is generated by the Schubert cycle of lines meeting a fixed line, $A^2(G)$ by the cycles $\Omega(0,3)$, i.e. the lines passing through a fixed point (this Schubert cycle is classically called an α-plane), and $\Omega(1,2)$, i.e. the lines contained in a fixed plane (classically called a β-plane). Finally $A^3(G)$ is generated by the cycle of lines contained in a plane and passing through a point in this plane and $A^4(G)$ by the class of a line.

All the Chow groups have only one generator, except $A^2(G)$. Hence a variety $X \subset G(1,3)$ has only one degree, except the surfaces, which have a bidegree. Let us then study the case of a surface X.

A surface $X \subset G$ is classically called a *congruence* of lines in $\mathbb{P}^3$. We have that $[X] = a\Omega(0,3) + b\Omega(1,2)$. From Theorem 1.3 we have

$$a = [X] \cdot \Omega(0,3)$$
$$b = [X] \cdot \Omega(1,2),$$

which means that a is the number of lines of X passing through a general fixed point (it is called the *order* of X), while b is the number of lines contained in a general plane (the *class* of X).

We denote by d the degree of X after the Plücker embedding in $\mathbb{P}^5$. Then d is the number of intersection points of X with 2 general hyperplanes, or the product of $[X]$ with the second power of the hyperplane section of $G(1,3)$. Hence $d = \sigma_1^2 \cdot [X]$ and, by Schubert calculus

$$d = [X] \cdot (\Omega(0,3) + \Omega(1,2)) = a + b.$$

REMARK. If we take the dual Grassmannian of $G(1,3)$, it is again a $G(1,3)$. Let X be a congruence of bidegree (a,b), and call X' its dual. Then, since points are the dual of planes, the lines passing through a point correspond to the lines contained in a plane. Hence the bidegree of X' is simply $(a',b') = (b,a)$.

We also recall the following

DEFINITIONS. Let X be a surface of $G(1,N)$ and p a point of $\mathbb{P}^N$. We say that p is a *fundamental point* for X if there exist infinitely many lines of X passing through p.

We say that a curve $C \subset \mathbb{P}^N$ is a *fundamental curve* for X if every point of C is fundamental for X.

Since X is a surface, equivalently we can say that a fundamental curve is a curve meeting all the lines of X.

EXAMPLE 2. Let us take a plane $\Pi \subset \mathbb{P}^3$, a conic $C \subset \Pi$, and a line L not intersecting C. The family of lines joining C and L is a congruence $X \subset G(1,3)$, and C and L are fundamental curves for X. Let us compute the bidegree of X.

If we take a general point $p \in \mathbb{P}^3$, the plane $\Pi_p = \langle p, L\rangle$ intersects Π in a line and then the conic C in two points p_1, p_2. The lines $L_1 = \langle p, p_1\rangle$ and $L_2 = \langle p, p_2\rangle$ intersect C and, since they are contained in Π_p, they meet the line L too. Hence L_1 and L_2 give rise to two points of our congruence X. We have then found that the order of X is $a = 2$. In order to compute the class b, we have to find the number of lines of X contained in a fixed plane Π_0. But Π_0 intersects L in one point q and C in two points p_1 and p_2, and then we find the two lines $\langle q, p_1\rangle$ and $\langle q, p_2\rangle$ inside Π_0. Hence the class b is 2. Then the bidegree of X is $(a,b) = (2,2)$.

The bidegree can also be computed observing that the congruence X is the intersection of the family of lines meeting M, whose class is the Schubert cycle $\Omega(1,3)$, and the family of lines meeting C, whose class is $2\Omega(1,3)$. Hence $[X] = 2\Omega(1,3)^2 = 2(\Omega(0,3) + \Omega(1,2))$.

If we take the dual congruence (which is again a $(2,2)$), it can be described as the family of lines meeting a line M (dual of L) and tangent to a quadratic cone $Q \subset \mathbb{P}^{3^*}$.

EXAMPLE 3. We can specialize the example above to the case in which L intersects C in a point p. If we take the lines joining L and C, we still have a $(2,2)$ congruence, but it is clearly reducible. In fact there exist two irreducible components: the α-plane $\Omega(p, \mathbb{P}^3)$ (the lines passing through p), and a congruence X of bidegree $(1,2)$.

We can describe the dual X' (which is now a $(2,1)$ congruence) in the following way. We take a quadratic cone $Q \subset \mathbb{P}^{3^*}$ and a line M contained in a plane Π, tangent to Q. Then we take the family of lines tangent to Q and meeting M. As before we have to remove the β-plane of lines contained in Π (which is the dual of the α-plane $\Omega(p, \mathbb{P}^3)$).

1.3. The Universal Bundles. Throughout the thesis we will use the usual convention of freely interchange the notions of vector bundle on a projective variety and its associated locally free sheaf.

Let us consider the following incidence diagram

$$\begin{array}{ccc} & I & \\ {}^{p}\swarrow & & \searrow^{q} \\ \mathbb{P}^N & & G(1,N), \end{array} \tag{4}$$

where I is the incidence variety $I = \{(x,l) \in \mathbb{P}^N \times G(k,N) \mid x \in L\}$. Then, from the Euler exact sequence

$$0 \to \Omega_{\mathbb{P}^N}(1) \to V \otimes \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_{\mathbb{P}^N}(1) \to 0$$

if we take the pull-back via the map p and then the push-forward via q, we obtain the so-called universal exact sequence on $G(k,N)$

$$0 \to \mathcal{S}^* \to V \otimes \mathcal{O}_G \xrightarrow{\pi} \mathcal{Q} \to 0, \tag{5}$$

where

$$\begin{aligned} \mathcal{S}^* &= q_* p^*(\Omega_{\mathbb{P}^N}(1)), \\ \mathcal{Q} &= q_* p^*(\mathcal{O}_{\mathbb{P}^N}(1)), \end{aligned} \tag{6}$$

are called the *universal subbundle* and *universal quotient bundle* respectively. They have rank $N-k$ and $k+1$ (we can also view the vector bundle $\mathcal{Q}$ as the dual of the tautological bundle whose fiber on a point $l \in G(k,N)$ is the vector space $\mathcal{L} \subset V^*$ corresponding to l). From the identities (6) one can prove that there exists a natural identification $H^0(G,\mathcal{Q}) = V$, and considering dual Grassmannians it follows that $H^0(G,\mathcal{S}) = V^*$. In particular giving a nonzero section of $\mathcal{Q}$ is the same as giving a hyperplane of $\mathbb{P}^N$, while a section of $\mathcal{S}$ corresponds to a point of $\mathbb{P}^N$. Moreover the zero locus of a section of $\mathcal{Q}$ is the set of k-planes contained in the corresponding hyperplane, and the zero locus of a section of $\mathcal{S}$ is the set of k-planes passing through the corresponding point.

More generally, the dependency locus of $N-k-a+1$ sections of $\mathcal{S}$ is the family of k-planes meeting an $(N-k-a)$-dimensional linear space, i.e. $\Omega(N-k-a, N-k+1, \dots, N) = \sigma_a$. From Porteous formula (see the Appendix) we then have that σ_a is the Chern class $c_a(\mathcal{S})$.

EXAMPLE 4. Let us take again the Grassmannian $G = G(1,3)$. In this case a section $s_1 \in H^0(G, \mathcal{Q})$ corresponds to a plane Π, and then the zero locus of s_1 is the β-plane of lines contained in Π. On the other hand, a section $s_2 \in H^0(G, \mathcal{S})$ corresponds to a point p, and its zero locus is the α-plane of lines passing through p.

REMARK. Clearly the Plücker embedding $\phi_{k,N} : G(k,N) \to \mathbb{P}(\bigwedge^{k+1} V)$ is the map associated to the line bundle $\mathcal{O}_{G(k,N)}(1) = \bigwedge^{k+1} \mathcal{Q} = \bigwedge^{N-k} \mathcal{S}$.

DEFINITION. Let $\mathcal{F}$ be a vector bundle on an n-dimensional projective variety X. We say that $\mathcal{F}$ *has not intermediate cohomology* if $H^i(X, \mathcal{F}(l)) = 0$ for $i = 1, \dots, n-1$ and $l \in \mathbb{Z}$.

It is easy to see from their definition that the universal bundles $\mathcal{Q}$ and $\mathcal{S}$ have not intermediate cohomology.

REMARK. Let us take a hyperplane $H \cong \mathbb{P}^{N-1} \subset \mathbb{P}^N$. We have an inclusion $G(1,H) \cong G(1,N-1) \subset G(1,N)$, identifying the Schubert variety $\Omega(N-2,H)$ with $G(1,H)$. We denote by $\mathcal{S}$, $\mathcal{Q}$ and $\bar{\mathcal{S}}$, $\bar{\mathcal{Q}}$ the universal bundles of $G(1,N)$ and $G(1,N-1)$ respectively. From the fact that $\Omega_{\mathbb{P}^N}(1)|_{\mathbb{P}^{N-1}} = \Omega_{\mathbb{P}^{N-1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{N-1}}$, while $\mathcal{O}_{\mathbb{P}^N}(1)|_{\mathbb{P}^{N-1}} = \mathcal{O}_{\mathbb{P}^{N-1}}(1)$, and from (6) we can easily see that

$$\begin{aligned} \mathcal{S}|_{G(1,N-1)} &= \bar{\mathcal{S}} \oplus \mathcal{O}_{G(1,N-1)}, \\ \mathcal{Q}|_{G(1,N-1)} &= \bar{\mathcal{Q}}. \end{aligned} \tag{7}$$

1.4. The Tangent Bundle. Let l be a point of $G(k,N)$. We are going to use the following

GENERAL NOTATION. $T_{G(k,N),l}$ is the Zariski tangent space to $G(k,N)$ at l;
$\mathbb{T}_{G(k,N),l} \subset \mathbb{P}(\bigwedge^{k+1} V)$ is the embedded tangent space;
$\mathcal{T}_{G(k,N)}$ is the tangent bundle.

Once we have covered $G(k,N)$ by affine pieces we can easily describe the tangent space at a point $l \in G(k,N)$. It is well-known that the tangent space $T_{G(k,N),l}$ can be canonically identified with $\mathrm{Hom}(\mathcal{L}, V^*/\mathcal{L})$. Globally we have the following

THEOREM 1.4. *There exists a natural isomorphism*

$$\mathcal{T}_{G(k,N)} \cong \mathcal{H}om(\mathcal{Q}^*, \mathcal{S}) \cong \mathcal{Q} \otimes \mathcal{S}.$$

1.5. Projections. Let us consider the linear projection $\pi_h : \mathbb{P}^N \to \mathbb{P}^{N-r-1}$ with center $H \cong \mathbb{P}^r \subset \mathbb{P}^N$. If we take a k-plane $l \in G(k,N)$ which does not intersect the center of projection H, then its image by π_h is again a k-plane $l' \in G(k, N-r-1)$. We then have a rational map from $G(k,N)$ to $G(k,N-r-1)$ which is not defined on the Schubert variety of k-planes meeting H. We are going to use the same symbol π_h for this projection in Grassmannians.

Let X be an n-dimensional subvariety of the Grassmannian $G(k,N)$.

GENERAL NOTATION. Throughout the thesis we are going to denote by Y the union inside $\mathbb{P}^N$ of the k-planes of X, i.e.

$$Y = \bigcup_{l \in X} L \subset \mathbb{P}^N.$$

We are also going to use the following:

DEFINITIONS. X is *nondegenerate* if it is not contained in any $G(k, N-1)$;
X is *linearly normal* if it is not projected from any nondegenerate subvariety of $G(k, N+1)$;
X is *uncompressed* if $\dim Y = n + k$, i.e. Y has the expected dimension.

EXAMPLE 5. Let us consider an example of n-dimensional variety that can be projected from $G(1, 2n+1)$ to $G(1, n+1)$ (see [**Arr99**]).

Let us take two n-dimensional linear spaces H_1 and H_2 inside $\mathbb{P}^{2n+1}$, and an isomorphism between them. We call X the family of lines joining one point on H_1 with the corresponding one on H_2.

In other words, we can take the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^n \to \mathbb{P}^{2n+1}$. If we call Y the image of this map, we know that there exists a 1-dimensional family of $\mathbb{P}^n$'s inside Y, and an n-dimensional family of lines transversal to those $\mathbb{P}^n$'s. This family of lines is our variety $X \subset G(1, 2n+1)$.

Let us fix coordinates on $\mathbb{P}^{2n+1}$ in such a way that we can write $H_1 = \{x_1 = x_3 = \ldots = x_{2n+1} = 0\}$ and $H_2 = \{x_0 = x_2 \ldots = x_{2n} = 0\}$. Then we have an embedding of $\mathbb{P}^n$ in $G(1, 2n+1)$ that can be given in the following way

$$(t_0, \ldots, t_n) \to \begin{pmatrix} t_0 & 0 & t_1 & 0 & \ldots & t_n & 0 \\ 0 & t_0 & 0 & t_1 & \ldots & 0 & t_n \end{pmatrix},$$

where $(t_0 : \ldots : t_n)$ are homogeneous coordinates for $\mathbb{P}^n$. We take the projection from $\mathbb{P}^{2n+1}$ to $\mathbb{P}^{n+1}$ defined by

$$(x_0 : \ldots x_{2n+1}) \to (x_0 : x_1 + x_2 : \ldots : x_{2n-1} + x_{2n} : x_{2n+1}).$$

This projection induces a projection from $G(1, 2n+1)$ to $G(1, n+1)$. We can now describe the new map from $\mathbb{P}^n$ to $G(1, n+1)$ as

$$(t_0, \ldots, t_n) \to \begin{pmatrix} t_0 & t_1 \ldots & t_n & 0 \\ 0 & t_0 \ldots & t_{n-1} & t_n \end{pmatrix}.$$

Since the minors of this matrix form a basis of the space of homogeneous polynomials of degree two, when we compose our map $\mathbb{P}^n \to G(1, n+1)$ with the Plücker embedding of $G(1, n+1)$ in $\mathbb{P}^{\binom{n+2}{2}-1}$, we obtain the double Veronese embedding of $\mathbb{P}^n$. In particular the first map must be an embedding, and hence the projection restricted to X is an isomorphism.

NOTATION. Following [**Arr99**], we will refer to the variety X we have just introduced, as the *n-dimensional Veronese variety.*

In particular for $n = 1$ we have that if we take one of the rulings of a quadric in $\mathbb{P}^3$, this is a curve in $G(1, 3)$ that can be projected to $G(1, 2)$. The image can be viewed as the family of tangent lines to a conic in $\mathbb{P}^2$.

EXAMPLE 6. Let us go back now to the $(2, 1)$-congruence $X \subset G(1, 3)$ of Example 3, i.e. the family of lines tangent to a quadratic cone $Q \subset \mathbb{P}^3$ and meeting a fixed line M (tangent to the same cone Q). This congruence is projected from $G(1, 4)$.

In fact, let us fix a plane Π such that the vertex q of the cone does not lie on Π. Then the tangent planes to Q can be viewed as the planes spanned by q and by the tangent lines to the conic $C := \Pi \cap Q$. We have seen that this 1-dimensional family of lines is the image of one ruling of a quadric in $\mathbb{P}^3$, say $\mathcal{F}_1 \subset G(1, 3)$. Hence the family of tangent planes to Q is the image of the planes in $\mathbb{P}^4$ spanned by a point q' and the lines of $\mathcal{F}_1$, i.e. it is one of the two 1-dimensional families of planes contained in the cone on the quadric, with vertex q'. In particular this shows that the surface X is the image of a surface in $G(1,4)$, namely the surface consisting of the lines contained in this family of planes and meeting a fixed line contained in one of them.

1.6. Maps to $G(k,N)$. Let us recall a universal property for maps to Grassmannians (which in fact characterizes the Grassmannian as a universal object)

PROPOSITION 1.5. *Let X be a smooth algebraic variety. Then giving a map $\varphi : X \to G(k,N)$ is equivalent to giving a locally free sheaf $\mathcal{E}$ on X of rank $k+1$ and an epimorphism $\phi : V \otimes \mathcal{O}_X \to \mathcal{E}$, where V is an $(N+1)$-dimensional vector space. Moreover ϕ is the pull back by φ of the universal epimorphism $\pi : V \otimes \mathcal{O}_G \to \mathcal{Q}$ appearing in* (5).

The geometric idea is the following. Every time we fix a point $x \in X$, we have an epimorphism $\phi_x : V \to \mathcal{E}_x$, and taking the hyperplanes we obtain the inclusion $\mathbb{P}(\mathcal{E}_x) \subset \mathbb{P}(V) = \mathbb{P}^N$, defining the k-plane corresponding to x. In particular, if φ is an embedding, then $\mathcal{E}$ is just the restriction of $\mathcal{Q}$ to X.

We remark that, since the Plücker embedding of $G(k,N)$ in $\mathbb{P}(\bigwedge^{k+1} V)$ is given by the line bundle $\bigwedge^{k+1}\mathcal{Q}$, if we take the map $\bigwedge^{k+1}\phi : \bigwedge^{k+1} V \to \bigwedge^{k+1}\mathcal{E}$ we obtain the Plücker embedding of the variety X.

From the map $\phi : V \otimes \mathcal{O}_X \to \mathcal{Q}|_X$, taking the associated map on global sections $\bar{\phi} : V = H^0(G,\mathcal{Q}) \to H^0(X,\mathcal{Q}|_X)$, one can prove that:

i) the map $\bar{\phi}$ is injective if and only if X is nondegenerate;

ii) the map $\bar{\phi}$ is surjective if and only if X is linearly normal.

Then, if we take the exact sequence of X

$$0 \to \mathcal{I}_X \to \mathcal{O}_G \to \mathcal{O}_X \to 0, \tag{8}$$

tensoring by $\mathcal{Q}$ and taking cohomology

$$0 \to H^0(G,\mathcal{I}_X \otimes \mathcal{Q}) \to H^0(G,\mathcal{Q}) \to H^0(X,\mathcal{Q}|_X) \to H^1(G,\mathcal{I}_X \otimes \mathcal{Q}) \to 0 \tag{9}$$

we can say that X is linearly normal if and only if $h^1(\mathcal{I}_X \otimes \mathcal{Q}) = 0$.

REMARK. When we have a map $\varphi : X \to G(k,N)$ given by a vector bundle which is the direct sum of $k+1$ line bundles, it is easy to give a description of X in $G(k,N)$.

Indeed, let $\mathcal{E} = \mathcal{L}_0 \oplus \cdots \oplus \mathcal{L}_k$ be a decomposable bundle of rank $k+1$. Then every line bundle $\mathcal{L}_i$ gives a map $\varphi_i : X \to \mathbb{P}^{n_i}$, where $n_i = h^0(X,\mathcal{L}_i) - 1$. Let us call X_i the image of X by φ_i. We have $k+1$ varieties X_i inside $\mathbb{P}^N$ (where $N = (\sum_i n_i) - 1$, and a correspondence between their points. Every time we take a point $x \in X$, its image in $G(k,N)$ is simply the k-plane spanned by the images of x by the maps φ_i, i.e. $\varphi(x) = \langle \varphi_0(x), \dots, \varphi_k(x) \rangle$.

Let us remark that the map φ can be an embedding even if some of the maps φ_i is not injective. For instance, if we take the rank-2 vector bundle $\mathcal{E} = \mathcal{O}_X \oplus \mathcal{L}$, where $\mathcal{L}$ is a very ample line bundle, we obtain an embedding of X in the Grassmannian $G(1,N)$, where $N = h^0(X,\mathcal{L})$. Indeed the summand $\mathcal{O}_X$ is sending X to a point $p \in \mathbb{P}^N$, while $\mathcal{L}$ gives an embedding of X in $\mathbb{P}^{N-1}$, and hence the lines of the image of X are the ruling lines of the cone on the image of X in $\mathbb{P}^{N-1}$, with vertex p.

More generally, if we take the vector bundle $\mathcal{E} = \mathcal{O}_X^{\oplus i+1} \oplus \mathcal{L}_{i+2} \cdots \oplus \mathcal{L}_k$, we obtain the k-planes of a cone with vertex the i-dimensional linear space $M := \mathbb{P}(\mathcal{O}_X^{\oplus i+1}) \cong \mathbb{P}^i$.

EXAMPLE 7. If we take $X = \mathbb{P}^n$ and the rank-2 vector bundle $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(1) \oplus \mathcal{O}_{\mathbb{P}^n}(1)$, we obtain an embedding of $\mathbb{P}^n$ in $G(1, 2n+1)$. In particular each of the summands of $\mathcal{E}$ yields an embedding of $\mathbb{P}^n$ as a linear subspace of $\mathbb{P}^{2n+1}$. Hence we have two disjoint $\mathbb{P}^n$'s and an isomorphism between them. The image of a point $x \in \mathbb{P}^n$ by the embedding in $G(1, 2n+1)$, is the line joining the two corresponding points on the $\mathbb{P}^n$'s. Then $X \subset G(1, 2n+1)$ is the n-dimensional Veronese variety of Example 5.

EXAMPLE 8. Let us denote by X the blow up of $\mathbb{P}^2$ in one point p, by L the pull back of a line, and by E the exceptional divisor. If we take the rank-2 vector bundle $\mathcal{O}_X(L-E) \oplus \mathcal{O}_X(L)$, we have that the first summand is sending X on a line $M \subset \mathbb{P}^4$, while the second one gives us a pencil of lines in a plane. Then we have the following geometric description. We have the line M, the pencil of lines $\Omega(q, \Pi)$ in a plane Π not intersecting M and an isomorphism $\sigma : M \to \Omega(q, \Pi)$. To each point $p \in M$ we simply associate the pencil of lines passing through p and contained in the plane $\langle p, \sigma(p) \rangle$.

This surface of $G(1,4)$ can be projected to $G(1,3)$, and here we have the same geometric description (see [**AS92**]).

Let us spend a few words to see that in fact the congruence X' of Example 3 coincides with the projection of this last one.

We take the quadratic cone Q with vertex q and the line M tangent to Q at a point p_0. Let us call Π_0 the tangent plane containing M (then $\Pi_0 = \langle q, M \rangle$). We fix Π, a tangent plane to Q, different from Π_0. Let us take $\Omega(q, \Pi)$, the pencil of lines in Π passing through the vertex q. Every time we take a point $p \in M$, $p \neq p_0$, there exists exactly one tangent plane different from Π_0 and containing p. We denote by Π_p this plane. The lines of X passing through p are the pencil $\Omega(p, \Pi_p)$. Since Π and Π_p intersect in a line L_p of the pencil $\Omega(q, \Pi)$, if we associate to the point p the line L_p, we have find the isomorphism $\sigma : M \to \Omega(q, \Pi)$. Moreover, from the construction, the plane Π_p is spanned by p and $\sigma(p)$ and hence we have found exactly the construction of [**AS92**].

We just remark that in fact the pencil of lines in the construction is not uniquely determined, because we can chose any of the tangent planes to the quadratic cone, hence there exist a one dimensional family of such pencils. This corresponds to the fact that there is not a unique epimorphism $\mathcal{O}_X(L) \oplus \mathcal{O}_X(L-E) \to \mathcal{O}_X(L)$ (up to a constant), because giving such an epimorphism is equivalent to give a section of the bundle $\mathcal{O}_X \oplus \mathcal{O}_X(E)$, and $h^0(X, \mathcal{O}_X \oplus \mathcal{O}_X(E)) = 2$.

Given $X \subset G(k, N)$, if we know the restriction to X of the universal quotient bundle, doing some calculation we can derive the multidegree of X. We are now going to see it with some examples.

EXAMPLE 9. Let $X \subset G = G(1, N)$ be a smooth surface. Then we can write $[X] = a\Omega(0,3) + b\Omega(1,2) \in A^{2N-4}(G)$, where $a = [X] \cdot \Omega(N-3, N)$ is the number of lines of X meeting a general $\mathbb{P}^{N-3}$ and $b = [X] \cdot \Omega(N-2, N-1)$ is the number of lines contained in a general hyperplane. The zero locus of a section of $\mathcal{Q}$ corresponds to the Schubert cycle $\Omega(N-2, N-1)$. Hence b is the zero locus of a section of $\mathcal{Q}|_X$ or, by Porteous,

$$b = c_2(\mathcal{Q}|_X).$$

Moreover, the cycle $\Omega(N-3, N)$ can be viewed as the dependency locus of $N-2$ sections of the rank-$(N-1)$ universal subbundle $\mathcal{S}$, and hence a is the class of the dependency locus of $N-2$ sections of $\mathcal{S}|_X$. Finally, Porteous formula tells us that

$$a = c_2(\mathcal{S}|_X).$$

For instance, if we take the Veronese surface $X \subset G(1,3)$, we know that $\mathcal{Q}|_X$ is the rank-2 vector bundle $\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$, and hence $b = c_2(\mathcal{Q}|_X) = 1$. From the universal sequence of $G(1,3)$ restricted to $X \cong \mathbb{P}^2$

$$0 \to \mathcal{Q}^*|_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} \to \mathcal{S}|_{\mathbb{P}^2} \to 0, \tag{10}$$

we can easily calculate the Chern classes of $\mathcal{S}|_{\mathbb{P}^2}$ and in particular

$$\begin{aligned} a &= c_2(\mathcal{S}|_{\mathbb{P}^2}) \\ &= c_1(\mathcal{Q}|_{\mathbb{P}^2})^2 - c_2(\mathcal{Q}|_{\mathbb{P}^2}) \\ &= 3 \end{aligned}$$

Hence the Veronese surface has bidegree $(3,1)$.

Let us see it in the case of the congruence $X \subset G(1,3)$ given in Example 8. The surface X is the blow up of $\mathbb{P}^2$ in a point p, and the rank-2 vector bundle $\mathcal{Q}|_X$ is $\mathcal{O}_X(L) \oplus \mathcal{O}_X(L-E)$, where L is the pull back of a line in $\mathbb{P}^2$ and E is the exceptional divisor. Then the Chern polynomial of $\mathcal{Q}|_X$ is $c_\lambda(\mathcal{Q}|_X) = 1 + (2L - E)\lambda + L \cdot (L-E)\lambda^2$, and hence $b = c_2(\mathcal{Q}|_X) = L^2 - L \cdot E = 1$. As before, if we take the universal sequence of $G(1,3)$ restricted to X, we find the Chern classes of $\mathcal{S}|_X$. We then have $a = c_2(\mathcal{S}|_X) = c_1(\mathcal{Q}|_X)^2 - c_2(\mathcal{Q}|_X) = (2L-E)^2 - 1 = 2$, and hence (as we already knew) X has bidegree $(2,1)$.

EXAMPLE 10. Let $X \subset G(1,4)$ be a variety of dimension 3. Then $[X] = a\Omega(0,4) + b\Omega(1,3)$, where $a = [X] \cdot \Omega(0,4)$ and $b = [X] \cdot \Omega(1,3)$. As before, the Schubert cycle $\Omega(0,4)$ is the zero locus of one section of the rank-3 vector bundle $\mathcal{S}$. Hence

$$a = c_3(\mathcal{S}|_X).$$

The Schubert cycle $\Omega(1,3)$ can be viewed as the product of $\Omega(2,3)$ (which is the zero locus of one section of $\mathcal{Q}$) with the special Schubert cycle $\sigma_1 = \Omega(2,4)$. The last one is a hyperplane section of $G(1,4)$ i.e. a section of $\mathcal{O}_{G(1,4)}(1)$. Hence $\Omega(1,3)$ is the zero locus of one section of the rank-3 vector bundle $\mathcal{Q} \oplus \mathcal{O}_{G(1,4)}(1)$. Then Porteous formula tells us that

$$b = c_3(\mathcal{Q}|_X \oplus \mathcal{O}_{G(1,4)}(1)|_X).$$

As a concrete example we are now going to calculate the bidegree of the Veronese threefold $X \subset G(1,4)$. We have seen that the restriction of the universal bundle $\mathcal{Q}$ to X is $\mathcal{Q}|_X = \mathcal{O}_X(1) \oplus \mathcal{O}_X(1)$, while $\mathcal{O}_{G(1,4)}(1)$ is the line bundle giving the Plücker embedding and hence we have $\mathcal{O}_{G(1,4)}(1) = \bigwedge^2 \mathcal{Q}$. In particular when we restrict to X the rank-3 vector bundle $\mathcal{Q} \oplus \mathcal{O}_{G(1,4)}(1)$ we obtain $\mathcal{O}_X(1) \oplus \mathcal{O}_X(1) \oplus \bigwedge^2(\mathcal{O}_X(1) \oplus \mathcal{O}_X(1)) = \mathcal{O}_X(1)^{\oplus 2} \oplus \mathcal{O}_X(2)$, and since b is the third Chern class of this bundle, $b = 2$.

From the universal exact sequence restricted to X we find that $a = c_3(\mathcal{S}|_X) = c_1(\mathcal{Q}|_X)^3 - 2c_1(\mathcal{Q}|_X) \cdot c_2(\mathcal{Q}|_X) = 8 - 4 = 4$. Hence X has bidegree $(4,2)$.

2. The Grassmannian of Lines

From now on, we are going to focus our attention on the case $k = 1$, i.e. the Grassmannian of lines in $\mathbb{P}^N$, which will be the main topic of this thesis.

2.1. The Plücker Embedding. We know that the Grassmannian $G(1,N)$ can be embedded in $\mathbb{P}(\bigwedge^2 V) \cong \mathbb{P}^{\frac{N^2+N-2}{2}}$. This projective space can be viewed as the projectivization of the space of skew-symmetric $(N+1)\times(N+1)$ matrices

$$\mathbb{P}^{\frac{N^2+N-2}{2}} = \left\{ \begin{pmatrix} 0 & p_{0,1} & \cdots & p_{0,N-1} & p_{0,N} \\ -p_{0,1} & 0 & \cdots & p_{1,N-1} & p_{1,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -p_{0,N-1} & -p_{1,N-1} & \cdots & 0 & p_{N-1,N} \\ -p_{0,N} & -p_{1,N} & \cdots & -p_{N-1,N} & 0 \end{pmatrix} \right\}$$

or the space of null-correlations from $\mathbb{P}^{N^*}$ to $\mathbb{P}^N$,

$$\mathbb{P}^{\frac{N^2+N-2}{2}} = \{\phi : \mathbb{P}^{N^*} \to \mathbb{P}^N \mid \forall \alpha \in \mathbb{P}^{N^*},\ \phi(\alpha) \in H_\alpha\},$$

where H_α is the hyperplane corresponding to the point $\alpha \in \mathbb{P}^{N^*}$. Under these assumptions, the points of the Grassmannian $G(1,N)$ after the Plücker embedding in $\mathbb{P}^{\frac{N^2+N-2}{2}}$ correspond to rank-2 skew-symmetric matrices or rank-2 null-correlations. Specifically, the point $l \in G(1,N)$ corresponds to the null-correlation

$$\begin{aligned} \phi_l : & \mathbb{P}^{N*} \to \mathbb{P}^N \\ & \alpha \mapsto H_\alpha \cap L \end{aligned}$$

defined on the hyperplanes not containing L and sending such a hyperplane to its intersection with the line L. Reciprocally a rank-2 null-correlation determines a line by just considering its image.

2.2. The Tangent Space. Let us fix a point $l \in G(1,N)$ and consider the tangent space to $G(1,N)$ in such a point. We can assume that L is the line $\{x_2 = x_3 = \cdots x_N = 0\} \subset \mathbb{P}^N$. Then, as a point of $G(1,N)$ its coordinates are the 2×2 minors of the following matrix

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix}.$$

It corresponds to the point $(1:0:0:\ldots:0) \in \mathbb{P}^{\frac{N^2+N-2}{2}}$ or the skew-symmetric matrix

$$\begin{pmatrix} 0 & 1 & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

The tangent space to $G(1,N)$ at the point l, is isomorphic to the space of homomorphisms from the vector space $\mathcal{L} = \langle e_0, e_1\rangle \subset V^*$ to the quotient $V^*/\mathcal{L} = \langle \bar{e}_2, \bar{e}_3, \ldots, \bar{e}_N\rangle$,

$$\begin{aligned} T_{G(1,N),l} &= \operatorname{Hom}\{\mathcal{L}, V^*/\mathcal{L}\} \\ &= \left\{ \begin{pmatrix} \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \\ \alpha_4 & \beta_4 \\ \vdots & \vdots \\ \alpha_N & \beta_N \end{pmatrix} : \mathcal{L} \to V^*/\mathcal{L} \right\}. \end{aligned}$$

A tangent vector to $G(1,N)$ at l corresponds to a homomorphism $\varphi : \mathcal{L} \to V^*/\mathcal{L}$. If we consider the following inclusion of vector spaces

$$\mathrm{Hom}(\mathcal{L}, V^*/\mathcal{L}) \hookrightarrow \wedge^2 V^* / \wedge^2 \mathcal{L}$$
$$\varphi \mapsto [e_0 \wedge \varphi(e_1) + \varphi(e_0) \wedge e_1]$$

we see that the embedded tangent space $\mathbb{T}_{G(1,N),l} \subset \mathbb{P}^{\frac{N^2+N-2}{2}}$ is the projectivization of the following space of matrices (of rank less than or equal to 4)

$$\left\{ \begin{pmatrix} 0 & \lambda & \beta_2 & \cdots & \beta_N \\ -\lambda & 0 & -\alpha_2 & \cdots & -\alpha_N \\ -\beta_2 & \alpha_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\beta_N & \alpha_N & 0 & \cdots & 0 \end{pmatrix} \right\}$$

i.e. the linear space $\{p_{2,3} = p_{2,4} = \ldots = p_{N-1,N} = 0\}$.
Finally, inside the space of null correlations, the tangent space can be identified as

$$\mathbb{T}_{G(1,N),l} = \left\{ \begin{array}{ll} \phi : \mathbb{P}^{N*} \to \mathbb{P}^N / & i)\phi(\alpha) \in H_\alpha,\ \forall \alpha \in \mathbb{P}^{N*} \\ & ii)\phi(\alpha) \in L,\ \forall \alpha \,\mathrm{s.t.}\, H_\alpha \supset L \end{array} \right\}.$$

We finally prove the following easy but useful

PROPOSITION 2.1. *Given a line $\mathbb{P}^1_u$, tangent to $G(1,N)$ in l, one of the following holds:*

1. *the line $\mathbb{P}^1_u$ is contained in $G(1,N)$ (equivalent to L meeting the infinitely closed line in direction $\frac{\partial}{\partial u}$);*
2. *$\mathbb{P}^1_u \cap G(1,N) = l$, with multiplicity $N-1$ (equivalent to L not intersecting the infinitely closed line in direction $\frac{\partial}{\partial u}$).*

PROOF. Let us fix a tangent direction $\frac{\partial}{\partial u}$ (i.e. a vector in $\mathrm{Hom}\{\mathcal{L}, V^*/\mathcal{L}\}$) and call $\mathbb{P}^1_u$ the tangent line to $G(1,N)$ in this direction, embedded in $\mathbb{P}^{\frac{N^2+N-2}{2}}$.
The line $\mathbb{P}^1_u$ is contained in $\mathbb{T}_{G(1,N),l} \cong \mathbb{P}^{2N-2}$ and passes through the point l. But $\mathbb{T}_{G(1,N),l} \cap G(1,N) = \Omega(L, \mathbb{P}^N)$, is the Schubert variety of lines of $\mathbb{P}^N$ meeting L and, after the Plücker embedding it is a cone with vertex l on the variety $\mathbb{P}^1 \times \mathbb{P}^{N-2}$, embedded in $\mathbb{P}^{2N-3}$ via the Segre embedding. Then $\Omega(L, \mathbb{P}^N)$ has degree $N-1$ and a line contained in $\mathbb{T}_{G(1,N),l}$, passing through l, either is a ruling (and then it is contained in the cone and consequently in $G(1,N)$), or cuts that cone (and then the Grassmannian) only in the vertex l, with multiplicity $N-1$.

In the first case $\mathbb{P}^1_u$ is the pencil of lines contained in a plane $\Pi_u \supset L$ and passing through a point $p_u \in L$. This means that when we go in the direction $\frac{\partial}{\partial u}$, the line infinitely close to L is contained in Π_u and meets L in the point p_u. On the other hand, if the line infinitely close to L in such direction meets L in a point p_u, the tangent line $\mathbb{P}^1_u$ can only be a pencil of lines passing through p_u. □

Appendix

We recall here some classical result that we are going to use several times throughout the thesis.

THEOREM 2.2. *Let $Y \subset \mathbb{P}^N$ be an n-dimensional variety containing at least a $(2n-3)$-dimensional family of lines or, when $n \geq 4$, an $(n-1)$-dimensional family of $(n-2)$-planes. Then one of the following holds:*

1. *$Y \cong \mathbb{P}^n$;*
2. *Y is a scroll of $\mathbb{P}^{n-1}$'s (i.e. Y contains a 1-dimensional family of $\mathbb{P}^{n-1}$'s and a general point $y \in Y$ is contained in one and only one $\mathbb{P}^{n-1}$ of the family);*
3. *Y is a hyperquadric in $\mathbb{P}^{n+1}$.*

This is a particular case of a general result of B. Segre (see [**Seg48**]). The case of lines has been reproved recently by Rogora ([**Rog94**]), while the case of $(n-2)$-planes can be deduced from Rogora's result by observing that the fact of containing many $(n-2)$-planes implies that the variety contains many lines (see for instance [**Arr98**]).

Porteous Formula. Let us recall the following general construction of subvarieties of a projective variety W (further details can be found for instance in [**ACGH85**]). Let $\mathcal{F}$ be a rank-k locally free sheaf on W, generated by its global sections, and $s_1, s_2, \dots, s_m$, m general sections. They define a morphism $\mathcal{O}_W^m \to \mathcal{F}$, and their dependency locus is the locus X where the morphism has rank at most $m-1$. Porteous formula (or rather the particular case we are going to use) tells us that the locus X has the expected codimension , i.e. $k-m+1$, and its class in $A^{k-m+1}(W)$ is

$$[X] = c_{k-m+1}(\mathcal{F}).$$

Moreover the singular locus of X is the locus in which the above morphism has rank at most $m-2$, and its class in $A^{2(k-m+2)}(G(1,N))$ can be expressed as

$$[\operatorname{Sing} X] = c_{k-m+2}(\mathcal{F})^2 - c_{k-m+3}(\mathcal{F})c_{k-m+1}(\mathcal{F}).$$

CHAPTER 2

Projections in Grassmannians

In this chapter we introduce the tools that will be used in the rest of the thesis to study projections of subvarieties of the Grassmannian of lines. In particular we analyze the relations between the case of varieties in the projective space and the new problems arising in the case of Grassmannians of lines.

In the first section we relate the projection from $G(1, N)$ to $G(1, N-k-1)$ with a projection in the corresponding projective space after the Plücker embedding.

In the second section we study the problem of projecting a variety X from $G(1, N)$ to $G(1, N-1)$, defining the variety $\Sigma X \subset \mathbb{P}^N$ of points from which we cannot project. Then, considering the dimension of this variety we have that every smooth n-dimensional variety $X \subset G(1, 2n+3)$ can be isomorphically projected to $G(1, 2n+2)$.

A first big difference with the projective case is that when considering a subvariety of the Grassmannian of lines, even if we know that every n-dimensional variety can be projected to $G(1, 2n+2)$, in fact we expect that in general X can be projected more.

For instance we classify n-dimensional varieties that cannot be isomorphically projected from $G(1, 2n+2)$ to $G(1, 2n+1)$, and we will see that this problem is strictly related to the projection of n-dimensional projective varieties, from $\mathbb{P}^{2n+1}$ to $\mathbb{P}^{2n}$.

Another difference with the projective space is that, while in the case of a variety $X \subset \mathbb{P}^N$, if we want to know how much X can be isomorphically projected, we just have to compute the dimension of its secant variety, when we take a variety $X \subset G(1, N)$ first we have to choose the Grassmannian to which we want to project X, and then, depending on this Grassmannian, we have different definitions of secant variety.

In particular, given a smooth $X \subset G(1, N)$, we introduce the *k-th secant variety* $S_k X \subset G(k, N)$ and we see that X can be projected to $G(1, N-k-1)$ if and only if $S_k X$ is not the whole $G(k, N)$.

The third section is devoted to the study of the first secant variety (that we call simply the secant variety) because this is the tool we are going to use in the next chapter.

The fourth section studies the relation of the secant variety with the projectability from $G(1, N)$ to $G(1, N-2)$ -which will be our most typical situation throughout the thesis- while the fifth section deals with a general projection to $G(1, N-k-1)$.

1. Projections and Plücker Embedding

We are now going to relate the projection from $G(1, N)$ to a smaller Grassmannian, with linear projections in the projective space after the Plücker embedding. The following holds:

PROPOSITION 1.1. *The projection* $\pi_h : G(1,N) \to G(1,N-k-1)$ *with center* $H \cong \mathbb{P}^k$ *is the restriction to* $G(1,N)$ *of the projection*

$$\pi_{K_h} : \mathbb{P}^{\frac{N^2+N-2}{2}} \dashrightarrow \mathbb{P}^{\frac{(N-k)(N-k-1)}{2}-1}$$

with center $K_h = \langle \Omega(H, \mathbb{P}^N) \rangle \cong \mathbb{P}^{\frac{(2N-k)(k+1)}{2}-1}$.

PROOF. Let us choose the homogeneous coordinates $(x_0 : x_1 : \ldots : x_N)$ for the projective space $\mathbb{P}^N$, so that $H = \{x_{k+1} = x_{k+2} = \ldots = x_N = 0\} \cong \mathbb{P}^k \subset \mathbb{P}^N$, or

$$H = \begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix},$$

and consider the projection $p_h : \mathbb{P}^N \to \mathbb{P}^{N-k-1} = \{x_0 = x_1 = \ldots = x_k = 0\}$. Let us take the two Plücker embeddings of $G(1,N)$ in $\mathbb{P}^{\frac{N^2+N-2}{2}}$ and $G(1,N-k-1)$ in $\mathbb{P}^{\frac{(N-k)(N-k-1)}{2}-1}$. We indicate with $l \in G(1,N)$ the point corresponding to

$$L = \begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_N \\ b_0 & b_1 & b_2 & \ldots & b_N \end{pmatrix},$$

a general line in $\mathbb{P}^N$ which does not intersect the center of projection H. The coordinates of l inside $\mathbb{P}^{\frac{N^2+N-2}{2}}$ are the 2×2 minors of the above matrix. Then $l = (\ldots : p_{i,j} : \ldots)$, or

$$l = \begin{pmatrix} 0 & p_{0,1} & \cdots & p_{0,N} \\ -p_{0,1} & 0 & \cdots & p_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ -p_{0,N} & -p_{1,N} & \cdots & 0 \end{pmatrix},$$

where we put $p_{i,j} = a_i b_j - a_j b_i$. The image of l by π_h is the intersection of $\mathbb{P}^{N-k-1}$ with the $\mathbb{P}^{k+2}$ spanned by H and L. We have

$$\langle H, L \rangle = \left\{ \begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ a_0 & \cdots & a_k & a_{k+1} & \cdots & a_N \\ b_0 & \cdots & b_k & b_{k+1} & \cdots & b_N \end{pmatrix} \right\}$$

and then, when we intersect it with $\mathbb{P}^{N-k-1} = \{x_0 = x_1 = \ldots = x_k = 0\}$ we obtain the line

$$\pi_h(l) = \begin{pmatrix} 0 & \cdots & 0 & a_{k+1} & \cdots & a_N \\ 0 & \cdots & 0 & b_{k+1} & \cdots & b_N \end{pmatrix},$$

corresponding to the skew-symmetric matrix

$$\begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & p_{k+1,k+2} & \cdots & p_{k+1,N-1} & p_{k+1,N} \\ 0 & \cdots & 0 & -p_{k+1,k+2} & 0 & \cdots & p_{k+2,N-1} & p_{k+2,N} \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -p_{k+1,N-1} & -p_{k+2,N-1} & \cdots & 0 & p_{N-1,N} \\ 0 & \cdots & 0 & -p_{k+1,N} & -p_{k+2,N} & \cdots & -p_{N-1,N} & 0 \end{pmatrix}.$$

Thus when we project from H, we are just forgetting the first $\frac{(2N-k)(k+1)}{2}$ coordinates. Moreover, we can easily see that K_h, linear span of $\Omega(H,\mathbb{P}^N)$ in $\mathbb{P}^{\frac{N^2+N-2}{2}}$ is given by $K_h \cong \mathbb{P}^{\frac{(2N-k)(k+1)}{2}-1} = \{p_{i,j}=0,\ k\le i\le N-1,\ i+1\le j\le N\}$, or

$$K_h = \left\{ \phi:\mathbb{P}^{N*}\to\mathbb{P}^N/ \begin{array}{l} i)\phi(\alpha)\in H_\alpha,\ \forall\alpha\in\mathbb{P}^{N*} \\ ii)\phi(\alpha)\in H,\ \forall\alpha \text{ s.t.} H_\alpha\supset H \end{array} \right\}$$

$$= \left\{ \begin{pmatrix} 0 & p_{0,1} & \cdots & p_{0,k} & p_{0,k+1} & \cdots & p_{0,N} \\ -p_{0,1} & 0 & \cdots & p_{1,k} & p_{1,k+1} & \cdots & p_{1,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ -p_{0,k} & -p_{1,k} & \cdots & 0 & p_{k,k+1} & \cdots & p_{k,N} \\ -p_{0,k+1} & -p_{1,k+1} & \cdots & -p_{k,k+1} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ -p_{0,N} & -p_{1,N} & \cdots & -p_{k,N} & 0 & \cdots & 0 \end{pmatrix} \right\},$$

and then when we forget the first $\frac{(2N-k)(k+1)}{2}$ coordinates, we are just projecting from $K_h \cong \mathbb{P}^{\frac{(2N-k)(k+1)}{2}-1}$. □

REMARK. We have seen in Example 5 that the n-dimensional Veronese variety $X\subset G(1,2n+1)$ can be isomorphically projected to $G(1,n+1)$. In this case, after the Plücker embedding of $G(1,2n+1)$ in $\mathbb{P}^{\binom{2n+1}{2}-1}$, we can see that in fact X is already contained in a $\mathbb{P}^{\binom{n+1}{2}-1}$. Hence the projection after the Plücker embedding is simply the identity map.

But this is not always the case. For instance in [**AS92**] (type 10 in the list of page 44) can be found an example of a congruence $X\subset G(1,3)$ of bidegree $(3,3)$ that is projected from $G(1,4)$, while if we regard it as a surface in $\mathbb{P}^5$, it is a projection from a surface of $\mathbb{P}^6$.

On the other hand, the opposite situation is also possible. The congruence $X\subset G(1,3)$ of bidegree $(2,1)$ described in Examples 3, 6 and 8 (which corresponds to the dual of type 3 in [**AS92**]) is an isomorphic projection from a surface in $G(1,4)$. But, as a surface in $\mathbb{P}^5$, is degenerate, since it is contained in a hyperplane (precisely the one corresponding to the complex of lines meeting the fundamental line that X possesses).

2. Projecting from $G(1,N)$ to $G(1,N-1)$

We are going to recall briefly what is the tool to study projections in the case of subvarieties of the projective space (see for instance [**Zak93**]). Let $X\subset\mathbb{P}^N$ be a smooth nondegenerate n-dimensional variety. We call I the closure of the set $I_0 = \{(x_1, x_2, z) \in (X \times X \setminus \Delta_X) \times \mathbb{P}^N \mid z \in \langle x_1, x_2 \rangle\}$, inside $X \times X \times \mathbb{P}^N$ and consider the diagram

(11)
$$\begin{array}{ccc} & I & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ X \times X & & \mathbb{P}^N. \end{array}$$

DEFINITIONS. The image $SX = p_2(I)$ of the second projection is called *secant variety* of X.

The *defect* of X, denoted by $\operatorname{def}(X)$, is defined to be the dimension of the general fiber of the second projection.

Therefore the defect of X is the dimension of the family of secant lines passing through a general point $p \in SX$.

With these definitions we have that $\dim SX = 2n + 1 - \operatorname{def}(X)$. Moreover it is not hard to prove the following

PROPOSITION. *Let $X \subset \mathbb{P}^N$ be a smooth nondegenerate variety of dimension n. Then the projection π_p is an isomorphism if and only if $p \notin SX$.*

In particular, since $\dim SX = 2n + 1 - \operatorname{def}(X) \leq 2n + 1$, we have that every smooth n-dimensional variety can be isomorphically projected to $\mathbb{P}^{2n+1}$. Moreover X can be projected more if and only if the defect is positive (in fact the value of $\operatorname{def}(X)$ tells us how much X can be isomorphically projected).

A first interesting problem is then to find smooth n-dimensional varieties that can be isomorphically projected from $\mathbb{P}^{2n+1}$ to $\mathbb{P}^{2n}$ (i.e. varieties with positive defect).

In the case $n = 2$, Severi proved in [**Sev01**] that the only smooth nondegenerate surface that can be isomorphically projected from $\mathbb{P}^5$ to $\mathbb{P}^4$ is the double Veronese embedding of $\mathbb{P}^2$.

For $n = 3$ there exists the following classification due to Fujita (see [**Fuj82**])

THEOREM 2.1. *If $X \subset \mathbb{P}^7$ is a smooth, nondegenerate variety of dimension 3 which can be isomorphically projected to $\mathbb{P}^6$, then X is the projection of one of the following:*

1. *the Veronese three-fold $\mathbb{P}^3$, embedded by $\mathcal{O}_{\mathbb{P}^3}(2)$;*
2. *the blowing-up of $\mathbb{P}^3$ at a point p, embedded by the quadrics through p;*
3. *a hyperplane section of $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$.*

Finally, Zak extended Severi's result and, for $n \geq 2$, proved the following

THEOREM 2.2. *The only smooth nondegenerate n-dimensional subvariety of $\mathbb{P}^{\frac{n(n+3)}{2}}$ that can be isomorphically projected to $\mathbb{P}^{2n}$ is the double Veronese embedding of $\mathbb{P}^n$.*

Let now $X \subset G(1, N)$ be a smooth, nondegenerate subvariety of dimension n, and $p \in \mathbb{P}^N$ a point such that $X \cap \Omega(p, \mathbb{P}^N) = \emptyset$. We can easily see that if we take $l_1, l_2 \in X$, their image by π_p is the same if and only if they lie on a plane containing the point p.

More in general, let us consider the following definition. We take the incidence variety $I_0 = \{(l,\pi) \in X \times G(2,N) \mid L \subset \Pi\}$ and the following diagram

(12)
$$\begin{array}{ccc} & I_0 & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ X & & G(2,N). \end{array}$$

DEFINITION. We say that Π is a *bad plane* for X, if $p_2^{-1}(\pi)$ has length at least 2. We call *family of bad planes* the subvariety $\Sigma(X) = \{\pi \in G(2,N) \mid \operatorname{length}(p_2^{-1}(\pi)) \geq 2\} \subset G(2,N)$.

Roughly speaking, $\Sigma(X)$ is the family of planes containing at least two lines of X, maybe infinitely closed.

GENERAL NOTATION. Throughout the thesis we will denote by ΣX the union inside $\mathbb{P}^N$ of planes of $\Sigma(X)$.

PROPOSITION 2.3. *Let $X \subset G(1,N)$ be a smooth, nondegenerate variety of dimension n, such that $X \cap \Omega(p,\mathbb{P}^N) = \emptyset$. Then the projection π_p restricted to X is an isomorphism if and only if $p \notin \Sigma X$.*

PROOF. First of all, as we have already seen, π_p is not injective if and only if p belongs to a bad plane spanned by two distinct lines of X.

Now we are going to prove that the differential $d_l\pi_p$ at a point $l \in X$ is not injective if and only if p belongs to a bad plane spanned by the line L and a second line infinitely closed to L.

In fact the map $d_l\pi_p : \mathbb{T}_{X,l} \to \mathbb{T}_{X',l'}$ is just the projection from the linear space $\Omega(p,\mathbb{P}^N) \cong \mathbb{P}^{N-1}$, restricted to $\mathbb{T}_{X,l}$, and it is not injective if and only if $\mathbb{T}_{X,l} \cap \Omega(p,\mathbb{P}^N) \neq \emptyset$. This is equivalent to say that there exists a tangent direction $\frac{\partial}{\partial u}$ such that the tangent line in this direction, say $\mathbb{P}^1_u$, intersects the linear space $\Omega(p,\mathbb{P}^N)$. As we have seen in Proposition 2.1 , either the line $\mathbb{P}^1_u$ is contained in $G(1,N)$ or they meet only in the point l.

If $\mathbb{P}^1_u \cap G(1,N) = \{l\}$, then, since $\mathbb{P}^1_u$ meets $\Omega(p,\mathbb{P}^N)$, we must have $l \in \Omega(p,\mathbb{P}^N)$, which is not possible because we are supposing $X \cap \Omega(p,\mathbb{P}^N) = \emptyset$.

Then it must be $\mathbb{P}^1_u \subset G(1,N)$, i.e. $\mathbb{P}^1_u$ is a pencil of lines contained in a plane Π_u and passing through a point $p_u \in \Pi_u$. Thus Π_u is a bad plane, because it is spanned by the line L and the line infinitely closed to it in direction $\frac{\partial}{\partial u}$. In this case $\mathbb{P}^1_u \cap \Omega(p,\mathbb{P}^N)$ is not the empty set if and only if one of the lines in the ruling contains the point p, which is equivalent to p lying on the bad plane Π_u. $\square$

REMARK. The hypothesis $X \cap \Omega(p,\mathbb{P}^N) = \emptyset$ is necessary to guarantee that the map π_p is defined. In fact, a first difference with the projective case is that when we project $X \subset \mathbb{P}^N$, the points for which the projection is not defined are simply the points $x \in X$, and X is trivially contained in its secant variety SX. On the other hand, when we are in the Grassmannian of lines, the projection π_p is not defined on the points $p \in Y$, the union of lines of X inside $\mathbb{P}^N$. But in general Y is not contained in ΣX. Hence we have to put the extra hypothesis $X \cap \Omega(p,\mathbb{P}^N) = \emptyset$ in Proposition 2.3.

2.1. Dimensions. Now, using the result of Proposition 2.3, we can easily prove the following

PROPOSITION 2.4. *Every smooth n-dimensional subvariety $X \subset G(1,N)$, with $N \geq 2n+3$ can be isomorphically projected to $G(1,2n+2)$.*

PROOF. Indeed, every line $l \in X$ can be contained at least in an n-dimensional family of bad planes (this corresponds to the worst case in which every line of X meets all the other lines of X), so that the family of bad planes $\Sigma(X)$ cannot have dimension bigger that $2n$. This implies that $\dim \Sigma X \leq 2n+2$ and then, if $N > 2n+2$, the family of bad planes cannot fill up the whole $\mathbb{P}^N$. □

REMARK. It seems to be the analogous to the fact that if we have a smooth n-dimensional projective variety $X \subset \mathbb{P}^N$, with $N \geq 2n+2$, it can be projected to $\mathbb{P}^{2n+1}$. But in fact there exists a big difference because in the projective case we expect that in general an n-dimensional variety cannot be projected to a smaller projective space. In case of the subvarieties of the Grassmannian of lines, we expect that an n-dimensional variety X in general can be projected to a Grassmannian smaller than $G(1,2n+2)$.

The fact is that we know that the dimension of the variety $\Sigma(X)$ cannot be bigger than $2n$, but in general we expect this dimension to be much smaller. We are going to characterize the n-dimensional varieties that cannot be projected to $G(1,2n+1)$. But before doing that, we recall the following

LEMMA 1. *Let $X \subset G(1,N)$ a smooth n-dimensional variety such that two general lines $l_1, l_2 \in X$ meet. Then one of the following holds:*
i) $n=2$ and X is a β-plane;
ii) the points of X correspond to the ruling lines of a cone on a smooth variety X'.

The proof is straightforward (see for instance [**Arr99**]).

PROPOSITION 2.5. *Let $X \subset G(1,2n+2)$ be a smooth, nondegenerate variety of dimension n. Then X cannot be isomorphically projected to $G(1,2n+1)$ if and only if the points of X correspond to the ruling lines of a cone on a variety $X' \subset \mathbb{P}^{2n+1}$ that cannot be isomorphically projected to $\mathbb{P}^{2n}$.*

PROOF. If $X \subset G(1,2n+2)$ is a smooth nondegenerate variety of dimension n that cannot be isomorphically projected to $G(1,2n+1)$, then the union of bad planes ΣX is the whole $\mathbb{P}^{2n+2}$. But this can be the case only if $\dim \Sigma(X) = 2n$ and a general point on a bad plane is contained in a finite number of bad planes. From the incidence diagram

$$\begin{array}{ccccc} & & J_\Sigma & & \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ \Sigma(X) & & & & X, \end{array} \tag{13}$$

where $J_\Sigma = \{(\pi, l) \in \Sigma(X) \times X \mid L \subset \Pi\}$, we find that the second projection must have n-dimensional fiber. This means that every line of X is contained in an n-dimensional family of bad planes and then, in particular it meets an n-dimensional family of lines of X. Then every line of X meets all the other lines of the variety. From the lemma above X is either a β-plane (but in this case X would be degenerate) or the family of the ruling lines of a cone on an n-dimensional variety $X' \subset \mathbb{P}^{2n+1}$. In this last case a bad plane is spanned by two ruling lines or equivalently by the vertex $p \in \mathbb{P}^{2n+2}$ and a secant line of the variety X'. Then the union of bad planes ΣX is simply the cone with vertex p, on the secant variety

$SX' \subset \mathbb{P}^{2n+1}$. The variety X' cannot be isomorphically projected to $\mathbb{P}^{2n}$ if and only if $SX' = \mathbb{P}^{2n+1}$, i.e. $\Sigma X = \mathbb{P}^{2n+2}$. □

REMARK. The proposition above is saying that the problem of projecting an n-dimensional variety $X \subset G(1, 2n+2)$ to $G(1, 2n+1)$ is equivalent to the problem of projecting a projective n-dimensional variety X' from $\mathbb{P}^{2n+1}$ to $\mathbb{P}^{2n}$.

As we have already seen, there exists a complete classification of such varieties only for $n = 2, 3$.

2.2. Expected Dimensions. We have seen that every smooth n-dimensional variety $X \subset G(1, N)$, with $N \geq 2n+3$ can be isomorphically projected to $G(1, 2n+2)$, and in general X can be projected to $G(1, 2n+1)$. We are going to see that in fact we expect X to be projected more.

Let us see that in general an n-dimensional variety is not expected to have a family of bad planes of dimension bigger than n. In fact, if we denote as usual by $Y \subset \mathbb{P}^N$ the union of lines of X, we have that in general a bad plane for X corresponds to a singular point of Y. Since $\dim Y \leq n+1$, then $\dim(\operatorname{Sing} Y) \leq n$ (and in fact one could expect $\dim(\operatorname{Sing} Y) \leq 2n+2-N$). Hence $\dim \Sigma(X) \leq n$, which implies that in general we expect that an n-dimensional variety can be isomorphically projected to $G(1, n+2)$. We will see that in general X cannot be projected more.

CLAIM. *In general we expect that a smooth nondegenerate n-dimensional subvariety $X \subset G(1, n+2)$ cannot be isomorphically projected to $G(1, n+1)$.*

Since $\Omega(1, n+2)$, the Schubert cycle of lines meeting a fixed line, has codimension n, while $[X]$ has dimension n, the intersection product $[X] \cdot \Omega(1, n+2)$ is expected to be different from zero (this number is one of the multidegrees of X). Then, when we fix a general line $l_0 \in X$, we expect that $X \cap \Omega(L_0, \mathbb{P}^N)$ is not the empty set. This is equivalent to say that a general line $l_0 \in X$ intersects at least another line of X, and hence we have an n-dimensional family of bad planes and in general their union fill up the whole $\mathbb{P}^{n+2}$. This prove the claim.

Summarizing, we have seen that every smooth n-dimensional variety $X \subset G(1, N)$, with $N \geq 2n+3$ can be isomorphically projected to $G(1, 2n+2)$, and in general we expect that it can be projected to $G(1, n+2)$. Then it is an interesting problem to classify the n-dimensional varieties $X \subset G(1, k+1)$ that cannot be isomorphically projected to $G(1, k)$, for $n+2 \leq k \leq 2n+1$. In Proposition 2.5 we have already given a classification in case $k = 2n+1$ and in Chapter 5 we will complete the case $n = 2$.

Let us see an example of a surface that cannot be projected from $G(1, 5)$ to $G(1, 4)$ (we are then in the case $n = 2$ and $k = n+2$).

EXAMPLE 11. Let C be a smooth, nondegenerate curve contained in $H \cong \mathbb{P}^3$ and $L \cong \mathbb{P}^1$ a line that does not intersect H. We take the variety $X = L \times C$ embedded in $G(1, 5)$ by the rank-2 vector bundle $\mathcal{Q}|_X = \mathcal{O}_X(1, 0) \oplus \mathcal{O}_X(0, 1)$, where $\mathcal{O}_X(1, 0)$ and $\mathcal{O}_X(0, 1)$ are the pull back of $\mathcal{O}_L(1)$ and $\mathcal{O}_C(1)$ respectively. We can then view X as the family of lines joining a point on L and a point on C. Clearly X is a smooth surface in $G(1, 5)$.

Every time we take a point $p \in L$ we find a 2-dimensional family of bad planes containing p, namely the planes spanned by p and a secant line of C. Then the union of bad planes passing through p contains the cone on the secant variety SC. Since C is nondegenerate, the secant variety SC is the whole H. Hence if we let p move on L we find a 3-dimensional family $\Sigma_L(X)$ of bad planes such that their union $\Sigma_L X \subset \mathbb{P}^5$ is the cone with vertex L, on $H \cong \mathbb{P}^3$, i.e. $\Sigma_L X = \mathbb{P}^5$. Thus X cannot be isomorphically projected to $G(1,4)$.

Moreover in this example we can see that in fact the family of bad planes $\Sigma(X)$ is not irreducible. Let us take a point $q \in C$. Clearly the plane $\Pi_q = \langle q, L\rangle$ is a bad plane containing a 1-dimensional family of lines of X (the pencil $\Omega(q, \Pi_q)$). Then if we let q move on the curve C we find a 1-dimensional family of bad planes $\Sigma_C(X)$. Clearly a plane $\pi \in \Sigma_C(X)$ cannot be an element of $\Sigma_L(X)$, because it contains the line L, while every plane of $\Sigma_L(X)$ cuts L in only one point.

Example 12. More in general, let us take inside $\mathbb{P}^5$ two curves C_1 and C_2 such that at least one of them (say C_1) is not plane. We consider the family of lines joining C_1 and C_2, i.e. the surface $X = C_1 \times C_2 \subset G(1,5)$, where the inclusion is given by 6 sections of the rank-2 vector bundle $\mathcal{O}_X(0,1) \oplus \mathcal{O}_X(1,0)$.

When we fix a point $p \in C_2$, the lines of X passing through p are the rulings of the cone on C_1. Therefore they give rise to a 2-dimensional family of bad planes such that their union is the cone on the secant variety of C_1, and then this union has dimension 4. Now, if we let p move on C_2 we find a 3-dimensional family of bad planes of this kind, say $\Sigma_2(X) \subset \Sigma(X)$, such that their union is the whole $\mathbb{P}^5$. Hence X cannot be isomorphically projected to $G(1,4)$.

In the next section we are going to see that if we consider projections from a line, the construction of the secant variety (i.e. the variety of lines from which we cannot project) is much more natural and in some sense similar to the construction of secant varieties in the projective space.

Moreover this secant variety will be an important tool in the study of varieties that can be projected more than one expects.

3. Secant Variety

3.1. Definition and Notations. We have seen in Lemma 1 that if two general lines of a nondegenerate variety X meet, then X is the family of ruling lines of a cone. Since we have already studied this case, in the remainder of this section we make the assumption that two general lines of X are skew.

Let $X \subset G(1,N)$ be a nondegenerate subvariety of dimension n, and

$$I^0 := \{(l_1, l_2, l) \in (X \times X \setminus \Delta_X) \times G(1,N) \mid \dim\langle L_1, L_2, L\rangle \leq 3\}.$$

Let I denote the closure of I^0 inside $X \times X \times G(1,N)$ and consider the following diagram

$$\begin{array}{ccc} & I & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ X \times X & & G(1,N). \end{array} \tag{14}$$

Definition. The *secant variety* of X, denoted by SX, is defined to be the image of the second projection p_2.

Roughly speaking, SX is the set of lines contained in a $\mathbb{P}^3$ which contains at least two lines of X, maybe infinitely closed to each other.

REMARK. It is easy to see that in fact there exist two distinct "pieces" in SX, i.e. the irreducible component $S'X$ of lines contained in a $\mathbb{P}^3$ spanned by two skew lines of X (or, to be precise, the closure of this kind of $\mathbb{P}^3$'s), and the set $S''X$ of lines meeting a bad plane. In general the latter is a union of irreducible components (see Example 11). Let us formalize it by studying separately both pieces.

3.2. $S'X$. Let $Z = \{(l_1, l_2) \in X \times X \mid L_1 \cap L_2 \neq \emptyset\}$ be the set of pairs of lines meeting and consider the map

$$g : (X \times X) \setminus Z \to G(3, N)$$

which associate to each pair of skew lines their linear span. Let us call $S'(X) = \overline{\operatorname{Im} g}$ the closure of the image of this map. Hence $S'(X)$ is the variety of $\mathbb{P}^3$'s spanned by two skew lines of X (and their limits). Let us recall the following definition (see [**Arr99**])

DEFINITION. The *secant defect* of X is the dimension δ of the family of lines $m \in X$, contained in a general $\mathbb{P}^3$ corresponding to a point $h \in S'(X)$.

With this definition the fiber of g has dimension 2δ and then $\dim S'(X) = 2n - 2\delta$.

EXAMPLE 13. Let us go back to the n-dimensional Veronese variety $X \subset G(1, 2n+1)$. We have seen that this is the set of lines joining the corresponding points on two disjoint copies of $\mathbb{P}^n$. Let us take $l_1, l_2 \in X$, two general lines. Then there exist p_1, p_2 on the first $\mathbb{P}^n$ and q_1, q_2 on the second one, such that $L_i = \langle p_i, q_i \rangle$, for $i = 1, 2$. Let $h \in S'(X)$ be the point corresponding to $H = \langle L_1, L_2 \rangle \cong \mathbb{P}^3$. Clearly H cuts the two $\mathbb{P}^n$'s along the lines $L = \langle p_1, p_2 \rangle$ and $M = \langle q_1, q_2 \rangle$ respectively. Hence all the lines joining one point on L with the corresponding one on M are contained in H. These lines correspond to one of the two rulings of a quadric. In particular we have seen that the defect of X is $\delta = 1$.

Indeed the following holds

LEMMA 2. *Let $X \subset G(1, N)$ be a smooth, nondegenerate n-dimensional subvariety such that one of the following holds:*
i) $N \geq n + 2$;
ii) X can be isomorphically projected to $G(1, N-1)$.
Then $\delta \leq 1$.

PROOF. By definition, δ can assume a value from 0 to 4. But if $\delta \geq 2$, then in a general $\mathbb{P}^3$ spanned by two skew lines of X there exist a 2-dimensional family of lines of X which must fill up the whole $\mathbb{P}^3$ (otherwise they would fill something of dimension two, which is necessarily a plane and hence the two lines would meet). Let us take two general points $y_1, y_2 \in Y$. There exist two skew lines $l_1, l_2 \in X$ such that $y_i \in L_i$, for $i = 1, 2$. As we have seen, $H = \langle L_1, L_2 \rangle$ is covered by lines of X, and then $H \subset Y$. In particular the line $\langle y_1, y_2 \rangle$ is contained in Y, and hence Y must be linear.

Now, if we are in case i), since $\dim Y \leq n + 1$, X is degenerate. In case ii), since we can isomorphically project to $G(1, N-1)$, it must be $\dim Y \leq N - 1$ and then X is again degenerate. □

We are now going back to the construction of the component $S'X$. Let us take the following incidence diagram

(15)
$$\begin{array}{ccc} & J & \\ {\scriptstyle q_1}\swarrow & & \searrow{\scriptstyle q_2} \\ S'(X) & & G(1,N), \end{array}$$

where $J = \{(h,l) \in S'(X) \times G(1,N) \mid L \subset H\}$, and call $S'X$ the image of the second projection. It is an irreducible variety and is clearly contained in SX. This is in some sense the nice component of the secant variety SX, i.e. the component of lines contained in a $\mathbb{P}^3$ spanned by two skew lines of X.
Let us recall the following definition (see [**Bal97**]).

DEFINITION. The *complementary defect* of X is the dimension ρ of the general fiber of the map q_2.

Roughly speaking, ρ is the dimension of the family of $h \in S'(X)$ containing a general element $l \in S'X$. From the diagram (15), since the fiber of the first projection is the Grassmannian of lines in a $\mathbb{P}^3$, we can derive that

$$\dim S'X = 2n + 4 - 2\delta - \rho.$$

In the case of a surface in $G(1,5)$, we have only the secant defect, as the following lemma shows.

LEMMA 3. *Let $X \subset G(1,5)$ be a smooth nondegenerate surface, then $\rho = 0$.*

PROOF. We know from Lemma 2 that the secant defect can be 0 or 1. Let us distinguish the two cases.

1. $\delta = 0$.
   Then $\dim S'(X) = 4$. Let us consider the diagram

(16)
$$\begin{array}{ccc} & J & \\ {\scriptstyle q_1}\swarrow & & \searrow{\scriptstyle q_2} \\ S'(X) & & S'X, \end{array}$$

where $J = \{(h,l) \in S'(X) \times G(1,5) \mid L \subset H\}$. Since $\dim J = 8$, if $\rho > 0$ then $\dim S'X \leq 7$, which is equivalent to say that every time we fix a general $l \in S'X$, it is contained in at least a 1-dimensional family of $\mathbb{P}^3$'s of $S'(X)$. We fix now $h \in S'(X)$ and take the diagram

(17)
$$\begin{array}{ccc} & I_H & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ S'(X) & & G(1,H), \end{array}$$

where $I_H = \{(h',l) \in S'(X) \times G(1,H) \mid L \subset H'\}$. The fiber of p_2 has dimension bigger than zero (this dimension is the defect ρ), and then $\dim I_H \geq 5$.

Let denote by Z_H the image of the projection p_1. If we take $h' \in Z_H$, the fiber $p_1^{-1}(h')$ is the intersection $G(1,H) \cap G(1,H')$, and hence it can be a point or a β-plane, depending on the intersection of H and H'.

If $\dim p_2^{-1}(l) \geq 2$, then $\dim I_H \geq 6$. Hence necessarily p_1 is surjective, $\dim I_H = 6$ and $\dim p_1^{-1}(h') = 2$. The latter is equivalent to say that a general $h' \in S'(X)$ intersects H in a plane. If we dualize, we obtain a 4-dimensional variety $S'(X)^* \subset G(1,5^*)$ such that every line meets all the other lines, and hence, by Lemma 1, the points of $S'(X)^*$ are the ruling lines of a cone. In particular all the lines pass through a point, which is equivalent to say that all the $\mathbb{P}^3$'s of $S'(X)$ are contained in a $\mathbb{P}^4$, i.e. X is degenerate. This is not possible.

Let us then suppose $\dim p_2^{-1}(l) = 1$. In this case we have $\dim I_H = 5$, which implies that the dimension of the fiber of p_1 is 2, and the image $Z_H \subset S'(X)$ has dimension 3. Therefore every time we fix $h \in S'(X)$ we find a 3-dimensional family of $\mathbb{P}^3$'s of $S'(X)$ which intersect H in a plane. If we dualize we find a 4-dimensional variety $S'(X)^* \subset G(1,5^*)$ such that every line $m \in S'(X)^*$ meets a 3-dimensional family of other lines.

Let us distinguish two cases.

(a) Through a general point of M there pass a 2-dimensional family of lines of $S'(X)^*$. But in this case the union of lines of $S'(X)^*$ would be a 3-dimensional variety containing a 4-dimensional family of lines, and hence a linear space $K^* \cong \mathbb{P}^3$. In particular $S'(X)^* = G(1,K^*)$ which is equivalent to $S'(X) = \Omega(0,K,\mathbb{P}^3,\mathbb{P}^4,\mathbb{P}^5)$, i.e. the family of $\mathbb{P}^3$'s containing the line K. But then in this case every time we fix a line $l \in G(1,5)$, the $\mathbb{P}^3$ spanned by K and L is contained in $S'(X)$ and hence $l \in S'X$. Therefore $S'X = G(1,5)$, which is a contradiction.

(b) Through a general point of M there pass no other lines of $S'(X)^*$. Hence there exists a point $p \in M$ such that through p there pass a 3-dimensional family of lines of $S'(X)^*$. Globally we have a curve C such that through every $p \in C$ there pass a 3-dimensional family of lines of $S'(X)^*$. Dualizing back we find a 1-dimensional family of $\mathbb{P}^4$'s containing each a 3-dimensional family of $\mathbb{P}^3$'s of $S'(X)$. But in each $\mathbb{P}^4$ we must have at most a 1-dimensional family of lines of X, and they cannot give rise to a 3-dimensional family of $\mathbb{P}^3$'s. We get again a contradiction

2. $\delta = 1$

Then $\dim S'(X) = 2$ and, from the diagram (17), if $\rho > 0$ we find that p_1 is surjective and the dimension of $p_1^{-1}(h')$ must be at least 3. But we have seen that this dimension can be only 0 or 2.

□

3.3. $S''X$. To describe the other piece of the secant variety, we can just take the family $\Sigma(X) \subset G(2,N)$ of bad planes, which in general is not irreducible, and then the diagram

(18)
$$\begin{array}{ccc} & I_\Sigma & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \Sigma(X) & & G(1,N), \end{array}$$

where we set $I_\Sigma = \{(\pi,l) \in \Sigma(X) \times G(1,N) \mid L \cap \Pi \neq \emptyset\}$. We call $S''X$ the image of the second projection. Hence $S''X$ is the variety of lines meeting a bad plane (in general this is not an irreducible component of SX).

REMARK. Let us see that in general $S''X$ is not expected to be contained in the component $S'X$. Indeed, let $X \subset G(1,N)$ be an n-dimensional subvariety such that there exists a bad plane Π for X. Then the Schubert variety $\Omega(\Pi, \mathbb{P}^N)$ (the lines meeting Π) is contained in $S''X$. In particular $\dim S''X \geq \dim \Omega(\Pi, \mathbb{P}^N) = N+1$, while $\dim S'X \leq 2n+4$. Then if $N \geq 2n+4$, clearly $S''X$ cannot be contained in $S'X$.

4. Secant Variety and Projections

We are now going to relate the secant variety SX and the problem of projecting subvarieties of the Grassmannian of lines.

PROPOSITION 4.1. *Let $X \subset G(1,N)$ be a smooth, nondegenerate subvariety of dimension n, and $m \in G(1,N)$ be a line such that $X \cap \Omega(M, \mathbb{P}^N) = \emptyset$. Then the projection $\pi_m : G(1,N) \to G(1,N-2)$ restricted to X is an isomorphism if and only if $m \notin SX$.*

PROOF. We are going to use the following notations:

$$S^0X := p_2(I^0),$$
$$TX := p_2(p_1^{-1}(\Delta_X)),$$

where p_1 and p_2 are the projections in diagram (14). Then S^0X is going to be the set of lines contained in a $\mathbb{P}^3$ which contains at least two distinct lines of X, while m is a point of TX if and only if there exists a pair of infinitely closed lines of X such that their linear span is a $\mathbb{P}^3$ containing M or it is a plane meeting the line M. We then divide the proof in two steps.

1. π_m is injective if and only if $m \notin S^0X$.
   Indeed, if π_m is not injective, then there exist two distinct lines $l_1, l_2 \in X$ such that $\pi_m(l_1) = \pi_m(l_2)$, i.e. $\langle M, L_1\rangle = \langle M, L_2\rangle$. Hence $\dim\langle L_1, L_2, M\rangle \leq 3$, which is equivalent to $m \in S^0X$. On the other hand, if $\dim\langle L_1, L_2, M\rangle \leq 3$, then $\pi_m(l_1) = \pi_m(l_2)$, or l_1 and l_2 belong to $\Omega(M, \mathbb{P}^N)$, but it is not possible because we are assuming $X \cap \Omega(M, \mathbb{P}^N) \neq \emptyset$.
2. $d_l\pi_m : \mathbb{T}_{X,l} \to \mathbb{T}_{X',l'}$ is injective if and only if $m \notin TX$.
   In particular we will prove that there exists $l \in X$ such that $d_l\pi_m$ is not injective, if and only if $m \in p_2(p_1^{-1}(l,l))$. Indeed, since π_m is the restriction to $G(1,N)$ of the projection from $K_m := \langle\Omega(M, \mathbb{P}^N)\rangle$, the differential of π_m at l is the restriction of the same projection to the tangent space $\mathbb{T}_{X,l}$. This differential is not injective if and only if

   $$\mathbb{T}_{X,l} \cap \langle\Omega(M, \mathbb{P}^N)\rangle \neq \emptyset,$$

   i.e., there exists a tangent direction $\frac{\partial}{\partial u}$ such that the tangent line to X in this direction, say $\mathbb{P}^1_u$, intersects the linear space K_m. Then it is sufficient to prove that $\mathbb{P}^1_u \cap K_m \neq \emptyset$ if and only if $m \in p_2(p_1^{-1}(l,l))$.

   Let us suppose that $\mathbb{P}^1_u \cap K_m \neq \emptyset$. As we have proved in Proposition 2.1, the line $\mathbb{P}^1_u$ is either contained in $G(1,N)$, or its intersection with $G(1,N)$ is only the point l, with multiplicity $N-1$.
   (a) $\mathbb{P}^1_u \subset G(1,N)$.
   This means that the line L intersects the infinitely closed line in the direction $\frac{\partial}{\partial u}$ and there exists a point $p_u \in L$ and a plane $\Pi_u \supset L$, such

that $\mathbb{P}^1_u = \Omega(p_u, \Pi_u)$ (the pencil of lines meeting p_u and contained in Π_u). But then we have that

$$\Omega(p_u, \Pi_u) \cap K_m \neq \emptyset,$$

which is equivalent to the existence of $l' \in \Omega(p_u, \Pi_u) \cap \Omega(M, \mathbb{P}^N)$. In other words, $\exists\ L' \subset \Pi_u$ such that $L' \cap M \neq \emptyset$, equivalent to $\langle M, \Pi_u \rangle \cong \mathbb{P}^3$. Therefore $m \in p_2(p_1^{-1}(l, l))$.

(b) $\mathbb{P}^1_u \cap G(1, N) = \{l\}$, with multiplicity $N - 1$.
This is equivalent to say that the line L does not intersect the infinitely closed line in the direction $\frac{\partial}{\partial u}$. Let H_u be the $\mathbb{P}^3$ spanned by L and this infinitely close line, and $\mathbb{P}^5_u = \langle G(1, H_u) \rangle$. Since $\mathbb{P}^1_u \cap K_m \neq \emptyset$, in particular $\mathbb{P}^5_u \cap K_m \neq \emptyset$, which can happen if and only if $H_u \cap M \neq \emptyset$. Let us distinguish two cases.
If $M \subset H_u$ then $m \in p_2(p_1^{-1}(l, l))$.
If H_u and M meet in only one point p_u, one can easily prove that

$$\begin{aligned} K_m \cap \mathbb{P}^5_u &= \left\{ \begin{array}{ll} \phi : \mathbb{P}^{N*} \to \mathbb{P}^N \mid & i)\phi(\alpha) \in H_\alpha,\ \forall \alpha \in \mathbb{P}^{(N)*} \\ & ii)\phi(\alpha) = p_u,\ \forall \alpha\, \mathrm{s.t.}\, H_\alpha \supset L \\ & iii) Im(\phi) \subset H_u \end{array} \right\} \\ &= \left\{ \begin{array}{ll} \phi : H_u^* \to H_u \mid & i)\phi(\pi) \in \Pi,\ \forall \pi \in H_u^* \\ & ii)\phi(\pi) = p_u,\ \forall \pi\, \mathrm{s.t.}\, \Pi \ni p_u \end{array} \right\} \\ &= \Omega(p_u, H_u). \end{aligned}$$

In particular this intersection is contained in $G(1, N)$, and then

$$\emptyset \neq \mathbb{P}^1_u \cap K_m = \{l\} \cap \Omega(M, \mathbb{P}^N),$$

which means $L \cap M \neq \emptyset$, a contradiction.

On the other hand, if $m \in p_2(p_1^{-1}(l, l))$, there exists a tangent direction $\frac{\partial}{\partial u}$ such that either M meets the bad plane Π_u spanned by L and the infinitely closed line in this direction, or $M \subset H_u$, the $\mathbb{P}^3$ spanned by L and the infinitely closed line. In the first case we have that $\mathbb{P}^1_u = \Omega(p_u, \Pi_u)$ intersects $\Omega(M, \mathbb{P}^N)$ and hence $\mathbb{P}^1_u \cap K_m \neq \emptyset$.

If $M \subset H_u$, then $\mathbb{T}_{G(1,H_u),m} = \langle \Omega(M, H_u) \rangle \subset \langle \Omega(M, \mathbb{P}^N) \rangle$, and this tangent space is a hyperplane inside $\mathbb{P}^5_u$. Therefore $\mathbb{P}^1_u \cap \mathbb{T}_{G(1,H_u),m} \neq \emptyset$, and this intersection is contained in $\mathbb{P}^1_u \cap \langle \Omega(M, \mathbb{P}^N) \rangle$. In particular $\mathbb{P}^1_u \cap K_m \neq \emptyset$, which completes the proof.

□

4.1. Dimensions. From Proposition 4.1 we can easily see that a smooth variety $X \subset G(1, N)$ can be isomorphically projected to $G(1, N-2)$ if and only if the dimension of its secant variety SX is smaller than $\dim G(1, N) = 2N - 2$. Let us make some remarks about the dimension of SX.

We start from the component $S''X$, i.e. the family of lines meeting a bad plane. From the diagram (18) we can say that the dimension of $S''X$ is not bigger than $\dim \Sigma(X) + \dim \pi_1^{-1}(\pi)$. The fiber $\pi_1^{-1}(\pi)$ is the Schubert variety $\Omega(\Pi, \mathbb{P}^N)$, i.e. the family of lines meeting Π, whose dimension is $N + 1$. Therefore $\dim S''X \leq \dim \Sigma(X) + N + 1$.

Let us consider now the nice component $S'X$, which is in some sense similar to the secant variety in the projective space. We have seen that $\dim S'X = 2n +$

$4 - 2\delta - \rho$, and in particular it is not bigger than $2n + 4$. If we take N such that $2N - 2 > 2n + 4$, we guarantee that $S'X \subsetneq G(1, N)$. So

$$\text{if } N \geq n + 4, \text{ then } S'X \subsetneq G(1, N). \tag{19}$$

Moreover, in the general case the defects are zero, thus the component $S'X$ has not dimension smaller than $2n + 4$, which means that if $N = n + 3$, we expect $S'X$ to be the whole Grassmannian $G(1, n + 3)$.

Summarizing, we have found again that every smooth n-dimensional variety $X \subset G(1, N)$, with $N \geq 2n + 4$ can be isomorphically projected to $G(1, 2n + 2)$, and in general we expect that it can be projected to $G(1, n + 2)$.

The difference is that now we have a nice component that we can study to find a characterization of varieties that can be projected more than one expects. We are going to see it in the next chapter. Now we spend a few words about a natural generalization of this construction.

5. A Generalization

A big difference between the projective space and the Grassmannians of lines is that in the case of an n-dimensional variety $X \subset \mathbb{P}^N$, if we want to know how much X can be isomorphically projected, we just have to compute the dimension of its secant variety $SX \subset \mathbb{P}^N$.

We have already seen that in the case of a subvariety $X \subset G(1, N)$, if we are interested in projecting to $G(1, N - 1)$ we have just to consider the union of bad planes, while in case of projections to $G(1, N - 2)$ we have to take care also of the $\mathbb{P}^3$'s spanned by pairs of skew lines. The fact is that when we are projecting from a line $M \subset \mathbb{P}^N$, this is the same as projecting from a point $p \in M$ and then from the image of M by the first projection. Then we have to pay attention to the bad planes already contained in $\mathbb{P}^N$ and also to the bad planes arising from the first projection to $\mathbb{P}^{N-1}$.

Then if we are interested in projections from a k-dimensional linear space, i.e. if we want to know whether an n-dimensional variety $X \subset G(1, N)$ can be isomorphically projected to $G(1, N - k - 1)$, we have to make the following more general definition.

Let $X \subset G(1, N)$ be a smooth nondegenerate subvariety of dimension n. We put

$$I_k^0(X) := \{(l_1, l_2, h) \in (X \times X \setminus \Delta_X) \times G(k, N) \mid \dim\langle L_1, L_2, H\rangle \leq k + 2\},$$

and $I_k(X)$ will be its closure in $X \times X \times G(k, N)$. Let us consider the diagram

$$\begin{array}{ccc} & I_k(X) & \\ {\scriptstyle p_{1,k}} \swarrow & & \searrow {\scriptstyle p_{2,k}} \\ X \times X & & G(k, N). \end{array} \tag{20}$$

Definition. The *k-th secant variety* of X, denoted by S_kX, is defined to be the image of the second projection $p_{2,k}$.

Clearly with this definition, the secant variety $SX \subset G(1, N)$ we introduced before, coincides with the first secant variety S_1X, while the union of bad planes $\Sigma X \subset \mathbb{P}^N$ corresponds to the case $k = 0$.

As before it is not hard to see that in fact there exist two pieces, namely the component $S'_k X$ of k-planes meeting a $\mathbb{P}^3$ of $S'(X)$ in at least a line, and the piece $S''_k X$, i.e. the k-planes meeting a bad plane.

Using the same arguments of Proposition 4.1 one can then prove the following

PROPOSITION 5.1. *Let $X \subset G(1,N)$ be a smooth, nondegenerate n-dimensional subvariety and $h \in G(k,N)$ a k-plane such that $X \cap \Omega(H,\mathbb{P}^N) = \emptyset$. Then the projection $\pi_h : G(1,N) \to G(1,N-k-1)$ restricted to X is an isomorphism if and only if $h \notin S_k X$.*

From the incidence diagram (20) we can easily find the following bounds for the dimensions of the two pieces of $S_k X$:

$$\dim S'_k X \leq 2n - 2\delta + 4 + (k-1)(N-k),$$
$$\dim S''_k X \leq \dim \Sigma(X) + 2 + k(N-k),$$

where $\Sigma(X) \subset G(2,N)$ is the family of bad planes.

EXAMPLE 14. We can now see another way of proving that the n-dimensional Veronese variety $X \subset G(1,2n+1)$ of Example 5 can be projected to $G(1,n+1)$.

In fact let us consider the variety $S_{n-1}X \subset G(n-1,2n+1)$. First of all we know that inside X there are no pairs of lines meeting each other, and hence $\Sigma(X) = \emptyset$. Then we have just to take care of the component $S'_{n-1}X$. But since X has defect $\delta = 1$, we can write down

$$\dim S'_{n-1}X \leq 2n - 2 + 4 + (n-2)(n+2) = n^2 + 2n - 2,$$

while $\dim G(n-1,2n+1) = n^2 + 2n$. Hence $S_{n-1}X \subsetneq G(n-1,2n+1)$, and X can be isomorphically projected to $G(1,n+1)$, as we already knew.

CHAPTER 3

A Structure Theorem for Projectable Varieties

We have said that in general we expect that an n-dimensional variety $X \subset G(1,N)$, with $N \geq n+4$ can be isomorphically projected to $G(1,n+2)$. In this chapter we study varieties that can be projected more than one expects. In particular we consider projections with center of dimension at least one, which is equivalent to restrict ourselves to the problem of n-dimensional varieties $X \subset G(1,n+3)$ that can be projected to $G(1,n+1)$.

Let us just remark that this is the problem analogous to the study of n-dimensional varieties $X \subset \mathbb{P}^{2n+1}$ that can be isomorphically projected to $\mathbb{P}^{2n}$. As we have seen in the previous chapter, this problem is completely solved only in the cases $n=2$ and 3.

In the case of Grassmannians we will give in this chapter a structure theorem for n-dimensional varieties $X \subset G(1,n+3)$ that can be isomorphically projected to $G(1,n+1)$ (Theorem 2.1). To prove the theorem, we will use the fact that, as we have seen in the previous chapter (see Proposition 4.1), the variety X can be isomorphically projected if and only if its secant variety SX is not the whole $G(1,n+3)$. We will focus ourselves on the component $S'X$, i.e. the lines contained in a $\mathbb{P}^3$ spanned by two skew lines of X, characterizing the varieties X for which $S'X$ has dimension smaller than $2n+4$.

In all this chapter we will consider, as usual, only varieties X for which two general lines are skew.

We will start with a first section in which we prove some technical lemmas we are going to use in the second section for the proof of our structure theorem.

1. Some Lemmas

In this section we are going to study the tangent space to the component $S'X$ of the secant variety.

First of all we express the tangent space to $S'X$ at a point corresponding to a line $L \subset H = \langle L_1, L_2\rangle \cong \mathbb{P}^3$, in terms of the tangent space to $S'(X)$ at h and the tangent space to the Grassmannian $G(1,H)$ at l. In this direction, we prove the following

LEMMA 4. *Let $X \subset G(1,N)$ be an n-dimensional variety such that $S'X$ is not the empty set (i.e., the points of X are not the ruling lines of a cone). Let $l_1, l_2 \in X$ be two skew lines, $H \cong \mathbb{P}^3$ their linear span and $L \subset H$ a general line. Then*

$$T_{S'X,l} = T_{G(1,H),l} \oplus \gamma_l(T_{S'(X),h}) \tag{21}$$

where γ_l is the restriction map from $T_{S'(X),h} \subset \mathrm{Hom}(\mathcal{H}, V^*/\mathcal{H})$ to $\mathrm{Hom}(\mathcal{L}, V^*/\mathcal{H})$ (and we chose an arbitrary lifting from this to $\mathrm{Hom}(\mathcal{L}, V^*/\mathcal{L})$).

PROOF. Let us take the incidence diagram 15 of the previous chapter, defining $S'X$. If we differentiate it, we obtain the following

$$\begin{array}{ccc} & T_{J,(l,h)} & \\ \swarrow^{dq_{1,(l,h)}} & & \searrow^{dq_{2,(l,h)}} \\ T_{S'(X),h} & & T_{G(1,N),l}, \end{array} \tag{22}$$

where $T_{J,(l,h)} = \{(\alpha, \beta) \in T_{S'(X),h} \times T_{G(1,N),l} \mid \alpha|_{\mathcal{L}} \equiv \beta \mod \mathcal{H}\}$ (see [**Har95**] for the tangent space to an incidence variety).

The tangent space $T_{S'X,l}$ is the image of $dq_{2,(l,h)}$, differential of q_2 at the point $(l, h) \in J$, thus

$$T_{S'X,l} = \{\beta \in T_{G(1,N),l} \mid \exists \alpha \in T_{S'(X),h}, \text{and}\, \alpha|_{\mathcal{L}} \equiv \beta \mod \mathcal{H}\}.$$

Let us consider the following exact sequence of vector spaces

$$0 \to \mathcal{H}/\mathcal{L} \to V^*/\mathcal{L} \to V^*/\mathcal{H} \to 0,$$

from which

$$0 \to \text{Hom}(\mathcal{L}, \mathcal{H}/\mathcal{L}) \to \text{Hom}(\mathcal{L}, V^*/\mathcal{L}) \to \text{Hom}(\mathcal{L}, V^*/\mathcal{H}) \to 0.$$

Choosing an arbitrary splitting of the above exact sequence, can write

$$\text{Hom}(\mathcal{L}, V^*/\mathcal{L}) \cong \text{Hom}(\mathcal{L}, \mathcal{H}/\mathcal{L}) \oplus \text{Hom}(\mathcal{L}, V^*/\mathcal{H}),$$

or

$$T_{G(1,N),l} \cong T_{G(1,H),l} \oplus \text{Hom}(\mathcal{L}, V^*/\mathcal{H}). \tag{23}$$

Then, if β is an element of $T_{G(1,N),l}$, we can write

$$\beta = \beta' \oplus \beta'',$$

where $\beta' \in T_{G(1,H),l}$ and $\beta'' \in \text{Hom}(\mathcal{L}, V^*/\mathcal{H})$. We have that such β is an element of $T_{S'X,l}$ if and only if there exists $\alpha \in T_{S'(X),h}$ such that $\alpha|_{\mathcal{L}} \equiv (\beta \mod \mathcal{H}) = \beta''$. Then if we introduce the restriction map

$$\begin{aligned} \gamma_l : & T_{S'(X),h} \to \text{Hom}(\mathcal{L}, V^*/\mathcal{H}) \\ & \alpha \mapsto \alpha|_{\mathcal{L}}, \end{aligned}$$

we can say that β is an element of $T_{S'X,l}$ if and only if β'' belongs to the image of γ_l, and hence $T_{S'X,l} = T_{G(1,H),l} \oplus \gamma_l(T_{S'(X),h})$. □

From this lemma we can see that the dimension of $T_{S'X,l}$ depends only on the dimension of $T_{S'(X),h}$ (and then on the defect δ) and on the map γ_l.

We are now going to fix a basis $\{e_0, e_1, \ldots, e_N\}$ for the $(N+1)$-dimensional vector space V^* and then coordinates $(x_0, x_1, \ldots, x_N)$ for the projective space $\mathbb{P}^N = \mathbb{P}(V)$ to describe the variety $S'X$ locally.

Let L_1, L_2 be two general skew lines of X. Since the Grassmannian of lines is a homogeneous variety, we can suppose that

$$L_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

i.e. it is the line $\{x_2 = x_3 = \cdots = x_N = 0\}$ and

$$L_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$$

i.e. the line $\{x_0 = x_1 = x_4 = \cdots = x_N = 0\}$. Let

$$\left\{ \begin{pmatrix} 1 & 0 & a_2 & a_3 & a_4 & \cdots & a_N \\ 0 & 1 & b_2 & b_3 & b_4 & \cdots & b_N \end{pmatrix} \right\}$$

and

$$\left\{ \begin{pmatrix} c_0 & c_1 & 1 & 0 & c_4 & \cdots & c_N \\ d_0 & d_1 & 0 & 1 & d_4 & \cdots & d_N \end{pmatrix} \right\}$$

be two parametrizations of X in two open neighborhoods of l_1 and l_2 respectively,

$$\begin{aligned} a_i &:= a_i(u_1, u_2, \cdots, u_n) \\ b_i &:= b_i(u_1, u_2, \cdots, u_n) \\ c_j &:= c_j(v_1, v_2, \cdots, v_n) \\ d_j &:= d_j(v_1, v_2, \cdots, v_n) \end{aligned}$$

being functions of n parameters, vanishing at the origin. We can then parametrize the tangent spaces to X at the points l_1 and l_2 as

$$T_{X,l_1} = \left\{ \begin{pmatrix} \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \\ \alpha_4 & \beta_4 \\ \vdots & \vdots \\ \alpha_N & \beta_N \end{pmatrix} : \mathcal{L}_1 \to V^*/\mathcal{L}_1 \right\} \subset \mathrm{Hom}(\mathcal{L}_1, V^*/\mathcal{L}_1)$$

and

$$T_{X,l_2} = \left\{ \begin{pmatrix} \gamma_0 & \delta_0 \\ \gamma_1 & \delta_1 \\ \gamma_4 & \delta_4 \\ \vdots & \vdots \\ \gamma_N & \delta_N \end{pmatrix} : \mathcal{L}_2 \to V^*/\mathcal{L}_2 \right\} \subset \mathrm{Hom}(\mathcal{L}_2, V^*/\mathcal{L}_2)$$

where we have set

$$\alpha_i := \sum_{k=1}^{n} \lambda_k a_{i,u_k}|_0, \quad \beta_i := \sum_{k=1}^{n} \lambda_k b_{i,u_k}|_0,$$
$$\gamma_j := \sum_{k=1}^{n} \mu_k c_{j,v_k}|_0, \quad \delta_j := \sum_{k=1}^{n} \mu_k d_{j,v_k}|_0$$

(a subindex u_k or v_k indicates a partial derivative with respect to that variable, and we are evaluating at the origin). Let $H = \langle L_1, L_2 \rangle$ be the $\mathbb{P}^3$ spanned by the two skew lines. We can parametrize $S'(X) \subset G(3, N)$ in a neighborhood of the point h as

$$S'(X) = \left\{ \begin{pmatrix} 1 & 0 & a_2 & a_3 & a_4 & \cdots & a_N \\ 0 & 1 & b_2 & b_3 & b_4 & \cdots & b_N \\ c_0 & c_1 & 1 & 0 & c_4 & \cdots & c_N \\ d_0 & d_1 & 0 & 1 & d_4 & \cdots & d_N \end{pmatrix} \right\}$$

We can change the basis of the 4-dimensional vector spaces, and obtain the equivalent description

$$S'(X) = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & a'_4 & \cdots & a'_N \\ 0 & 1 & 0 & 0 & b'_4 & \cdots & b'_N \\ 0 & 0 & 1 & 0 & c'_4 & \cdots & c'_N \\ 0 & 0 & 0 & 1 & d'_4 & \cdots & d'_N \end{pmatrix} \right\},$$

where the functions a'_i, b'_i, c'_i and d'_i have the same linear terms as the corresponding a_i, b_i, c_i and d_i. We can then express the tangent space $T_{S'(X),h}$ as

$$(24) \qquad T_{S'(X),h} = \left\{ \begin{pmatrix} \alpha_4 & \beta_4 & \gamma_4 & \delta_4 \\ \alpha_5 & \beta_5 & \gamma_5 & \delta_5 \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_N & \beta_N & \gamma_N & \delta_N \end{pmatrix} : \mathcal{H} \to V^*/\mathcal{H} \right\} \subset \mathrm{Hom}(\mathcal{H}, V^*/\mathcal{H})$$

with respect to the bases $\{e_0, e_1, e_2, e_3\}$ and $\{[e_4], [e_5], \dots, [e_N]\}$ of $\mathcal{H}$ and $V^*/\mathcal{H}$ respectively.

We can now prove the following technical lemma.

LEMMA 5. *Let $X \subset G(1,N)$, with $N \leq n+3$, and $Y := \bigcup_{m \in X} M$ the union in $\mathbb{P}^N$ of the lines of X. Let $m \in X$ be a general point and Y_m the image of Y via the projection from the corresponding line, $\pi_m : \mathbb{P}^N \to \mathbb{P}^{N-2}$.*

If the map γ_l of Lemma 4 is not surjective for a general $l \in S'X$, then $\dim Y_m \leq N-3$ *(and in particular it is not bigger than n).*

PROOF. Let us fix the following set of homomorphisms spanning the tangent space $T_{S'(X),h}$,

$$\left\{ \begin{pmatrix} a_{4,u_i} & b_{4,u_i} & 0 & 0 \\ a_{5,u_i} & b_{5,u_i} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{N,u_i} & b_{N,u_i} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & c_{4,v_j} & d_{4,v_j} \\ 0 & 0 & c_{5,v_j} & d_{5,v_j} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & c_{N,v_j} & d_{N,v_j} \end{pmatrix} \right\},$$

with $1 \leq i, j \leq n$ (the partial derivatives are evaluated at the origin). Let us remark that it is a basis of the tangent space in the case $\delta = 0$, while it is just a set of linearly dependent generators if $\delta = 1$.

Concerning the vector space $\mathrm{Hom}(\mathcal{L}, V^*/\mathcal{H})$, we take the canonical base

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Now, taking a general line $L \subset H$ we know that the restriction map

$$\gamma_l : T_{S'(X),h} \subset \mathrm{Hom}(\mathcal{H}, V^*/\mathcal{H}) \to \mathrm{Hom}(\mathcal{L}, V^*/\mathcal{H})$$

is not surjective. In particular, we restrict ourselves to the set of lines in $G(1,H)$ meeting the two lines L_1, L_2 (and inside this closed set, we take the affine open set $\{p_{03} \neq 0\}$). Then

$$L = \begin{pmatrix} 1 & \lambda & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \mu & 1 & 0 & \cdots & 0 \end{pmatrix},$$

and when we restrict the generators of $T_{S'(X),h}$ to the vector space $\mathcal{L} = \langle e_0 + \lambda e_1, \mu e_2 + e_3 \rangle$, corresponding to l, we obtain respectively

$$\begin{pmatrix} a_{4,u_i} & b_{4,u_i} & 0 & 0 \\ a_{5,u_i} & b_{5,u_i} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{N,u_i} & b_{N,u_i} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \lambda & 0 \\ 0 & \mu \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{4,u_i} + \lambda b_{4,u_i} & 0 \\ a_{5,u_i} + \lambda b_{5,u_i} & 0 \\ \vdots & \vdots \\ a_{N,u_i} + \lambda b_{N,u_i} & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 & c_{4,v_j} & d_{4,v_j} \\ 0 & 0 & c_{5,v_j} & d_{5,v_j} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & c_{N,v_j} & d_{N,v_j} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \lambda & 0 \\ 0 & \mu \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \mu c_{4,v_j} + d_{4,v_j} \\ 0 & \mu c_{5,v_j} + d_{4,v_j} \\ \vdots & \vdots \\ 0 & \mu c_{N,v_j} + d_{N,v_j} \end{pmatrix}.$$

Thus the $2n \times 2(N-3)$ matrix of the map γ_l is the following

$$M_l = \begin{pmatrix} A_\lambda & 0 \\ 0 & B_\mu \end{pmatrix},$$

where A_λ and B_μ are the $n \times (N-3)$ matrices

$$A_\lambda = \begin{pmatrix} a_{4,u_1} + \lambda b_{4,u_1} & \cdots & a_{4,u_n} + \lambda b_{4,u_n} \\ \vdots & & \vdots \\ a_{N,u_1} + \lambda b_{N,u_1} & \cdots & a_{N,u_n} + \lambda b_{N,u_n} \end{pmatrix}$$

$$B_\mu = \begin{pmatrix} \mu c_{4,v_1} + d_{4,v_1} & \cdots & \mu c_{4,v_n} + d_{4,v_n} \\ \vdots & & \vdots \\ \mu c_{N,v_1} + d_{N,v_1} & \cdots & \mu c_{N,v_n} + d_{N,v_n} \end{pmatrix}.$$

If γ_l is not surjective, then $\operatorname{rank} M_l < 2(N-3)$ when l varies in the intersection $\Omega(L_1, H) \cap \Omega(L_2, H)$ (i.e. $\forall(\lambda, \mu) \in \mathbb{C}^2$), which implies

$$\begin{array}{c} \operatorname{rank} A_\lambda < N-3 \\ \text{or} \\ \operatorname{rank} B_\mu < N-3. \end{array} \tag{25}$$

Then for each pair $(\lambda, \mu) \in \mathbb{C}^2$, it happens that $\operatorname{rank} A_\lambda < N-3$ or $\operatorname{rank} B_\mu < N-3$. Let us set $Z_A = \{(\lambda, \mu) \in \mathbb{C}^2 \mid \operatorname{rank} A_\lambda < N-3\}$ and $Z_B = \{(\lambda, \mu) \in \mathbb{C}^2 \mid \operatorname{rank} B_\mu < N-3\}$. Thus $Z_A \cup Z_B = \mathbb{C}^2$. But Z_A and Z_B are closed subsets of $\mathbb{C}^2$, which implies $Z_A = \mathbb{C}^2$ or $Z_B = \mathbb{C}^2$.

Let us suppose $Z_A = \mathbb{C}^2$, which is equivalent to say that $\operatorname{rank} A_\lambda < N-3$, $\forall(\lambda, \mu) \in \mathbb{C}^2$.

In this case we call $\pi_{l_2} : \mathbb{P}^N \to \mathbb{P}^{N-2}$ the projection with center $L_2 \subset Y$, $\pi_2 : G(1, N) \to G(1, N-2)$ the corresponding projection of Grassmannians, $X_2 \subset G(1, N-2)$ the image of X under π_2 and $Y_2 = \bigcup_{l \in X_2} L = \pi_{l_2}(Y) \subset \mathbb{P}^{N-2}$. Let us take the incidence variety $I_2 = \{(l, y) \subset X_2 \times \mathbb{P}^{N-2} \mid y \in L\}$ and the diagram

$$\begin{array}{ccc} & I_2 & \\ {\scriptstyle p_{2,1}} \swarrow & & \searrow {\scriptstyle p_{2,2}} \\ X_2 & & \mathbb{P}^{N-2}. \end{array} \tag{26}$$

We put $l_1'' = \pi_2(l_1) \in X_2$, and with these definitions we prove the following

CLAIM. $\operatorname{rank} A_\lambda < N-3$, $\forall \lambda \in \mathbb{C}$, *is equivalent to say that the differential* $d_{(l_1'', y)} p_{2,2}$ *is not surjective* $\forall y \in L_1''$.

Indeed we can write

$$L_1'' = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

because projecting from L_2 is equivalent to forget the second and third coordinates. In particular, X_2 in a neighborhood of l_1'' can be parametrized as

$$X_2 = \left\{ \begin{pmatrix} 1 & 0 & a_4 & a_5 & \cdots & a_N \\ 0 & 1 & b_4 & b_5 & \cdots & b_N \end{pmatrix} \right\},$$

and Y_2, in the open set $\{x_0 \neq 0\}$,

$$Y_2 = \{(1 : \lambda : a_4 + \lambda b_4 : a_5 + \lambda b_5 : \ldots : a_N + \lambda b_N)\}.$$

Then $p_{2,2}$ in coordinates can be described as

$$(\lambda, u_1, \ldots, u_n) \mapsto (\lambda, a_4 + \lambda b_4, \ldots, a_N + \lambda b_N)$$

and its differential at a point $(l_1'', y) \in I^2$ is given by the $(n+1) \times (N-2)$ matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & a_{4,u_1} + \lambda b_{4,u_1} & \cdots & a_{N,u_1} + \lambda b_{N,u_1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{4,u_n} + \lambda b_{4,u_n} & \cdots & a_{N,u_n} + \lambda b_{N,u_n} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & A_\lambda \end{pmatrix}$$

which, under our hypothesis has rank smaller than $N-2$, $\forall \lambda \in \mathbb{C}$. Then the differential of $p_{2,2}$ is not surjective at the points (l_1'', y), $\forall y \in L_1''$. This ends the proof of the claim.

In the same way, if $\operatorname{rank} B_\mu < n$, $\forall \mu \in \mathbb{C}$, we can repeat the same argument taking the projection $\pi_1 : G(1,N) \to G(1,N-2)$ induced by π_{l_1}. Then, if we denote by $X_1 \subset G(1,N-2)$ the image of X and $I_1 = \{(l,y) \in X_1 \times \mathbb{P}^{N-2} \mid y \in L\}$, and take the incidence diagram

$$\begin{array}{ccc} & I_1 & \\ {}^{p_{1,1}}\swarrow & & \searrow^{p_{1,2}} \\ X_1 & & \mathbb{P}^{N-2}, \end{array}$$

it happens that the differential of $p_{1,2}$ is not surjective at the points $(l_2' := \pi_1(l_2), y)$, $\forall y \in L_2'$.

Let us recall that we set $Z = \{(l_1, l_2) \in X \times X \mid L_1 \cap L_2 \neq \emptyset\}$. Then we have proved that every time we fix a pair of lines $(l_1, l_2) \in X \times X \setminus Z$, it happens that either the differential $d_{(l_2', y)} p_{1,2}$ is not surjective $\forall y \in L_2'$, or the differential $d_{(l_1'', y)} p_{2,2}$ is not surjective $\forall y \in L_1''$.

In fact we have the following

CLAIM. *Neither the differential $d_{(l_2', y)} p_{1,2}$ nor $d_{(l_1'', y)} p_{2,2}$ is surjective.*

Indeed, let us set

$$W_1 := \{(l_1, l_2) \in (X \times X) \setminus Z \mid d_{(l_2', y)} p_{1,2} \text{ is not surjective}\},$$
$$W_2 := \{(l_1, l_2) \in (X \times X) \setminus Z \mid d_{(l_1'', y)} p_{2,2} \text{ is not surjective}\}.$$

We have that W_1 and W_2 are two closed subset of $(X \times X) \setminus Z$, which is irreducible. Since at least one of the two differentials is not surjective, we have that $W_1 \cup W_2 = (X \times X) \setminus Z$. Moreover W_1 and W_2 are naturally isomorphic (it is enough to take the restriction of the automorphism of $X \times X$, $(l_1, l_2) \mapsto (l_2, l_1)$) and in particular not empty. Then $W_1 = W_2 = (X \times X) \setminus Z$ which implies that in fact every time we fix a pair of lines in $(X \times X) \setminus Z$, the two differentials we have introduced are not surjective. This proves the claim.

Summarizing, we have that given a general line $m \in X$, if we consider the projection $\pi_m : X \to X_m \subset G(1, N-2)$, the incidence variety $I_m = \{(l, y) \in X_m \times \mathbb{P}^{N-2} \mid y \in L\}$, and the diagram

$$\begin{array}{ccc} & I_m & \\ {}^{p_{m,1}}\swarrow & & \searrow^{p_{m,2}} \\ X_m & & \mathbb{P}^{N-2}, \end{array}$$

we have that the differential $d_{(l,y)}p_{m,2}$ is not surjective for a general line $l \in X$ and $\forall y \in L$. But this can happen only if the image $Y_m = p_{m,2}(I_m)$ is not the whole $\mathbb{P}^{N-2}$. Hence, projecting the variety $Y \subset \mathbb{P}^N$ from a general internal line corresponding to a point $m \in X$, we obtain Y_m of dimension less than or equal to $N - 3$, which completes the proof. $\square$

REMARK. We have proved in the previous lemma that when the restriction map γ_l is not surjective for a general $l \in S'X$, then the image Y_m of Y by the projection $\pi_m : \mathbb{P}^N \to \mathbb{P}^{N-2}$ has dimension smaller than $n+1$. Then, if the variety X is uncompressed (i.e. $\dim Y = n + 1$), we have that the dimension of Y gets smaller when we project. If X has positive secant defect, then it is easy to see why the dimension of Y_m is smaller than the dimension of Y.

Indeed let us take the corresponding projection $\pi'_m : G(1, N) \to G(1, N-2)$, and denote by $X_m \subset G(1, N-2)$ the image of the variety X. It turns out that Y_m is the union in $\mathbb{P}^{N-2}$ of the lines of X_m. If X has positive defect δ (hence, for Lemma 2, $\delta = 1$), we have that in a $\mathbb{P}^3$ spanned by M and another skew line, there are infinitely many lines of X. When projecting X from M, all these lines have the same image in $G(1, N-2)$, and hence the variety X_m has dimension $n - 1$. In particular $\dim Y_m \leq n$.

In what follows, we are going to focus our attention on the case $N = n + 3$, i.e. on n-dimensional varieties $X \subset G(1, n+3)$.

LEMMA 6. *Let $X \subset G(1, n+3)$ a nondegenerate, uncompressed subvariety of dimension n, $\pi_m : \mathbb{P}^{n+3} \to \mathbb{P}^{n+1}$ the projection from a general line $m \in X$, Y_m the image of Y by π_m.*
If for a general $m \in X$ we have that $\dim Y_m \leq n$, then Y is a scroll of $\mathbb{P}^n$'s.

PROOF. For simplicity of notation we write π instead of $\pi_m|_Y$, the restriction to Y of the projection $\pi_m : \mathbb{P}^{n+3} \to \mathbb{P}^{n+1}$ (neither π_m nor π is defined on the points of M);

$$\begin{array}{ccc} Y & \hookrightarrow & \mathbb{P}^{n+3} \\ \downarrow \pi & & \downarrow \pi_m \\ Y' & \hookrightarrow & \mathbb{P}^{n+1}. \end{array}$$

Given a point $p \in \mathbb{P}^{n+1}$, the fiber of the projection is $\pi_m^{-1}(p) = \langle M, p \rangle \setminus M$ and then, given $y' \in Y'$, $\pi^{-1}(y') = (\langle M, y' \rangle \cap Y) \setminus M$. If $\dim Y' = n$, the fiber of π must be 1-dimensional. Let us consider the map

$$g : Y \setminus M \to G(2, n+3)$$

which associates the plane $\langle M, y\rangle$ to each $y \in Y \setminus M$. Let us denote by Γ the image of g in $G(2, n+3)$ (Γ is contained in the Schubert variety $\Omega(0, M, \mathbb{P}^{n+3})$, i.e. the variety of planes containing the line M). Given a general plane $\Pi_y = \langle M, y\rangle$, corresponding to $\pi_y \in \Gamma$, since $g^{-1}(\pi_y)$ coincides with the fiber of the projection π, we have $\dim(g^{-1}(\pi_y)) = 1$. Then, since $\dim(Y \setminus M) = n+1$, it will be $\dim \Gamma = n$, i.e. inside the variety $\Omega(0, M, \mathbb{P}^{n+3})$, which has dimension $n+1$, there exist an n-dimensional family of planes cutting $Y \setminus M$ along a curve. And this is true for a general line $m \in X$. Therefore we have proved that given a general $m \in X$, there exist an n-dimensional family $\mathcal{F}_m$ of curves meeting M and covering the whole Y (because they are the fibers of the projection π).

CLAIM. *The curves of $\mathcal{F}_m$ are lines.*

Indeed, let d be the degree of $Y \subset \mathbb{P}^{n+3}$. In general, when we project a variety $W \subset \mathbb{P}^{n+3}$, of degree d from a point $w \in W$ to a $\mathbb{P}^{n+2}$, if the image W_1 has the same dimension of W, then its degree is $d-1$; if $\dim W_1 = \dim W - 1$, then $\deg W_1 = \deg W$, because W is a cone over W_1, with vertex w.

Now, in our case, projecting from a line $M \subset Y$ is the same as projecting first from a point $y_1 \in M$ and then from the image of M by this first projection (which is another point y_2). Since the image of Y via the two projections is Y', of dimension $n = \dim Y - 1$, then $\deg Y' = d-1$.

Let $L \cong \mathbb{P}^1 \subset \mathbb{P}^{n+1}$ be a general line in $\mathbb{P}^{n+1}$, and $\{y_1, y_2, \dots, y_{d-1}\} = L \cap Y'$ the intersection points of Y' and L. We have that $\pi_m^{-1}(l) = \langle M, L\rangle \setminus M = H_l \setminus M$, where $H_l \cong \mathbb{P}^3$. Then $H_l \cap Y$ is a curve C of degree d. But

$$\begin{aligned} H_l \cap Y &= M \cup \pi_m^{-1}(L \cap Y') \\ &= M \cup \bigcup_{i=1}^{d-1} \pi^{-1}(y_i) \\ &= M \cup \bigcup_{i=1}^{d-1} C_i \end{aligned}$$

and then $\deg C_i = 1$, for $1 \le i \le d-1$. But $C_i = \pi^{-1}(y_i)$ are the fibers of the projection π, i.e. the curves we were considering. This ends the proof of the claim.

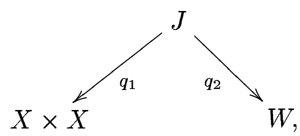
We have then proved that, given a general line $m \in X$, there exist an n-dimensional family $\mathcal{F}_m \subset \Omega(M, \mathbb{P}^{n+3}) \subset G(1, n+3)$, of lines meeting M and such that their union is the whole Y.

Let us consider the incidence set $I := \{(m, r) \in X \times G(1, n+3) \mid r \in \mathcal{F}_m\}$, and the corresponding diagram

$$\begin{array}{ccc} & I & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ X & & W, \end{array}$$

where we denote by W the image of the second projection inside $G(1, n+3)$. For a general line $m \in X$, the fiber of p_1 is isomorphic to the family $\mathcal{F}_m$, and then $\dim I = 2n$. Let us study the fiber of the second projection. There are three possible cases.

1. $\dim(p_2^{-1}(r)) = 0$.
   Then $\dim W = 2n$, and Y is an $(n+1)$-dimensional variety containing a $2n$-dimensional family of lines, which means that Y is linear. This is impossible, because X is nondegenerate.
2. $\dim(p_2^{-1}(r)) = \beta \geq 2$.
   Then $\dim W = 2n - \beta$. Let us consider the new incidence diagram



   where we set $J := \{(m_1, m_2, r) \in X \times X \times G(1, n+3) \mid r \in \mathcal{F}_{m_1} \cap \mathcal{F}_{m_2}\}$. Clearly $\dim(q_2^{-1}(r)) = 2\dim(p_2^{-1}(r)) = 2\beta$, which implies $\dim J = 2n + \beta$ and $\dim(q_1^{-1}(m)) = \beta \geq 2$. But then, given two general points $m_1,\ m_2 \in X$, all the lines joining M_1 and M_2 belong to W and hence they are contained in Y. This means that the linear span of M_1 and M_2 is contained in Y. Therefore, as before, Y is linear.
3. $\dim(p_2^{-1}(r)) = 1$.
   In this case, $\dim W = 2n - 1$. Then, from the classification of B.Segre and Rogora, (see Theorem 2.2, Chapter 1), Y must be one of the following:
   i) $Y \cong \mathbb{P}^{n+1}$;
   ii) Y is a quadric in $\mathbb{P}^{n+2}$;
   iii) Y is a scroll of $\mathbb{P}^n$'s.
   Since we are supposing X nondegenerate, the only possible case is the last one, i.e. Y is the union of the $\mathbb{P}^n$'s of a 1-dimensional family corresponding to a curve $C \subset G(n, n+3)$, such that the general point $y \in Y$ is contained in one and only one $\mathbb{P}^n$ of C.

□

EXAMPLE 15. Let us analyze the concrete example of the Veronese surface $X \subset G(1,5)$. In this case we know that the defect is $\delta = 1$, and hence when we project from a general line $m \in X$ we have that the image $X_m \subset G(1,3)$ has dimension 1, which implies $\dim Y_m \leq 2$. Thus we are in the hypothesis of Lemma 6.

We have seen in Example 5 that X can be viewed as the set of lines in $Y = \mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$, transversal to the 1-dimensional family of planes contained in Y. Therefore Y is clearly a scroll of planes. We call $C \subset G(2,5)$ the curve corresponding to the 1-dimensional family of planes. A general line $m \in X$ intersects every plane of C in a point. Hence the 2-dimensional family of lines $\mathcal{F}_m$ we introduced in the proof of Lemma 6 is just the union of the pencils of lines contained in a plane $\pi \in C$ and passing through the point $M \cap \Pi$.

2. The Structure Theorem

In this last section, with the help of the above technical lemmas, we are going to prove the following theorem giving some structure conditions for the n-dimensional varieties which can be projected from $G(1, n+3)$ to $G(1, n+1)$.

THEOREM 2.1. *Let $X \subset G(1, n+3)$ be a smooth, nondegenerate and uncompressed subvariety of dimension n that can be isomorphically projected to $G(1, n+1)$.*

Then Y is the union of the $\mathbb{P}^n$'s of a 1-dimensional family corresponding to a curve $C \subset G(n, n+3)$ and one of the following holds:

1. *C is rational, $Y = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus n-r+1}\right)$, with $r \geq 3$, and the lines of X correspond to sections of this projective bundle;*
2. *all the $\mathbb{P}^n$'s of the family intersect in a linear space $K \cong \mathbb{P}^k$, with $1 \leq k \leq n-1$, all the lines of X are contained in these $\mathbb{P}^n$'s and meet the space K.*

PROOF. Since X can be isomorphically projected to $G(1, n+1)$, we know that its secant variety SX is not the whole $G(1, n+3)$. In particular, $\dim S'X < 2n+4$, which implies that, for a general $l \in S'X$ we have $\dim T_{S'X,l} < 2n+4$. We have seen in Lemma 4 that

$$T_{S'X,l} = T_{G(1,H),l} \oplus \gamma_l(T_{S'(X),h}),$$

where H is a $\mathbb{P}^3$ spanned by two skew lines of X, and L a general line in H. Since $\dim T_{G(1,H),l} = 4$, it must be $\dim \gamma_l(T_{S'(X),h}) < 2n$, which is equivalent to say that the map $\gamma_l : T_{S'(X),h} \to \mathrm{Hom}(\mathcal{L}, V^*/\mathcal{H})$ is not surjective for a general $l \in S'X$. We can then apply Lemma 5 and Lemma 6 and find that $Y \subset \mathbb{P}^{n+3}$ is the union of a 1-dimensional family of $\mathbb{P}^n$'s, such that a general point $y \in Y$ is contained in one and only one $\mathbb{P}^n$ of the family.

Let $m \in X$ be a general line. In the proof of Lemma 6 we have seen that there exist an n-dimensional family $\mathcal{F}_m$ of lines meeting M and such that their union $\bigcup_{l \in \mathcal{F}_m} L$ is the whole Y. This can happen only if the line M meets all the $\mathbb{P}^n$'s of the family. Moreover a general point $y \in M$ is a general point on Y, and then there exists only one $\mathbb{P}^n$ containing y. We can then construct a rational map from $M \cong \mathbb{P}^1$ to the Grassmannian $G(n, n+3)$, sending a point $y \in M$ to the $\mathbb{P}^n$ of the family containing y. This map can be extended to a morphism

$$\varphi_m : \mathbb{P}^1 \longrightarrow G(n, n+3).$$

We study separately two possible cases.

1. *The morphism φ_m is not constant.*
   Then the image of φ_m is the curve $C \subset G(n, n+3)$ which is therefore rational. Hence there exists a rank-$(n+1)$ vector bundle $\mathcal{E}$ on $\mathbb{P}^1$ such that the $(n+1)$-dimensional variety Y is the projective bundle $\mathbb{P}(\mathcal{E})$. Since vector bundles on $\mathbb{P}^1$ are decomposable, we can write

$$\mathcal{E} = \bigoplus_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}^1}(a_i),$$

   with $a_i \geq 0$.

   CLAIM. *The integers a_i are not bigger than 1, for each $i = 1, \ldots, n+1$.*

   A general line $m \in X$ meets all the $\mathbb{P}^n$'s of the family and through each point there passes exactly one $\mathbb{P}^n$, then the lines of X are sections of $\mathbb{P}(\mathcal{E})$. Therefore the line M gives rise to an embedding

$$i_m : M \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1)) \hookrightarrow \mathbb{P}(\mathcal{E}),$$

   or a surjective map between vector bundles

$$\psi_m : \mathcal{E} \to \mathcal{O}_{\mathbb{P}^1}(1).$$

Now, let us suppose that for some indices $i = 1, \ldots, s$ we have $a_i \geq 2$, and write

$$\mathcal{E}' = \bigoplus_{i=1}^{s} \mathcal{O}_{\mathbb{P}^1}(a_i),$$
$$\mathcal{E}'' = \bigoplus_{j=s+1}^{n+1} \mathcal{O}_{\mathbb{P}^1}(a_j).$$

Then the map ψ_m restricted to $\mathcal{E}'$ is the zero map (the only morphism from $\mathcal{O}_{\mathbb{P}^1}(a)$ to $\mathcal{O}_{\mathbb{P}^1}(1)$, with $a \geq 2$ is the zero morphism), and then, going back to the map i_m, its image is contained in $Y'' := \mathbb{P}(\mathcal{E}'')$. This means that the general line of X is contained in Y'', which has dimension $n+1-s$, and since we are supposing that the union of lines of X has dimension $n+1$, it must be $s = 0$. This ends the proof of the claim.

Then we can write

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus n+1-r},$$

from which follows $h^0(\mathbb{P}^1, \mathcal{E}) = n+1+r$. Finally, since X is nondegenerate it must be $n+1+r \geq n+4$, which is equivalent to $r \geq 3$.
These are varieties corresponding to case 1 of the statement.

2. *The morphism φ_m is constant.*
In this case the line M is contained in one of the $\mathbb{P}^n$'s of the family. But, since M meets all the $\mathbb{P}^n$'s, there must exist a point $p \in M$ contained in every $\mathbb{P}^n$. The union of all these points, as m varies on X, is contained in the linear space $K \cong \mathbb{P}^k$, intersection of all the $\mathbb{P}^n$'s (then in particular $k \leq n-1$). Moreover, if $k = 0$, we would have that all the lines of X pass through a point, and this means that there exists a $2n$-dimensional family of bad planes, whose union fills up the whole $\mathbb{P}^{n+3}$. Then X could not be projected to $G(1, n+2)$ (and in particular to $G(1, n+1)$), which is a contradiction. Then it must be $1 \leq k \leq n-1$, and hence we are in case 2 of the statement.

□

CHAPTER 4

Compressed Varieties

We have already studied the varieties that can be isomorphically projected from $G(1, n+3)$ to $G(1, n+1)$. The latter is the first Grassmannian in which we do not expect to be able to project n-dimensional varieties. In the first section we want to study when it is possible to project into some smaller Grassmannian. Specifically, we are going to investigate varieties that can be projected to $G(1, n)$.

Clearly, if $X \subset G(1, N)$ is an n-dimensional variety that can be isomorphically projected to $G(1, n)$, then there exists an $(N-n-1)$-dimensional linear space which does not intersect the variety Y. Hence $\dim Y \leq n$, which is equivalent to say that X is compressed. In this chapter we first study compressed varieties of dimension n that can be projected to $G(1, n)$, for n small.

First of all, for $n = 2$ the problem is trivial, because if X is a surface in $G(1, 2)$, X can only be the whole Grassmannian, and in particular it is a β-plane. It cannot be projected from a bigger Grassmannian.

In the case $n = 3$, let $X \subset G = G(1, 3)$ be a 3-dimensional variety. Then X is a hypersurface and $[X] = d\Omega(1, 3) \in A^1(G)$. In particular the ideal sheaf of X is $\mathcal{I}_X = \mathcal{O}_G(-d)$, and hence, tensoring by $\mathcal{Q}$ and taking cohomology we deduce that $H^0(G, \mathcal{I}_X \otimes \mathcal{Q}) = H^0(G, \mathcal{Q}(-d)) = 0$, because $\mathcal{Q}$ has no intermediate cohomology. Therefore X is linearly normal.

So we will study the case $n = 4$. In particular, in the first section we construct a four-fold $X' \subset G(1, 4)$, with a singular point, which is an extension of the congruence of bidegree $(2, 1)$ given in Example 3 and is projected from $G(1, 5)$.

Then in the second section we prove that if $X' \subset G(1, 4)$ is projected from $G(1, 5)$ then X' must be an extension of the $(2, 1)$ congruence.

In the last section we give a generalization of the structure theorem for compressed varieties.

1. A Non Linearly Normal Four-fold in $G(1, 4)$

Let s_1 and s_2 be two general sections of the rank-3 vector bundle $\bigwedge^2 \mathcal{S} = \mathcal{S}^*(1)$ on $G = G(1, 4)$. Since $\bigwedge^2 \mathcal{S}$ is generated by its sections, we can apply Porteous formula (see the appendix to the first chapter) and hence the dependency locus X' of s_1 and s_2 has the expected codimension 2, i.e. X' is a four-fold of $G(1, 4)$, and its class is $[X'] = c_2(\mathcal{S}^*(1))$. Moreover its singular locus has codimension 6, which means that there exist finitely many singular points. We can also calculate the class of this finite set, $[\mathrm{Sing}\, X'] = c_3(\mathcal{S}^*(1))^2$ (in fact $\mathrm{Sing}\, X'$ is the intersection of the zero locus of s_1 and s_2). We know that the Chern classes of $\mathcal{S}$ are the special Schubert cycles, namely $c_i(\mathcal{S}) = \sigma_i$, $i = 1, 2, 3$. Therefore, using the splitting principle (see for instance [**Har77**]) we can easily compute

$$c_1(\mathcal{S}^*(1)) = 2\sigma_1,$$
$$c_2(\mathcal{S}^*(1)) = \sigma_1^2 + \sigma_2,$$
$$c_3(\mathcal{S}^*(1)) = \sigma_1\sigma_2 - \sigma_3.$$

From which we obtain that $[X'] = \sigma_1^2 + \sigma_2 = 2\Omega(1,4) + \Omega(2,3)$ and $[\operatorname{Sing} X'] = (\sigma_1\sigma_2 - \sigma_3)^2 = \Omega(0,1)$, the class of one point.

From the description of X' as the dependency locus of two sections of $\mathcal{S}^*(1)$ we can find the Koszul exact sequence

$$0 \to \mathcal{O}_G^{\oplus 2} \to \mathcal{S}^*(1) \to \mathcal{I}_{X'}(2) \to 0,$$

and, tensoring by $\mathcal{Q} \otimes \mathcal{O}_G(-2)$ we obtain

$$0 \to \mathcal{Q}(-2)^{\oplus 2} \to \mathcal{S}^* \otimes \mathcal{Q}^* \to \mathcal{I}_{X'} \otimes \mathcal{Q} \to 0,$$

because $\mathcal{Q}(-1) = \mathcal{Q}^*$. Since $\mathcal{Q}$ has no intermediate cohomology we can write

$$H^1(G, \mathcal{I}_{X'} \otimes \mathcal{Q}) \cong H^1(G, \mathcal{S}^* \otimes \mathcal{Q}^*).$$

But this cohomology group is known to be one dimensional. In fact $H^1(G, \mathcal{S}^* \otimes \mathcal{Q}^*) = \operatorname{Ext}^1(\mathcal{S}^*, \mathcal{Q})$ and the non-trivial extension corresponds to the universal exact sequence on $G(1,4)$

$$0 \to \mathcal{S}^* \to V \otimes \mathcal{O}_G \to \mathcal{Q} \to 0.$$

Hence $X' \subset G(1,4)$ is the projection of a four-fold $X \subset G(1,5)$.

Moreover, if we take a general hyperplane $H \subset \mathbb{P}^4$ and intersect the variety X' with the Schubert variety $\Omega(2,H) = G(1,H)$, we obtain a smooth surface $S' \subset G(1,3)$ (it is smooth because the singular locus of X' has dimension at most 1). We have seen in Chapter 1 that if we denote by $\bar{\mathcal{S}}$ the universal subbundle of the Grassmannian $G(1,3) = G(1,H)$, then the restriction to $G(1,H)$ of the subbundle $\mathcal{S}$ is given by $\mathcal{S}|_{G(1,3)} = \bar{\mathcal{S}} \oplus \mathcal{O}_{G(1,3)}$. Hence

$$\mathcal{S}^*(1)|_{G(1,3)} = \bar{\mathcal{S}}^*(1) \oplus \mathcal{O}_{G(1,3)}(1) = \bar{\mathcal{S}} \oplus \mathcal{O}_{G(1,3)}(1),$$

which means that X' is an extension of the surface $S' \subset G(1,3)$ given by the dependency locus of two sections of $\bar{\mathcal{S}} \oplus \mathcal{O}_{G(1,3)}(1)$. This is the $(2,1)$ congruence of Example 3.

We are going to see in the next section that in fact, if a four-fold $X' \subset G(1,4)$ is the projection of $X \subset G(1,5)$, then it must be an extension of the $(2,1)$ congruence.

2. On Four-folds Projected from $G(1,5)$ to $G(1,4)$

Let $X \subset G(1,5)$ be a 4-dimensional nondegenerate subvariety with singular locus of dimension at most 1. Since $\dim G(1,5) = 8$, the class $[X]$ belongs to the group $A^4(G(1,5))$ (the Chow group of codimension-4 cycles), generated by the Schubert cycles $\Omega(0,5)$, $\Omega(1,4)$ and $\Omega(2,3)$. We can then write

$$[X] = d_1\Omega(1,4) + d_2\Omega(2,3) + d_3\Omega(0,5),$$

and hence Schubert calculus provides $d_1 = [X] \cdot \Omega(1,4)$, $d_2 = [X] \cdot \Omega(2,3)$, $d_3 = [X] \cdot \Omega(0,5)$. Therefore d_1 is the number of lines of X contained in a $\mathbb{P}^4$ and intersecting a line in this $\mathbb{P}^4$, d_2 is the number of lines of X contained in a $\mathbb{P}^3$ and d_3 the number of lines of X passing through a general point of $\mathbb{P}^5$. If X can be projected to $X' \subset G(1,4)$, then through a general point in $\mathbb{P}^5$ does not pass any line of X, which is equivalent to $d_3 = 0$.

Let us fix a general hyperplane $H \cong \mathbb{P}^4 \subset \mathbb{P}^5$ and take the lines contained in H, i.e. the codimension-2 Schubert variety $\Omega(3,H) \cong G(1,4) \subset G(1,5)$. If we cut X with this variety we obtain a smooth surface (because we are supposing that the singular locus of X has dimension at most one) in $G(1,4)$. Moreover the class of S in the Chow ring $A^4(G(1,4))$ is given by

$$[S] = d_1\Omega(0,3) + d_2\Omega(1,2).$$

REMARK. In the case of the projective space, when we take an n-dimensional variety $X \subset \mathbb{P}^N$, and we call X_H its intersection with a general hyperplane $H \cong \mathbb{P}^{N-1}$, we have that if X is nondegenerate, then also X_H is nondegenerate in H.

Let us take now an n-dimensional nondegenerate variety $X \subset G(1,N)$ and a general hyperplane H. We denote by X_H the intersection of X with the Grassmannian of lines in H, i.e. $X_H = X \cap \Omega(N-2,H) = X \cap G(1,H)$. In this case, even if X is nondegenerate, it can happen that X_H is degenerate in $G(1,H)$. In fact the sequence defining X_H as a subvariety of X is the following

$$0 \to \mathcal{O}_X \to \mathcal{Q}_X \to \mathcal{O}_X(1) \to \mathcal{O}_{X_H}(1) \to 0,$$

and hence, tensoring by $\mathcal{Q}_X(-1)$ and taking cohomology we find that the kernel of the map $H^0(X,\mathcal{Q}_X) \to H^0(X_H,\mathcal{Q}_{X_H})$ can have dimension bigger than 1. This is equivalent to say that X_H can be degenerate in $G(1,H)$.

EXAMPLE 16. Let us take the Veronese three-fold $X \subset G(1,7)$, i.e. the embedding of $\mathbb{P}^3$ given by the rank-2 vector bundle $\mathcal{Q}|_{\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(1)$. As we have seen in Example 5, we can give the following description of X. We take two disjoint $\mathbb{P}^3$'s, say H_1 and H_2 in $\mathbb{P}^7$, and an isomorphism $\phi : H_1 \to H_2$. Then X is the set of lines joining a point $p \in H_1$ with $\phi(p) \in H_2$. Now, if we take a hyperplane $H \cong \mathbb{P}^6$ and intersect X with the Schubert cycle $\Omega(5,H) = G(1,H)$, we have the following situation. The hyperplane H meets H_1 and H_2 in the planes Π_1 and Π_2 respectively. Let us call $\Pi_1' \subset H_2$ the image of Π_1 by the isomorphism ϕ. The plane Π_1' intersects Π_2 in a line L_2, which is the image by ϕ of a line $L_1 \subset \Pi_1$. Hence the points of X contained in H correspond to the lines joining a point $p \in L_1$ with $\phi(p) \in L_2$, which is equivalent to say that X_H is contained in the Grassmannian of lines in $\langle L_1, L_2\rangle \cong \mathbb{P}^3$. Therefore X_H is very degenerate (it is contained in a $G(1,3)$). This is because $H^0(\mathcal{Q}|_{\mathbb{P}^3} \otimes \mathcal{Q}|_{\mathbb{P}^3}(-1))$ is four-dimensional.

We can however prove the following lemma, stating that, in the case we are interested in, X_H cannot be degenerate.

LEMMA 7. *Let $X \subset G(1,5)$ be a nondegenerate compressed variety of dimension 4. Then the intersection of X and the Schubert variety of lines contained in a general hyperplane $H \cong \mathbb{P}^4 \subset \mathbb{P}^5$ is nondegenerate in $G(1,H)$.*

PROOF. As usual we denote by $Y \subset \mathbb{P}^5$ and by $Y_H \subset H$ the union of the lines of X and $S := X \cap G(1,H)$ respectively. Since X is compressed, we have $\dim Y \leq 4$, but if this dimension were strictly smaller than 4, Y would be linear, and hence X degenerate. Therefore the dimension of Y is 4. Let us suppose that there exists a linear space $H' \cong \mathbb{P}^3 \subset H$ such that S is contained in the Grassmannian $G(1,H')$. Then, either the lines of S fill up the whole H', or they are contained in a plane. This is equivalent to $Y_H = H'$ or $Y_H \cong \mathbb{P}^2$. Since the variety Y_H is clearly the intersection of Y and the hyperplane H, in the first case we would have that when we intersect Y with a general hyperplane, we obtain a $\mathbb{P}^3$, and hence Y would be a $\mathbb{P}^4$. This is a contradiction, because X is nondegenerate.

Finally, if Y_H were a plane, then Y would have dimension 3, again a contradiction. $\square$

Then, if X is a nondegenerate four-fold in $G(1,5)$ that can be projected to $G(1,4)$, when we intersect X with the variety $G(1,H)$, we obtain a nondegenerate surface $S \subset G(1,H) = G(1,4)$ that can be isomorphically projected to $G(1,3)$.

There exists a classification of smooth congruences $S \subset G(1,4)$ which can be projected to $S' \subset G(1,3)$ (see [**AS92**]). Let us give the list, recalling some properties of these congruences that we are going to use in the remainder of this chapter.

1. S is $\mathbb{P}^2$ blown up in a point p and embedded by the vector bundle $\mathcal{Q}|_S = \mathcal{O}_S(L) \oplus \mathcal{O}_S(L-E)$, where L is the pull-back of a line and E is the exceptional divisor. Then, as we have seen in Chapter 1, S is the congruence of bidegree $(2,1)$ given in Example 3. In particular there exist a 1-dimensional family of planes containing infinitely many lines of S. They are the planes spanned by a point $y \in \mathbb{P}^4$ and the lines of one of the two rulings of a quadric. Hence the family is defined by the morphism $\mathbb{P}^1 \to G(2,4)$ given by the rank-3 vector bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.
2. S is a rational normal scroll of bidegree $(2,2)$. Also in this case there exist a 1-dimensional family of planes each of them containing infinitely many lines of S, and this family is given by the same vector bundle as before.
3. S is the Veronese surface, i.e. $\mathbb{P}^2$ embedded by the vector bundle $\mathcal{Q}|_S = \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$. This surface has bidegree $(3,1)$.
4. S is a $(3,2)$ congruence, such that the dual of $S' \subset G(1,3)$ is a Del Pezzo surface given by $\mathbb{P}^2$ blown up in 4 points, and in particular it has no fundamental curve (but there exist finitely many fundamental points, corresponding to the lines of the Del Pezzo surface).
5. S is a Del Pezzo surface of bidegree $(3,3)$. In this case S' is selfdual (in the sense that its dual congruence belongs to the same family), and again contains finitely many fundamental points.

With the help of this classification we can now prove the following

PROPOSITION 2.1. *Let $X \subset G(1,5)$ be a 4-dimensional variety with singular locus of dimension at most one, such that the projection to $G(1,4)$ from a general point of $\mathbb{P}^5$ is an isomorphism. Then X is an extension of the $(2,1)$ congruence given in Example* 3.

PROOF. Since the projection from a general point of $\mathbb{P}^5$ is an isomorphism, the union of bad planes ΣX cannot be the whole $\mathbb{P}^5$. Therefore $\dim \Sigma X \leq 4$. Moreover we have seen that X must be compressed, which is equivalent to $\dim Y \leq 4$. In fact we have $\dim Y = 4$, because if this dimension were smaller than 3, Y would be linear and hence X degenerate. From the diagram

$$\begin{array}{ccc} & I_X & \\ \swarrow & & \searrow \\ X & & Y, \end{array}$$

where we set $I_X = \{(l,p) \in X \times \mathbb{P}^5 \mid p \in L\}$, we can see that through a general point $y \in Y$ there pass a 1-dimensional family of lines of X. This implies that every line of X meets a 2-dimensional family of lines of X. In particular every line is contained in at least one bad plane, i.e. $Y \subset \Sigma X$. Let us consider the following incidence diagram

$$\begin{array}{ccc} & I & \\ {\scriptstyle p_1} \swarrow & & \searrow {\scriptstyle p_2} \\ X & & X, \end{array}$$

where we put $I = \{(l_1, l_2) \mid l_2$ is contained in a 2-dimensional component of lines meeting $L_1\} \subset X \times X$. The dimension of the general fiber of p_1 is 2, so that $\dim I = 6$. There must exist an irreducible component $I_0 \subset I$ such that $p_1|_{I_0}$ is still surjective. Then as before $\dim I_0 = 6$. Let us consider the map

$$\rho : I_0 \to \Sigma(X),$$

which associates to each pair (l_1, l_2) the linear span $\langle L_1, L_2 \rangle$, and denote by $\Sigma_0(X)$ the image of ρ.

If we set $\Sigma_0 X = \bigcup_{\pi \in \Sigma_0(X)} \Pi$, the variety Y is contained in $\Sigma_0 X$. Since $\Sigma_0 X$ is irreducible and has dimension 4, we have $Y = \Sigma_0 X$.

Hence $\dim \Sigma_0(X)$ can be $2, 3$ or 4. Notice that, considering a general hyperplane of $\mathbb{P}^5$ and looking at the list we gave before the statement of this proposition, we can easily find a big family of bad planes (of dimension 3 or 4), but we do not know a priori that their union contains Y, which is crucial for our proof.

Let us study now the different cases for the dimension of $\Sigma_0(X)$.

i) $\dim \Sigma_0(X) = 2$.

In this case we are going to prove that the union of bad planes of X is the whole $\mathbb{P}^5$. From the incidence diagram

$$\begin{array}{ccc} & I_1 & \\ {\scriptstyle q_2} \swarrow & & \searrow {\scriptstyle q_1} \\ X & & \Sigma_0(X), \end{array} \tag{27}$$

where $I_1 = \{(l, \pi) \in X \times \Sigma_0(X) \mid L \subset \Pi\}$, it turns out that $\pi \in \Sigma_0(X)$ contains a 2-dimensional family of lines of X -i.e. every line contained in Π corresponds to a point of X- and every line of X is contained in a finite number of planes of $\Sigma_0(X)$. Hence the points of X are the lines contained in the planes of the surface $\Sigma_0(X) \subset G(2,5)$.

In this situation, if two planes Π_1 and Π_2 corresponding to two points of $\Sigma_0(X)$, meet in a point p, then their linear span is a $\mathbb{P}^4$ covered by bad planes. Indeed, the two pencils of lines $\Omega(p, \Pi_1)$ and $\Omega(p, \Pi_2)$ are contained in X. Every time we fix a line L_1 of the first pencil, the bad planes spanned by L_1 and a line of the second pencil fill up the linear space $\langle L_1, \Pi_2 \rangle \cong \mathbb{P}^3$. If we now let the line L_1 move on the first pencil, these $\mathbb{P}^3$'s fill up $\langle \Pi_1, \Pi_2 \rangle \cong \mathbb{P}^4$.

If we take the class $[\Sigma_0(X)] \in A^7(G(2,5))$, and the special Schubert cycle $\sigma_1 = \Omega(2,4,5) \in A^1(G(2,5))$, their product is the class of the family of planes of $\Sigma_0(X)$ meeting a fixed plane in at least one point. This product is an element of $A^8(G(2,5))$, and in particular it has dimension one (the dimension of the Grassmannian $G(2,5)$ is 9). Hence, if we take a point $\pi_0 \in \Sigma_0(X)$, the intersection of the Schubert variety $\Omega(\Pi_0, \mathbb{P}^4, \mathbb{P}^5)$ (the family of planes meeting Π_0) and the surface $\Sigma_0(X)$ must be a curve. But this is equivalent to say that a general plane Π_0 corresponding to a point of $\Sigma_0(X)$ meets a 1-dimensional family of planes of this surface, and hence Π_0 is contained at least in a $\mathbb{P}^4$ covered by bad planes. Finally, when we let π_0 move on the surface $\Sigma_0(X)$, these $\mathbb{P}^4$'s fill up the whole $\mathbb{P}^5$, which is our claim.

ii) $\dim \Sigma_0(X) = 3$.
This is equivalent to $\dim \rho^{-1}(\pi) = 3$. From the diagram (27), we find that the only possibility is $\dim q_2^{-1}(\pi) = 2$, which means that also in this case every line in a plane $\pi \in \Sigma_0(X)$ corresponds to a point of X. Moreover $\dim q_1^{-1}(l) = 1$, then every line of X is contained in a 1-dimensional family of planes of $\Sigma_0(X)$. In particular every line of X is the intersection of two planes of $\Sigma_0(X)$. But every time that two planes of $\Sigma_0(X)$ meet in a line, it corresponds to a singular point for X, so that in this case every point of X is singular. A contradiction.

iii) $\dim \Sigma_0(X) = 4$.
It follows that $\Sigma_0 X$ is a 4-dimensional variety containing a 4-dimensional family of planes. Since X is nondegenerate, by Theorem 2.2, Chapter 1, the only possibility is $\Sigma_0 X$ to be a scroll of $\mathbb{P}^3$'s, i.e. the union of the $\mathbb{P}^3$'s corresponding to a curve $C \subset G(3,5)$.

From the diagram (27), if the dimension of the general fiber of p_2 is zero, then also p_1 has a 0-dimensional general fiber. But this means that every line of X is contained in a finite number of planes of $\Sigma_0(X)$ and every plane of $\Sigma_0(X)$ contains only a finite number of lines of X, which is not possible because we know that every line of X meets a 2-dimensional family of lines of X.

If $\dim p_2^{-1}(\pi) = 2$, all the lines contained in a plane of the family $\Sigma_0(X)$ correspond to points of X. But then every line in a $\mathbb{P}^3$ of the curve C is a point of X, which implies $\dim X = 5$, a contradiction.

Then it must be $\dim p_2^{-1}(\pi) = 1$, which means that every plane of $\Sigma_0(X)$ contains a 1-dimensional family of lines of X, and every line is contained in a 1-dimensional family of planes of $\Sigma_0(X)$. In particular, when we cut with a hyperplane $H \cong \mathbb{P}^4$, we obtain a surface $S \subset G(1,4)$, and every $\mathbb{P}^3$ of the curve C gives a plane of $\Sigma_0(X)$, containing a 1-dimensional family of lines of S. Thus, projecting to $G(1,3)$, we find a congruence S' such that its dual has a fundamental curve.

We are now going to analyze the 5 surfaces S appearing in the list above, subdividing them in three different groups.

a) $(d_1, d_2) = (3,2)$ or $(3,3)$.

In this case, projecting to $S' \subset G(1,3)$ we know that the dual congruence of S' has not a fundamental curve, a contradiction.

b) $(d_1, d_2) = (3,1)$.

Then $S \subset G(1,4)$ is the Veronese surface. We know from Example 5 that S can be described by taking two planes Π_1 and Π_2 intersecting in a point y, an isomorphism $\varphi : \Pi_1 \to \Pi_2$, and then the family of lines $\{\langle q, \varphi(q)\rangle \mid q \in \Pi_1\}$.

If we consider y as a point of Π_2, we can set $y_1 := \varphi^{-1}(y) \in \Pi_1$, while if we take y as a point of Π_1, we set $y_2 := \varphi(y) \in \Pi_2$. Clearly the image of the line $M_1 := \langle y, y_1\rangle$ by φ, is $M_2 := \langle y, y_2\rangle$, and hence the plane $\Pi := \langle M_1, M_2\rangle = \langle y, y_1, y_2\rangle$ is a bad plane containing infinitely many lines of S (namely the lines joining a point $q \in M_1$ with the corresponding one $\varphi(q) \in M_2$).

Let us see that there are no other bad planes for S.

If we take two lines $l, l' \in S$, then there exist $p, p' \in \Pi_1$ such that $L = \langle p, \varphi(p)\rangle$ and $L' = \langle p', \varphi(p')\rangle$. Let us suppose that $L \cap L' \neq \emptyset$ and denote by Π' the plane $\langle L, L'\rangle$, and by L_i the lines cut by Π' on Π_i, for $i = 1, 2$. Therefore $L_1 = \langle p, p'\rangle$, $L_2 = \langle \varphi(p), \varphi(p')\rangle$ and hence L_2 is the image of L_1 by φ. Moreover L_1 and L_2 must intersect in the point $y = \Pi_1 \cap \Pi_2$. But the only line in Π_1 passing through y and such that its image in Π_2 passes through y too, is the line M_1.

Therefore Π is the only bad plane for S, and hence we do not have a 1-dimensional family of bad planes. This is again a contradiction.

c) $(d_1, d_2) = (2, 1)$ or $(2, 2)$.

In this case we have seen in the list above that there exists a rational curve $C' \subset G(2, 4)$ such that each plane of C' contains infinitely many lines of S. This curve is defined by a morphism $\mathbb{P}^1 \to G(2, 4)$ given by the rank-3 vector bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Moreover, we know that the planes of C' are the intersections of the $\mathbb{P}^3$'s of the curve C with the hyperplane H. Hence we find that C is defined by a morphism $\mathbb{P}^1 \to G(3, 5)$ given by the rank-4 vector bundle $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Therefore

$$\Sigma_0 X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) = \mathbb{P}(E). \tag{28}$$

If we call W the set of all lines contained in $\Sigma_0 X$, we have that $\dim W = 5$ and since $X \subset W \subset G(1, 5)$, X can be seen as a divisor in $W = G(1, \mathbb{P}(E))$.

Let us consider the diagram

$$\begin{array}{ccc} p^*E & & E \\ \downarrow & & \downarrow \\ W & \xrightarrow{p} & \mathbb{P}^1 \end{array}$$

and the relative exact sequence

$$0 \to \mathcal{S}^*_{W/\mathbb{P}^1} \to p^*E \to \mathcal{Q}_{W/\mathbb{P}^1} \to 0$$

or

$$0 \to \mathcal{S}^*_{W/\mathbb{P}^1} \to \mathcal{O}_Y \oplus \mathcal{O}_Y \oplus \mathcal{O}_Y(t) \oplus \mathcal{O}_Y(t) \to \mathcal{Q}_{W/\mathbb{P}^1} \to 0 \tag{29}$$

where t is the class of a fiber of p and $\mathcal{S}_{W/\mathbb{P}^1}$, $\mathcal{Q}_{W/\mathbb{P}^1}$ are the universal relative bundles of $W \to \mathbb{P}^1$.

Let $c_\lambda(\mathcal{Q}_{W/\mathbb{P}^1}) = 1 + c_1\lambda + c_2\lambda^2$ be the Chern polynomial of $\mathcal{Q}_{W/\mathbb{P}^1}$. The Chow ring $A(W)$ is generated by t, c_1, c_2. From the exact sequence (29) and the splitting principle we can derive the Chern polynomial of $\mathcal{S}_{W/\mathbb{P}^1}$ and, setting $c_3(\mathcal{S}_{W/\mathbb{P}^1}) = c_4(\mathcal{S}_{W/\mathbb{P}^1}) = 0$ we find relations holding in $A(W)$. First of all,

$$\begin{aligned} c_1(\mathcal{S}_{W/\mathbb{P}^1}) &= c_1 - 2t, \\ c_2(\mathcal{S}_{W/\mathbb{P}^1}) &= c_1^2 - c_2 - 2tc_1 \\ c_3(\mathcal{S}_{W/\mathbb{P}^1}) &= c_1^3 - 2c_1c_2 + 2c_2t - 2c_1^2t, \\ c_4(\mathcal{S}_{W/\mathbb{P}^1}) &= c_2^2 - c_1^2c_2 + 2tc_1c_2, \end{aligned}$$

so that the Chow ring of W can be written as

$$A(W) = \frac{\mathbb{Z}[t, c_1, c_2]}{(t^2, c_1^3 - 2c_1c_2 + 2c_2t - 2c_1^2t, c_2^2 - c_1^2c_2 + 2tc_1c_2)}. \tag{30}$$

In particular

$$\begin{aligned} A^0(W) &= \langle [W] \rangle, \\ A^1(W) &= \langle t, c_1 \rangle, \\ A^2(W) &= \langle c_1^2, c_2, c_1 t \rangle, \\ A^3(W) &= \langle c_1^2 t, c_2 t, c_1 c_2 \rangle, \\ A^4(W) &= \langle c_1 c_2 t, c_1^2 c_2 \rangle, \\ A^5(W) &= \langle c_1^2 c_2 t \rangle, \end{aligned}$$

where $c_1^2 c_2 t$ is the class of one point. Moreover in $A^5(W)$ the following relations hold

$$c_1^4 t = 2,\ c_2^2 t = 1,\ c_1 c_2^2 = 2,\ c_1^3 c_2 = 4,\ c_1^5 = 10. \tag{31}$$

Since X is a divisor in W, we can write down its class as $[X] = at + bc_1$ (while it was $[X] = d_1\Omega(1,4) + d_2\Omega(2,3)$ as an element of $A^4(G(1,5))$) and try to express the invariants of X in terms of a and b. Let us start from the bidegree (d_1, d_2);

$$d_2 = [X] \cdot \Omega(2,3) = [X] \cdot c_2(\mathcal{Q}_{W/\mathbb{P}^1}|_X)^2 = c_2^2(at + bc_1) = a + 2b,$$

and, since after the Plücker embedding it turns out that the degree of X is $d = 3d_1 + 2d_2$,

$$3d_1 = d - 2d_2 = H|_X^4 - 2d_2 = c_1^4(at + bc_1) - 2d_2 = 6b,$$

where $H = c_1(\mathcal{Q}_{W/\mathbb{P}^1})$ is the hyperplane section of W. Summarizing we have found

$$(d_1, d_2) = (2b, a + 2b). \tag{32}$$

Let us project X to $X' \subset G = G(1,4)$. We are going to use a double point formula but, since it can be used only in case of smooth varieties, we cut our variety X' with two general hyperplane sections H_1 and H_2. We denote by X'_H the intersection $X' \cap H_1 \cap H_2$, and by G_H the four dimensional variety $G \cap H_1 \cap H_2$. Since the dimension of the singular locus of X' is smaller than 2, X'_H is a smooth surface in G_H and we can use the double point formula

$$[X'_H]^2 = c_2(\mathcal{N}_{X'_H|G_H}). \tag{33}$$

Since $[X'] = d_1\Omega(1,4) + d_2\Omega(2,3)$, if we call H the class of a hyperplane section of $G(1,4)$ (i.e. $H = \Omega(2,4)$), we have

$$\begin{aligned} [X'_H]^2 &= [X']^2 \cdot H^2 \\ &= [d_1^2\Omega(1,4)^2 + d_2^2\Omega(2,3)^2 + 2d_1d_2\Omega(1,4)\Omega(2,3)] \cdot H^2 \\ &= [(d_1^2 + 2d_1d_2)\Omega(0,3) + (d_1^2 + d_2^2)\Omega(1,2)] \cdot (\Omega(1,4) + \Omega(2,3)) \\ &= 2d_1^2 + 2d_1d_2 + d_2^2 \\ &= 20b^2 + 8ab + a^2. \end{aligned} \tag{34}$$

On the other hand, to find the second Chern class of the relative normal bundle $\mathcal{N}_{X'_H|G_H}$ we can use the exact normal sequence defining it

$$0 \to \mathcal{T}_{X'_H} \to \mathcal{T}_{G_H}|_{X'_H} \to \mathcal{N}_{X'_H|G_H} \to 0 \tag{35}$$

from which

$$c_2(\mathcal{N}_{X'_H|G_H}) = c_2(\mathcal{T}_{G_H}|_{X'_H}) - c_2(\mathcal{T}_{X'_H}) + c_1(\mathcal{T}_{X'_H})^2 - c_1(\mathcal{T}_{X'_H})c_1(\mathcal{T}_{G_H}|_{X'_H}). \tag{36}$$

Since we know that the tangent bundle to the Grassmannian is $\mathcal{T}_G = Q \otimes S$, we can write down its Chern polynomial (up to terms of degree 2)

$$c_\lambda(\mathcal{T}_G) = 1 + 5\sigma_1\lambda + (12\sigma_1^2 - \sigma_2)\lambda^2 + ...$$

thus

$$\begin{aligned} c_1(\mathcal{T}_{G_H}|_{X'_H}) &= 5\sigma_1|_{X'_H} = 5c_1|_{X'_H} \\ c_2(\mathcal{T}_{G_H}|_{X'_H}) &= (12\sigma_1^2 - \sigma_2)|_{X'_H} = (11c_1^2 + c_2)|_{X'_H}, \end{aligned} \tag{37}$$

since on W, $\sigma_1 = \Omega(2,4) = c_1(\mathcal{Q}_{W/\mathbb{P}^1}) = c_1$ and $\sigma_2 = \Omega(1,4) = c_1^2 - c_2$.
Concerning the Chern classes of X'_H, we can use the fact that it is a divisor of W_H (i.e. W intersected with two hyperplane sections), so that the normal sequence of X'_H in W_H can be written as

$$0 \to \mathcal{T}_{X'_H} \to \mathcal{T}_{W_H}|_{X'_H} \to \mathcal{O}_{X'_H}(at + bc_1) \to 0, \tag{38}$$

from which

$$\begin{aligned} c_1(\mathcal{T}_{X'_H}) &= c_1(\mathcal{T}_{W_H}|_{X'_H}) - (at + bc_1)|_{X'_H} \\ c_2(\mathcal{T}_{X'_H}) &= c_2(\mathcal{T}_{W_H}|_{X'_H}) - c_1(\mathcal{T}_{W_H}|_{X'_H})(at + bc_1)|_{X'_H} \end{aligned} \tag{39}$$

Now, from the exact sequence

$$0 \to p^*\Omega_{\mathbb{P}^1} \to \Omega_W \to \Omega_{W/\mathbb{P}^1} \to 0$$

or

$$0 \to \mathcal{O}_W(-2t) \to \Omega_W \to \Omega_{W/\mathbb{P}^1} \to 0,$$

we can obtain that $c_\lambda(\Omega_W) = c_\lambda(\Omega_{W/\mathbb{P}^1})(1 - 2t\lambda)$. And, since $\Omega_{W/\mathbb{P}^1} = \mathcal{Q}^*_{W/\mathbb{P}^1} \otimes \mathcal{S}^*_{W/\mathbb{P}^1}$,

$$\begin{aligned} c_\lambda(\Omega_W) &= 1 + (2t - 4c_1)\lambda + (7c_1^2 - 6c_1t)\lambda^2 + ... \\ c_\lambda(\mathcal{T}_W) &= 1 + (4c_1 - 2t)\lambda + (7c_1^2 - 6c_1t)\lambda^2 + ... \end{aligned} \tag{40}$$

Finally, if we substitute it in (39) we find

$$\begin{aligned} c_1(\mathcal{T}_{X'_H}) &= (4c_1 - 2t)|_{X'_H} - (at + bc_1)|_{X'_H} = [(-2-a)t + (4-b)c_1]|_{X'_H}; \\ c_2(\mathcal{T}_{X'_H}) &= (7c_1^2 - 6c_1t)|_{X'_H} - [(-2-a)t + (4-b)c_1](at + bc_1)|_{X'_H} \\ &= [c_1^2(b^2 - 4b + 7) + c_1t(2ab + 2b - 4a - 6)]|_{X'_H}. \end{aligned} \tag{41}$$

Going back to (36) we obtain

$$\begin{aligned} c_2(\mathcal{N}_{X'_H|G_H}) &= [bc_1^2 + c_1t(a + 2b) + c_2]|_{X'_H} \\ &= [bc_1^2 + c_1t(a + 2b) + c_2]c_1^2(at + bc_1) \\ &= [(2ab + 2b^2)c_1^4t + ac_1^2c_2t + b^2c_1^5 + bc_1^3c_2] \end{aligned} \tag{42}$$

which, recalling (31), becomes

$$c_2(\mathcal{N}_{X'_H|G_H}) = 14b^2 + 4ab + 4b + a. \tag{43}$$

Equalizing (43) and (34) we found the expression

$$6b^2 + 4ab - 4b + a^2 - a = 0, \tag{44}$$

and, if we consider this as a polynomial in the variable a and write down its discriminant

$$\Delta = -8b^2 + 8b + 1,$$

it must be a perfect square. Then the only possibilities are $b = 0, 1$, and consequently

$$(a, b) = (0, 0), (1, 0), (-1, 1), (-2, 1),$$

which correspond to bidegrees

$$(d_1, d_2) = (0, 0), (0, 1), (2, 1), (2, 0). \tag{45}$$

But since we were considering only the bidegrees $(2, 1)$ and $(2, 2)$, (d_1, d_2) must be equal to $(2, 1)$. Therefore $S' \subset G(1, 3)$ is the surface given in Example 3 , and $X' \subset G(1, 4)$ is an extension of this surface.

□

3. A Generalization of the Structure Theorem

In this section we are going to see that in fact Theorem 2.1 of Chapter 3 can be generalized to compressed varieties. We recall that given a variety $X \subset G(1, N)$, we indicate by Y the union of lines of X inside $\mathbb{P}^N$. Let us make the following

Definition. Let $X \subset G(1, N)$ be a compressed n-dimensional variety. We call *compression degree* of X the number $\sigma(X) := n + 1 - \dim Y$.

Then, if we consider the incidence diagram

$$\begin{array}{ccc} & I_X & \\ {\scriptstyle p_2} \swarrow & & \searrow {\scriptstyle p_1} \\ X & & \mathbb{P}^N, \end{array}$$

where $I_X = \{(l, p) \in X \times G(1, N) \mid p \in L\}$, we have that $\sigma(X)$ is the dimension of the general fiber of p_2, i.e. the dimension of the family of lines of X passing through a general point of Y.

Let us consider now an n-dimensional variety $X \subset G(1, N)$, with $s := n + 3 - N > 0$. If X can be isomorphically projected to $X' \subset G(1, N - 2)$, then the dimension of Y cannot be bigger than $N - 2 = n + 1 - s$, which means that the compression degree of X is at least s. Now, using the same techniques of Theorem 2.1 of Chapter 3 we are going to prove the following

Theorem 3.1. *Let $X \subset G(1, n+3-s)$, with $s \geq 1$, be a smooth nondegenerate variety of dimension n. If X can be isomorphically projected to $G(1, n+1-s)$, and the dimension of Y is $n - s + 1$ (i.e. the compression degree is the expected one, $\sigma(X) = s$), then Y is the union of a 1-dimensional family of $\mathbb{P}^{n-s}$'s, corresponding to a curve $C \subset G(n - s, n + 3 - s)$ and one of the following holds:*

1. *the curve $C \subset G(n-s, n+3-s)$ is rational, $Y = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus z} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus n-s+1-z}\right)$, with $z \geq 3$, and the lines of X are sections of this projective bundle;*

2. *all the $\mathbb{P}^{n-s}$'s of the family intersect in a linear space $K \cong \mathbb{P}^k$, with $1 \leq k \leq n - s - 1$, all the lines of X are contained in the $\mathbb{P}^{n-s}$'s of the family and meet the space K.*

Proof. We know that if X can be projected, then $S'X$ is not the whole $G(1, n - s + 3)$. Let us take two skew lines $l_1, l_2 \in X$ and denote as usual by

H their linear span $\langle L_1, L_2\rangle \cong \mathbb{P}^3$. For a general line $L \subset H$ we have that $\dim T_{S'X,l} < 2n-2s+4$. By Lemma 4

$$T_{S'X,l} = T_{G(1,H),l} \oplus \gamma_l(T_{S'(X),h}),$$

where γ_l is the restriction map from $T_{S'(X),h} \subset \mathrm{Hom}(\mathcal{H}, V^*/\mathcal{H})$ to $\mathrm{Hom}(\mathcal{L}, V^*/\mathcal{H})$. Therefore, the dimension of the image of γ_l is smaller than $2n-2s$, which implies that γ_l is not surjective. We can still apply Lemma 5 and say that projecting from a general line $m \in X$, the image Y_m of Y has dimension not bigger than $n-s$. Since we are supposing $\dim Y = n-s+1$ (which is the expected dimension), the fibers of the projection from a line $m \in X$ must be 1-dimensional. Using the same arguments of Lemma 6 we can prove that these fibers are lines. Then, fixing a general line $m \in X$ we find an $(n-s)$-dimensional family $\mathcal{F}_m$ of lines meeting M and such that their union covers all the variety Y.

Let us consider the incidence set $I := \{(m,r) \in X \times G(1, n+3-s) \mid r \in \mathcal{F}_m\}$, and the corresponding diagram

$$\begin{array}{ccc} & I & \\ \swarrow^{p_1} & & \searrow^{p_2} \\ X & & W, \end{array}$$

where we denote by W the image of the second projection inside $G(n+3-s)$. For a general line $m \in X$, the fiber of p_1 is isomorphic to the family $\mathcal{F}_m$, and then $\dim I = 2n-s$. Let us study the fiber of the second projection. We distinguish three possible cases.

1. $\dim(p_2^{-1}(m)) = \alpha \leq s$.
   Then $\dim W = 2n-s-\alpha \geq 2n-2s$, i.e. Y is an $(n-s+1)$-dimensional variety containing at least a $(2n-2s)$-dimensional family of lines, which means that Y is linear. This is impossible, because we are assuming X nondegenerate.
2. $\dim(p_2^{-1}(m)) = \beta \geq s+2$.
   Then $\dim W = 2n-s-\beta$. Let us consider the new incidence diagram

$$\begin{array}{ccc} & J & \\ \swarrow^{q_1} & & \searrow^{q_2} \\ X \times X & & W, \end{array}$$

   where we set $J := \{(m_1, m_2, r) \in X \times X \times G(1, n+3) \mid r \in \mathcal{F}_{m_1} \cap \mathcal{F}_{m_2}\}$. Clearly $\dim(q_2^{-1}(r)) = 2\dim(p_2^{-1}(r)) = 2\beta$, which implies $\dim J = 2n-s+\beta$ and $\dim(q_1^{-1}(m)) = \beta - s \geq 2$. But then, given two general points $m_1,\ m_2 \in X$, all the lines joining M_1 and M_2 belong to W and hence they are contained in Y. But this implies again Y linear.
3. $\dim(p_2^{-1}(l)) = s+1$.
   In this case, $\dim W = 2n-2s-1 = 2\dim Y - 3$. Therefore, by Theorem 2.2, Chapter 1, Y must be one of the following;
   i) $Y \cong \mathbb{P}^{n-s+1}$;
   ii) Y is a quadric in $\mathbb{P}^{n-s+2}$;
   iii) Y is a scroll of $\mathbb{P}^{n-s}$'s.

   But since we are supposing X nondegenerate, the only possible case is the last one, i.e. Y is the union of the $\mathbb{P}^{n-s}$'s of a 1-dimensional family corresponding to a curve $C \subset G(n-s, n-s+3)$, such that the general point $y \in Y$ is contained in one and only one $\mathbb{P}^{n-s}$ of C.

Now, using exactly the same argument of Theorem 2.1, Chapter 3, we conclude.

□

3.1. An Example. We are now going to use the theorem above in the first interesting case, i.e. 4-folds projectable from G(1,6) to G(1,4). Although the result could be obtained from Proposition 2.1 we prefer to use this more conceptual proof.

PROPOSITION 3.2. *Let $X \subset G(1,6)$ be a smooth nondegenerate 4-dimensional variety. Then X cannot be isomorphically projected to $G(1,4)$.*

PROOF. Let us suppose that X can be isomorphically projected. Then we are in the hypothesis of Theorem 3.1, with $n = 4$ and $s = 1$. Hence there are three possibilities.

1. $\sigma(X) \geq 2$.
   This is equivalent to say that $\dim Y \leq 3$, which implies Y linear, because the only 3-dimensional variety containing a 4-dimensional family of lines is $\mathbb{P}^3$. But this is not possible because we are assuming X nondegenerate.
2. $Y = \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a rank-4 vector bundle on $\mathbb{P}^1$. Then Y is the union of the $\mathbb{P}^3$'s of a curve $C \subset G(3,6)$ and the lines of X are transversal to these $\mathbb{P}^3$'s. Since X is nondegenerate, there are only two possibilities.
   (a) $Y = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 4}\right)$. But in this case the transversal lines are only a 3-dimensional family, which is impossible.
   (b) $Y = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}\right)$. Under these hypothesis the variety $Y \subset \mathbb{P}^6$ is a cone with vertex a point $v \in \mathbb{P}^6$ on the Segre three-fold $Y' = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$.

   CLAIM. *In this case the bad planes for X fill up the whole $\mathbb{P}^6$.*

   Indeed the variety Y' contains a 1-dimensional family of planes $\mathcal{F}_1 \subset G(2,6)$ and a 2-dimensional family of lines $\mathcal{F}_2 \subset G(1,6)$ transversal to the planes of $\mathcal{F}_1$. Then a $\mathbb{P}^3$ of the curve C is simply the linear span of the vertex v and a plane of $\mathcal{F}_1$, while the points of X correspond to the lines contained in the 2-dimensional family of planes spanned by v and the lines of $\mathcal{F}_2$.
   Then, every time we fix a point $y \in Y'$, the line $L_y = \langle v, y\rangle$ corresponds to a point $l_y \in X$ (because there exists a line of $\mathcal{F}_2$ passing through y, and all the lines contained in the plane spanned by v and this line correspond to points of X).
   In particular, when we take a secant line to Y', the plane spanned by v and this line is a bad plane. Thus the union of bad planes $\Sigma X \subset \mathbb{P}^6$ is the cone with vertex v on the secant variety $SY' = \mathbb{P}^5$ and hence it is the whole $\mathbb{P}^6$, which is our claim.

   Therefore X cannot be projected to $G(1,5)$ and hence neither to $G(1,4)$.
3. Y is the union of the $\mathbb{P}^3$'s of a 1-dimensional family, all of them containing a linear space of dimension 1 or 2, and the lines of X are all contained in these $\mathbb{P}^3$'s. Let us distinguish the two cases.

(a) There exists a fixed line L_0.
Through a general point $p \in L_0$ there pass a 3-dimensional family of lines of X (otherwise there would be a point $p_0 \in L_0$ meeting a 4-dimensional family of lines, and hence the points of X would be the ruling lines of a cone and the bad planes would fill up the whole $\mathbb{P}^6$). In this case the family of lines of X passing through a point $p \in L_0$ is the union of the stars of lines passing through p and contained in the $\mathbb{P}^3$'s. Let us fix two $\mathbb{P}^3$'s of our family, say H_1 and H_2. Let $K = \langle H_1, H_2 \rangle \cong \mathbb{P}^5$ be their linear span. The union of planes spanned by a pair of lines $(l_1, l_2) \in H_1 \times H_2$ and passing through p, when p varies on L, fills up the whole K. But these are bad planes, and hence the union of bad planes is at least a 5-dimensional variety (in fact, when H_1 and H_2 vary, the bad planes fill up the whole $\mathbb{P}^6$), which means that X cannot be projected.

(b) There exists a fixed plane Π_0.
In this case it can happen that there exists a curve $C_0 \subset \Pi_0$ such that through each point $p_0 \in C_0$ there pass a 3-dimensional family of lines of X. But then the intersection $X_0 := X \cap \Omega(p_0, \mathbb{P}^6)$ is the union of the stars of lines passing through p_0 and contained in the $\mathbb{P}^3$'s of our family. Every time we fix two $\mathbb{P}^3$'s of the family, say H_1 and H_2, we find at least a 4-dimensional family of bad planes contained in the $\mathbb{P}^4$ spanned by H_1 and H_2, and hence this $\mathbb{P}^4$ is covered by bad planes. When we let the pair (H_1, H_2) move, we find that the bad planes fill up the whole $\mathbb{P}^6$.

Then we have that through a general point $p \in \Pi_0$ there pass a 2-dimensional family of lines of X.

CLAIM. *The lines of X passing through a general point $p \in \Pi_0$ are the lines of the star $\Omega(p, \mathbb{P}^3)$, for some $\mathbb{P}^3$ of the family.*

Indeed, if this is not the case, the lines passing through p give rise to a 4-dimensional family of bad planes and when the point p moves on Π_0, these planes fill up at least a 5-dimensional variety. Hence X cannot be projected.

Therefore for each $H \cong \mathbb{P}^3$ of the 1-dimensional family there exists a curve $C_H \subset \Pi_0$ such that the lines of X contained in H are the stars $\Omega(p, H)$, for $p \in C_H$. In particular X is the union of a 2-dimensional family of α-planes and every time we fix two of these α-planes, they meet in a singular point corresponding to a line contained in Π_0. A contradiction, because we are assuming X smooth.

□

CHAPTER 5

Projection of Surfaces

In this chapter we will use the techniques developed in the thesis to give a complete picture about the projection of surfaces in the Grassmannian.

More precisely, in section 1 we will characterize, for any k, which smooth surfaces can be isomorphically projected from G(1,k+1) to G(1,k). In section 2 we will also study the case of singular surfaces $X \subset G(1,5)$ with small secant variety (in the smooth case this is equivalent to find surfaces that can be projected to $G(1,3)$ and this problem was already solved in [**AS92**]).

1. Smooth Surfaces

Let us recall briefly what happens in the case of a smooth surface $X \subset \mathbb{P}^N$. First of all, if $N \geq 6$, we know that every surface X can be isomorphically projected to $\mathbb{P}^5$. Then, concerning surfaces that can be projected more, there exists the following well known result of Severi (see [**Sev01**]):

THEOREM (SEVERI) . *The only smooth nondegenerate surface that can be isomorphically projected from $\mathbb{P}^5$ to $\mathbb{P}^4$ is the Veronese surface.*

And it is trivial to prove that no surface can be isomorphically projected from $\mathbb{P}^4$ to $\mathbb{P}^3$.

In the case of the Grassmannian of lines, the picture is more complicated.

We have seen that every smooth surface $X \subset G(1, N)$, with $N \geq 7$, can be projected to $G(1,6)$, while in general we expect that X can be projected to $G(1,4)$. We are then left to classify surfaces $X \subset G(1, k+1)$ that cannot be projected to $G(1,k)$, with $k = 4, 5$.

In the case $k = 5$ we have the following easy corollary of Proposition 2.5 and Severi's Theorem.

PROPOSITION 1.1. *A smooth, nondegenerate surface $X \subset G(1,6)$ cannot be projected to $G(1,5)$ if and only if the points of X are the ruling lines of a cone on a smooth nondegenerate surface $X' \subset \mathbb{P}^5$, such that X' is not the Veronese surface.*

Concerning next case, i.e. surfaces $X \subset G(1,5)$ that cannot be isomorphically projected to $G(1,4)$, the classification is much more complicated. We will devote the rest of the section to analyze this case.

We have already seen some example of surfaces that cannot be projected to $G(1,4)$. First of all we know that if the points of X are the ruling lines of a cone on a smooth, nondegenerate surface $X' \subset \mathbb{P}^4$, then X cannot be projected to $G(1,4)$.

Then in Example 12 we have seen that also the join of two curves such that at least one of them is not plane is a non projectable surface. In particular this surface has a fundamental curve.

In fact, we will see in the following proposition that (with the exception of two surfaces) the ruling lines of a cone and some particular surfaces with fundamental curve, are the only surfaces that cannot be isomorphically projected from $G(1,5)$ to $G(1,4)$.

PROPOSITION 1.2. *Let $X \subset G(1,5)$ be a smooth nondegenerate surface. Then X cannot be isomorphically projected to $G(1,4)$ if and only if one of the following holds:*

1. *the points of X are the ruling lines of a cone on a nondegenerate surface $X' \subset \mathbb{P}^4$;*
2. *there exists a fundamental curve C such that the infinitely many lines passing through a general point of C are not contained in a $\mathbb{P}^3$;*
3. *X is the family of lines contained in the complete intersection of two hyperquadrics of $\mathbb{P}^5$;*
4. *X is the family of lines contained in the projection of $G(1,4) \cap \mathbb{P}^6$.*

PROOF. Let X be a smooth nondegenerate surface contained in the Grassmannian of lines $G(1,5)$. Let us suppose that X cannot be isomorphically projected to $G(1,4)$. Then the union of bad planes is the whole $\mathbb{P}^5$. If we consider the family of bad planes $\Sigma(X) \subset G(2,5)$ and the following diagram

$$\begin{array}{ccc} & I & \\ \swarrow^{p_1} & & \searrow^{p_2} \\ \Sigma(X) & & \mathbb{P}^5, \end{array} \tag{46}$$

where $I = \{(\pi, p) \in \Sigma(X) \times \mathbb{P}^5 \mid p \in \Pi\}$, we have that the projection on the second factor must be surjective. Then in particular, $\dim I \geq 5$ and, since the fiber of the first projection has dimension 2, we have that $\dim \Sigma(X) \geq 3$. Moreover, this dimension cannot be bigger than 4, so we just have to consider the two cases $\dim \Sigma(X) = 3, 4$.

1. $\dim \Sigma(X) = 4$.

   From the diagram

$$\begin{array}{ccc} & J & \\ \swarrow^{q_1} & & \searrow^{q_2} \\ \Sigma(X) & & X, \end{array} \tag{47}$$

   where $J = \{(\pi, l) \in \Sigma(X) \times X \mid L \subset \Pi\}$, we can see that in our hypothesis J has dimension at least 4. Then the general fiber of the second projection has dimension at least 2, i.e. the general line of X is contained in a two parameter family of bad planes. In particular, if we fix a general line $l \in X$ we can find a two dimensional family of lines of X meeting L. Hence a general line $l \in X$ meets all the other lines of X. But from Lemma 1, this can be the case only if X is either a β-plane, or the points of X correspond to the ruling lines of a cone. We can exclude the first case because we are supposing X nondegenerate.

   Then there exists a smooth surface X' contained in $K \cong \mathbb{P}^4$, such that the points of X are the ruling lines of the cone on X', with vertex $p \notin K$. This is case 1 in the statement.

In this case the union of bad planes is simply the cone on the secant variety $SX' \subset K$, with vertex p. Clearly, since X' is nondegenerate (because X is not), the secant variety SX' is the whole $\mathbb{P}^4$, and the cone on it is $\mathbb{P}^5$.

2. $\dim \Sigma(X) = 3$.

From the diagram (47), J must have dimension at least 3. Then the general fiber of the second projection will have dimension at least 1, i.e. the general line of X is contained in a one parameter family of bad planes. In particular, if we fix a general line $l \in X$ we can find a one parameter family of lines of X meeting L. We can distinguish two cases:

i) *through a general point of L there pass no other line of X but L;*

ii) *there exists at least another line of X passing through the general point of L.*

Let us analyze these two cases in detail.

i) Let l be a general line of X. We know that L meets a one dimensional family of lines of X so, if there is no other line passing through a general point of L, there must exist a finite number of fundamental points on L. Now, if we let l move on X, we find a one dimensional family of fundamental points, and then a fundamental curve C.

Let us see that C can have at most two irreducible components. In fact, if C_1 and C_2 are two components of C, then taking the join of C_1 and C_2 we find a 2-dimensional family of lines, contained in our surface X. Thus X must be this join, and the fundamental curve cannot have other components.

CLAIM. *Let $X \subset G(1,5)$ be a surface that admits a fundamental curve and such that a general line of X is not covered by intersection points with other lines of X. Then X cannot be isomorphically projected to $G(1,4)$ if and only if there exists a component C' of C such that the lines passing through a general point of C' are not contained in a $\mathbb{P}^3$.*

On one hand, let us suppose that for each irreducible component C_i of C, the lines passing through a general point are contained in a $\mathbb{P}^3$. Then we fix such a component C_i and a point $p \in C_i$. We call H_p the $\mathbb{P}^3$ containing all the lines of X passing through p. The bad planes spanned by pairs of lines passing through p are a two dimensional family, but they just fill up H_p. When p moves on C_i, we find a three dimensional family of bad planes $\Sigma_i \subset \Sigma(X)$ that can just cover a four-fold (the union of a one dimensional family of $\mathbb{P}^3$'s). Repeating the same argument for each irreducible component of C we then find at most two four-folds covered by bad planes, and this means that bad planes of this kind cannot fill up the whole $\mathbb{P}^5$.

Let us remark that there can exist other components of the family of bad planes $\Sigma(X)$, because on a general line L we can also find a finite number of points contained in finitely many lines of X. But in this case, every point gives rise to finitely many bad planes, and when we let l move on X we find at least a 2-dimensional family of bad planes, and hence their union cannot fill $\mathbb{P}^5$.

Therefore X can be projected to $G(1,4)$.

On the other hand, let us suppose that there exists an irreducible component C_0 of C such that the lines through a general point of C_0 are not contained in a $\mathbb{P}^3$. Let us fix a general point $p \in C_0$. There exist a two dimensional family of bad planes spanned by pairs of lines passing through p, and they must cover a four-fold W_p (because the only three-fold containing a family of planes of dimension bigger than or equal to 2, is $\mathbb{P}^3$).

Now, when p moves on C_0 we obtain a one parameter family of four-folds whose union must fill up the whole $\mathbb{P}^5$. The only case left is when the four-folds W_p are always the same, $W_p = W$, $\forall p \in C_0$. Let us see that it is not possible in our situation.

In fact we have a 4-dimensional variety W that contains a 3-dimensional family of planes. By the results of Theorem 2.2, Chapter 1, there are only three possibilities:

(a) $W \cong \mathbb{P}^4$. But since we are supposing X nondegenerate, this is not possible.

(b) W is the union of a one parameter family of $\mathbb{P}^3$'s.
Now, since we are assuming that the lines passing through p cannot be contained in a $\mathbb{P}^3$ of the family, there is a finite number of these lines in each $\mathbb{P}^3$. But then the bad planes are not contained in W, which is a contradiction.

(c) W is a hyperquadric.
If it is smooth, it is known that through each point of a smooth quadric there are two 1-dimensional families of planes, but there are no points contained in a 2-dimensional family of planes. However in our case, every point of the curve C_0 should be contained in a 2-dimensional family of planes, a contradiction.
If there exists a singular point q, then W is a cone on a quadric in a $\mathbb{P}^4$, with vertex q. In this case, all the planes of the 3-dimensional family (i.e. the bad planes) contain the vertex q.
Let us see that this is impossible. In fact, when we fix a point $p \in C_0$, we have that all the bad planes spanned by pairs of lines through p, contain q too. Then, all these planes contain the line $L = \langle p, q \rangle$. Let L_1, L_2, L_3 be three general lines through p. We must have that L is contained in the planes $\langle L_1, L_2 \rangle$, $\langle L_1, L_3 \rangle$, $\langle L_2, L_3 \rangle$; and this is possible if and only if L_1, L_2, L_3 are contained in a plane. But, since they are general, we find that all the lines passing through p are coplanar, a contradiction.
Finally, if the singular locus has dimension one, i.e. it is a line L_0, then W is a cone on a quadric Q contained in a $\mathbb{P}^3$, with vertex L_0. But then W is again a one dimensional family of $\mathbb{P}^3$'s, and we have already seen that this is not possible.
This ends the proof of our claim, which says that we are in case 2 of the statement.

ii) Let us suppose now that there exists at least another line of X passing through a general point $p \in L$. Since l was general in X, we then have a three-fold $Y = \bigcup_{l \in X} L \subset \mathbb{P}^5$ which is covered by the lines of a 2-dimensional family X, and such that through a general point $y \in Y$ there pass a finite number of lines of X.

Projecting to $\mathbb{P}^4$ we obtain a three-fold Y' birational to Y and with the same number of lines passing through a general point. There exists a classification of such three-folds $Y' \subset \mathbb{P}^4$, due to Mezzetti and Portelli (see [**MP00**]). We extract from their list the three-folds Y' for which there is an irreducible family of lines such that through a general point of Y' there pass $\mu > 1$ lines of the family.

(a) Y' is a *quadric bundle*, i.e. Y' is covered by a 1-dimensional family of quadrics such that there is one and only one quadric of the family passing through any general point of Y'. In this case we have $\mu = 2$;
(b) Y' is a cubic with singular locus of dimension at most one ($\mu = 6$);
(c) Y' is the projection of a complete intersection of two hyperquadrics in $\mathbb{P}^5$ ($\mu = 4$);
(d) Y' is the projection of a section of $G(1,4)$ with a $\mathbb{P}^6$ ($\mu = 3$).

Let us study these cases separately.

(a) Y' *is a quadric bundle.*

In this case we have a 1-dimensional family of $\mathbb{P}^3$'s, say $\{H_t\}$, such that in each of them is contained a quadric Q_t. Then each quadric Q_t gives rise to a 2-dimensional family of bad planes, but they just cover H_t, the $\mathbb{P}^3$ containing Q_t. Globally we find a 3-dimensional family of bad planes that just fill up a four-fold (the union of the $\mathbb{P}^3$'s of our family). To conclude, we just have to see if there can exist another 3-dimensional family of bad planes coming from the intersections between different quadrics.

In general two $\mathbb{P}^3$'s of the family, say H_1 and H_2, meet along a line M. Concerning the two corresponding quadrics Q_1 and Q_2, there are three possibilities.

If Q_1 and Q_2 are disjoint, then there are no new bad planes.

If Q_1 and Q_2 meet along a finite number of points, we just find a new family of bad planes of dimension 2, which cannot fill up $\mathbb{P}^5$.

Hence, if X cannot be projected to $G(1,4)$ we can assume that Q_1 and Q_2 meet along the line M. In this case we fix the first quadric Q_1 and let the second one move on the family, if the intersection lines are not always the same, then they cover the whole Q_1. But this means that through a general point of Y there pass at least two quadrics of the family, a contradiction. Then all the $\mathbb{P}^3$'s of the family meet along the same line M, and this line is contained in every quadric. Through a point $q \in M$ there pass infinitely many lines of X (one for each quadric) and then M is a fundamental curve. In particular the lines of X meeting M are a 2-dimensional family. But the line M is contained in one of the two rulings of each quadric, and hence M can meet only the lines of the other ruling. Therefore, since X contains all the lines of the quadrics, X must be reducible, a contradiction.

Finally, if two general $\mathbb{P}^3$'s H_1 and H_2 of the family meet along a plane Π, it can appear a new 3-dimensional family of bad planes if and only if Π cuts the same conic C on Q_1 and Q_2. As before, if we fix Q_1 and let Q_2 move in the family, the intersection must be always the same curve C (because through a general point there pass only one quadric). Then C is a fundamental curve.

In this last case, the bad planes that can fill up the whole $\mathbb{P}^5$ are only the planes spanned by pairs of lines passing through a point of C. Hence we can repeat the arguments of the claim above to prove that X cannot be projected to $G(1,4)$ only if the lines passing through a general point of C are not contained in a $\mathbb{P}^3$.

We are then again in case 2 of the statement.

(b) $Y' \subset \mathbb{P}^4$ *is a cubic hypersurface with singular locus of dimension at most one.*

Let us see that this case does not occur, since such a three-fold cannot come from a three-fold $Y \subset \mathbb{P}^5$ via a birational projection. For contradiction, assume there exists a cubic three-fold $Y \subset \mathbb{P}^5$ whose projection to $\mathbb{P}^4$ is Y' and consider the diagram

$$\begin{array}{ccc} & I & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ Y \times Y & & \mathbb{P}^5, \end{array}$$

where I is the closure in $Y \times Y \times \mathbb{P}^5$ of the set $I_0 = \{(x,y,z) \in (Y \times Y \setminus \Delta_Y) \times \mathbb{P}^5 \mid z \in \langle x, y\rangle\}$. From the first projection we can see that $\dim I = 7$. Moreover the second projection p_2 is surjective (if it were not the case, then the image of p_2 would be a four-fold containing a 6-dimensional family of lines, i.e. a $\mathbb{P}^4$, and then X would be degenerate). Then through a point of $\mathbb{P}^5$ there pass at least a 2-dimensional family of bisecant lines of Y, which means that projecting Y to $\mathbb{P}^4$, we obtain a three-fold with a singular locus of dimension 2, a contradiction.

(c) Y' *is the projection of a complete intersection of two hyperquadrics in* $\mathbb{P}^5$.

Then $Y \subset \mathbb{P}^5$ is the complete intersection of two hyperquadrics Q_1 and Q_2. We are going to prove that in this case the union of bad planes is the whole $\mathbb{P}^5$.

Let us calculate the class $[\Sigma(X)] \in A(G(2,5))$. We take the rank-6 vector bundle $S^2\mathcal{Q}$ (the second symmetric power of the universal bundle $\mathcal{Q}$ of $G(2,5)$). The two quadrics Q_1 and Q_2 defining Y correspond to two sections s_1 and s_2 of $S^2\mathcal{Q}$. The dependency locus of these two sections is then the set Z of planes in $\mathbb{P}^5$ whose intersection with Q_1 and Q_2 is the same conic. From Porteous formula, the class of Z is given by the Chern class $c_5(S^2\mathcal{Q}) \in A^5(G(2,5))$. Now we have to put the condition that the planes intersect the quadric Q_1 (and hence also the other one) along a singular conic (the union of two lines). The polarity associated to the quadric Q_1 is given by the map $\varphi_1 : \mathcal{Q}^* \to \mathcal{Q}$ corresponding to $s_1 \in H^0(S^2\mathcal{Q}) \subset H^0(\mathcal{Q} \otimes \mathcal{Q}) = \mathrm{Hom}(\mathcal{Q}^*, \mathcal{Q})$. Hence the planes intersecting Q_1 along a singular conic correspond to the locus in which the map φ_1 has rank smaller than 3. This is equivalent to say that the determinant $\wedge^3\varphi_1 : \mathcal{O}_G(-1) \to \mathcal{O}_G(1)$ is zero. Hence this zero locus is the Chern class $c_1(\mathcal{O}_G(2)) = 2c_1(\mathcal{Q})$.

We can then write

$$[\Sigma(X)] = 2c_5(S^2\mathcal{Q}) \cdot c_1(\mathcal{Q}) \in A^6(G(2,5)).$$

From this description we see that in fact $\Sigma(X)$ has dimension 3 and we can also find the number of planes of $\Sigma(X)$ passing through a general point of $\mathbb{P}^5$. This number is the intersection number of $\Sigma(X)$ and the special Schubert variety $\sigma_3 = c_3(\mathcal{S})$. An easy calculation gives

$$\begin{aligned}[\Sigma(X)] \cdot c_3(\mathcal{S}) &= 2c_5(S^2\mathcal{Q}) \cdot c_1(\mathcal{Q}) \cdot c_3(\mathcal{S}) \\ &= 8.\end{aligned}$$

This is equivalent to say that through a general point of $\mathbb{P}^5$ there pass 8 bad planes of $\Sigma(X)$, and then their union is the whole $\mathbb{P}^5$.

Hence we get case 3 in the statement.

(d) *Y' is the projection of a section of $G(1,4)$ with a $\mathbb{P}^6$.*

Let $Y'' \subset \mathbb{P}^6$ be the section of $G(1,4)$ with a $\mathbb{P}^6$. The family X'' of the lines contained in Y'' is isomorphic to our surface X (from Theorem 1.1 we know that a surface can be isomorphically projected from $G(1,6)$ to $G(1,5)$, if and only if its points are not the ruling lines of a cone).

Let us recall some properties of the variety $Y'' = G(1,4) \cap \mathbb{P}^6$ that we are going to use in what follows (see for instance [**DP00**] for more details). This is a three-fold of degree 5 contained in $G(1,4)$, and can be described as the family of trisecant lines to the Veronese surface in $\mathbb{P}^4$. Then Y'' is a smooth nondegenerate congruence which contains neither planes nor cones. Moreover, through a general point $y'' \in Y''$ there pass exactly 3 lines of X''.

We are going to prove that in this case the union of bad planes for X'' has dimension 5. First of all we prove that in fact the family of bad planes $\Sigma(X'')$ has dimension 3. Indeed, when we take the incidence variety $I = \{(y'', \pi) \in Y'' \times \Sigma(X'') \mid y'' \in \Pi\}$ and the diagram

$$\begin{array}{ccc} & I & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ Y'' & & \Sigma(X''), \end{array}$$

we know that the first projection has 1-dimensional fiber (because every point of Y'' is contained in a line of X'', and every line is contained in a 1-dimensional family of bad planes). Hence $\dim I = 4$, and the fiber of p_2 cannot have dimension bigger than 1, because if $\dim p_2^{-1}(\pi) = 2$, then the plane Π is contained in Y'', a contradiction (Y'' does not contain planes). Therefore $\dim \Sigma(X'') = 3$. Moreover, Y'' is nondegenerate and it is contained in the union of bad planes $\Sigma X'' = \bigcup_{\pi \in \Sigma(X'')} \Pi \subset \mathbb{P}^6$.

Let us suppose for contradiction $\dim \Sigma X'' \leq 4$. Since X is nondegenerate, this dimension cannot be smaller than or equal to 3 (by the results of B. Segre and Rogora, Theorem 2.2, Chapter 1) . Hence $\Sigma X''$ is a 4-dimensional variety which contains a 3-dimensional family of planes. Because of Theorem 2.2, Chapter 1, $\Sigma X''$ can be a $\mathbb{P}^4$, a quadric $Q \subset \mathbb{P}^5$ or a scroll of $\mathbb{P}^3$'s. Since X'' is nondegenerate, the only possibility would be $\Sigma X''$ to be a scroll. Let us see that this is not possible.

We have a 1-dimensional family of $\mathbb{P}^3$'s each of them containing a 1-dimensional family of lines of X'' and a 2-dimensional family of planes of $\Sigma(X'')$. Let us fix H_λ, a $\mathbb{P}^3$ of this family and call X_λ the family of lines of X'' contained in H_λ, $X_\lambda = X'' \cap \Omega(2, H_\lambda)$. Since they give rise to a 2-dimensional family of bad planes, we have that any line of X_λ meets a 1-dimensional family of lines of X_λ. Since these lines are not contained in a plane, there are only two possibilities: the lines of X_λ are the ruling lines of a cone, or the two rulings of a quadric.

In the first case we get a contradiction, because we know that the variety Y'' contains no cones.

In the second case, Y'' is the union of a 1-dimensional family of quadrics, $\{Q_\lambda = H_\lambda \cap Y''\}$. If we fix a general point $q \in Q_\lambda$, there are two lines of the quadric passing through it and since Y'' has $\mu = 3$, there must be a third line passing through q and not contained in H_λ. Then this third line is contained in another quadric, and hence Q_λ meets another quadric. On the other hand, through an intersection point of two quadrics there pass four lines of X'', two for each quadric; hence we have that two quadrics meet along a line. So, if we fix two $\mathbb{P}^3$'s of the family, say H_1 and H_2, they meet in a line, thus they span a hyperplane K of $\mathbb{P}^6$. The union of the two quadrics Q_1 and Q_2 is then contained in the intersection $Y'' \cap K$, which must be a surface of degree 5 ($= \deg Y''$). Then the extra intersection of Y'' and K must be a plane. Hence we get again a contradiction, because Y'' does not contain planes.

Then $\Sigma X''$ must have dimension 5, which implies that ΣX has dimension at least 5 (when projecting, the family of bad planes remains the same or get bigger), and hence $\Sigma X = \mathbb{P}^5$, and X cannot be projected to $G(1,4)$.

We are in case 4 of our statement.

□

Then we have solved the problem of classifying surfaces that cannot be projected as much as one expects.

In the next section we are going to study surfaces in $G(1,5)$ with small secant variety, considering also the singular case.

2. Singular Surfaces

2.1. The Projective Case. Let us first make a remark on singular varieties in the projective space.

REMARK. When a variety $X \subset \mathbb{P}^N$ is singular, the fact that its secant variety SX is not the whole $\mathbb{P}^N$ does not necessarily imply that the variety can be isomorphically projected to $\mathbb{P}^{N-1}$.

EXAMPLE 17. Let $X \subset \mathbb{P}^5$ be a nondegenerate cone with vertex q, on a curve C. Then the secant variety SX has dimension 4. In fact SX is simply the cone on the secant variety SC (every time we fix a secant line to C, say L, the plane spanned by q and L is covered by secant lines to X). Therefore SX is a cone on a 3-dimensional variety, and hence $\dim SX = 4$.

But, since the cone is nondegenerate, the Zariski tangent space to X at q is a linear space containing the tangent cone, hence it must be 5-dimensional.

This means that if we take the projection to $\mathbb{P}^4$, the Zariski tangent space of this projection at the image of q is not isomorphic to the one at q. Therefore X cannot be isomorphically projected to $\mathbb{P}^4$ even if its secant variety has dimension 4.

In fact in the case of surfaces, if we admit singularities, the following holds (see [**Moi77**])

THEOREM (MOISHEZON) . *The only nondegenerate surfaces X in $\mathbb{P}^5$ with isolated singular points and such that their secant variety SX is not the whole $\mathbb{P}^5$ are the Veronese surface and the cones.*

2.2. The Grassmannian $G(1,5)$. If the n-dimensional variety $X \subset G(1,N)$ is singular, we can give the same definition of secant variety that we gave in Chapter 2, taking the open subset $X_0 := X \setminus \operatorname{Sing} X$.

We take the diagram

$$\begin{array}{ccc} & I & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ X \times X & & G(1,N), \end{array}$$

where I is the closure inside $X \times X \times G(1,N)$ of the set

$$I^0 := \{(l_1, l_2, l) \in (X_0 \times X_0 \setminus \Delta_X) \times G(1,N) \mid \dim\langle L_1, L_2, L\rangle \leq 3\}$$

and define the secant variety SX as the image of the second projection, i.e. $SX = p_2(I) \subset G(1,N)$.

With this definition, the lemma's of Chapter 3 still hold, and hence, if X is a variety not necessarily smooth, we have the following

THEOREM 2.1. *Let $X \subset G(1, n+3)$ be a nondegenerate and uncompressed subvariety of dimension n, such that $\dim SX < 2n+4$. Then Y is the union of the $\mathbb{P}^n$'s of a 1-dimensional family corresponding to a curve $C \subset G(n, n+3)$ and one of the following holds:*

1. *C is rational, $Y = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus n-r+1}\right)$, with $r \geq 3$, and the lines of X correspond to sections of this projective bundle;*
2. *all the $\mathbb{P}^n$'s of the family intersect in a linear space $K \cong \mathbb{P}^k$, with $1 \leq k \leq n-1$, all the lines of X are contained in these $\mathbb{P}^n$'s and meet the space K.*

As in the projective case, if the secant variety of a singular variety $X \subset G(1,N)$ is not the whole $G(1,N)$, it does not necessarily implies that X can be projected to $G(1, N-2)$. Let us see the following example of a surface in $G(1,5)$ with small secant variety but that cannot be isomorphically projected to $G(1,3)$.

EXAMPLE 18. We take a line $M \subset \mathbb{P}^5$, a rational curve C of degree $d \geq 3$, contained in $H \cong \mathbb{P}^3$, disjoint from M, and an isomorphism $\varphi : C \to M$. For each $p_\lambda \in C$ we take the pencil of lines contained in the plane $\Pi_\lambda = \langle p_\lambda, M\rangle$ and passing through the point $\varphi(p_\lambda) \in M$. The set of the lines contained in the 1-dimensional family of these pencils is a surface $X \subset G(1,5)$. Moreover a pencil of lines in the Grassmannian is a line after the Plücker embedding, so that X contains a 1-dimensional family of lines, all passing through the point m (corresponding to the fixed line M). Therefore X is a cone.

Let us denote by $C' \subset G(1,5)$ the curve of the lines joining a point $p_\lambda \in C$ with the corresponding one $\varphi(p_\lambda) \in M$. Then C' is defined by the morphism

$\mathbb{P}^1 \to G(1,5)$ given by 6 sections of the rank-2 vector bundle $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(d)$. If we take a line $l \in C'$ we can easily see that the pencil spanned by L and M is contained in X, and hence X can be viewed as the cone on the rational curve C', with vertex m.

Let us see that the secant variety of X is not the whole $G(1,5)$. In fact we have that two lines of X can meet only on the line M, and since there exists an isomorphism between the family of planes $\{\Pi_\lambda\}$ and the line M, through a point $q \in M$ there pass only the lines of one pencil (this is true only for cones on rational curves). Hence the family of bad planes $\Sigma(X)$ is the 1-dimensional family $\{\Pi_\lambda\}$. Recalling that $\dim S''X \leq \dim \Sigma(X) + N + 1$, we find $\dim S''X \leq 7$, which implies $S''X \subsetneq G(1,5)$.

Concerning the component $S'X$, when we take two general skew lines, $l_1, l_2 \in X$, their span is a linear space $H \cong \mathbb{P}^3$, containing the line M and then the planes $\Pi_i = \langle M, L_i \rangle$, for $i = 1,2$. Thus H contains a 1-dimensional family of lines of X (namely, all the lines of the two pencils contained in Π_i), which is equivalent to say that X has secant defect $\delta = 1$. Hence $\dim S'(X) = 2n - 2\delta = 2$, and therefore $\dim S'X \leq 6$.

Let us suppose that the cone $X \subset G(1,5)$ can be isomorphically projected to $G(1,3)$. In this case, if we take the image $X' \subset G(1,3)$ and we embed it in $\mathbb{P}^5$, we obtain a cone which is contained in a hyperplane (because X' is contained in the hyperplane section of lines meeting the vertex M). Therefore we would have that X' is a cone in $\mathbb{P}^4$ that is projected from at least a $\mathbb{P}^5$, but this cannot be the case as we have seen in Example 17. Then X cannot be isomorphically projected from $G(1,5)$ to $G(1,3)$.

Finally we can calculate the bidegree of these cones. In fact, since the bidegree is invariant under rational maps, we can project the cone X to $G(1,3)$ (the projection is not an isomorphism, but it is a rational map). The geometric description is the same as in $G(1,5)$, but now the line M and the rational curve C are both contained in $\mathbb{P}^3$. Let us write $[X] = a\Omega(0,3) + b\Omega(1,2)$. As we have already seen, from Schubert calculus, $a = [X] \cdot \Omega(0,3)$ is the number of lines of X passing through a general point of $\mathbb{P}^3$. Let us fix a point $p \in \mathbb{P}^3$ and take the plane $\Pi = \langle M, p \rangle$ spanned by the point p and the line M. The plane Π intersects C in d points, namely $p_1, \ldots, p_d$. Let us denote by L_i the line joining the point $p_i \in C$ with the corresponding one on the line M, i.e. $L_i = \langle p_i, \varphi(p_i) \rangle$. Then $X \cap \Omega(p, \mathbb{P}^3) = \{l_1, \ldots, l_d\}$, i.e $a = d$. On the other hand, $b = [X] \cdot \Omega(1,2)$ is the number of lines of X contained in a general plane of $\mathbb{P}^3$. Let us fix a plane $\Pi \subset \mathbb{P}^3$. If we put $q = M \cap \Pi$, then the lines of X passing through q are in the pencil $\Omega(q, \Pi')$, where Π' is a plane containing M. Hence there exists only one line of this pencil contained in Π, which is equivalent to $b = 1$.
Therefore the cone on a rational curve of degree d has bidegree $(a,b) = (d,1)$.

In fact we are now going to prove that these cones and the Veronese surface are the only surfaces in $G(1,5)$ with small secant variety.

THEOREM 2.2. *The only nondegenerate surfaces $X \subset G(1,5)$ such that their secant variety SX is not the whole $G(1,5)$ are the Veronese surface and the cones on a rational curve.*

PROOF. We already know that the Veronese surface and the cones on rational curves have a secant variety of dimension < 8. We have to prove that they are the only surfaces with this property.

We take a nondegenerate surface $X \subset G(1,5)$ such that the secant variety SX is properly contained in $G(1,5)$, and denote as usual by Y the union in $\mathbb{P}^5$ of the lines of X. Then according to Theorem 2.1, one of the following holds:

i) X is compressed, which means $\dim Y \leq 2$. But then Y is a surface containing a 2-dimensional family of lines, i.e. a plane. This implies that X is degenerate (indeed it is a β-plane), which is a contradiction.

ii) Y is the union of the planes corresponding to a rational curve $\mathcal{G} \subset G(2,5)$, so that we can write $Y = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 3-r}\right)$. Since r must be at least 3, the only possibility is

$$Y = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3}\right).$$

Then Y is the variety $\mathbb{P}^1 \times \mathbb{P}^2$ embedded in $\mathbb{P}^5$ via the Segre embedding. Moreover, the lines of X are sections of this projective bundle, i.e. they are the lines joining the corresponding points on each plane of the family. But this is exactly the Veronese surface in $G(1,5)$.

iii) Y is the union of of a 1-dimensional family $\mathcal{G}$ of planes intersecting in a fixed line M, while the lines of X are all contained in these plains.

We then have a 1-dimensional family of lines of X on each plane of $\mathcal{G}$. Let us fix a plane $\pi_\lambda \in \mathcal{G}$ and the family of lines $X_\lambda := X \cap \Omega(1, \Pi_\lambda)$, i.e. the lines of X contained in the plane Π_λ.

CLAIM. *X_λ is a pencil of lines with center on the line M.*

If not, when we fix a point $p \in M$, we can always find a line $l_\lambda \in X_\lambda$ different from M and passing through the point p. Then, every time we fix a pair of planes $\pi_1, \pi_2 \in \mathcal{G}$ and a point $p \in M$, we can find a bad plane containing p and contained in the $\mathbb{P}^3$ spanned by Π_1 and Π_2 (i.e. the plane spanned by the two lines $l_i = X_i \cap \Omega(p, \mathbb{P}^5)$, $i = 1, 2$). If we now let the point p move on M, we find a 1-dimensional family of bad planes of this type, covering all the $\mathbb{P}^3 = \langle \Pi_1, \Pi_2 \rangle$. Finally, if we let the pair of planes (π_1, π_2) move on $\mathcal{G} \times \mathcal{G}$ we find a 2-dimensional family of $\mathbb{P}^3$'s covered by bad planes, which must fill up at least a 4-dimensional variety. Then the union of bad planes ΣX has dimension at least 4, which means that every line in $\mathbb{P}^5$ intersects ΣX and hence $S''X = G(1,5)$. This proves the claim.

Then X_λ is a pencil of lines with center on M, and the surface X must be a cone with vertex m (considered as a point of $G(1,5)$). In fact, every time we take a point $l \in X$, there exists a plane $\pi_\lambda \in \mathcal{G}$ such that $l \in X_\lambda$. In particular, the line joining l and m is the pencil X_λ and then is contained in X. Then there exists a curve C such that the family of planes $\mathcal{G}$ is defined by the morphism $C \to G(1,5)$ given by 6 sections of a rank-3 vector bundle of the form $\mathcal{E} = \mathcal{O}_C \oplus \mathcal{O}_C \oplus \mathcal{L}$ ($\mathcal{L}$ is a line bundle on C such that $h^0(C, \mathcal{L}) \geq 4$). Let us see that in fact $C \cong \mathbb{P}^1$.

There exists a map

$$\phi : C \to M,$$

which associates to every plane $\pi_\lambda \in C$, the center p_λ of the pencil X_λ. First of all, the map ϕ is not constant, because in this case we would have that all the lines of X pass through a fixed point. Hence X would be the set of ruling lines of a cone, and the bad planes would fill up the whole $\mathbb{P}^5$.

Therefore ϕ is a finite map from the curve C to $M \cong \mathbb{P}^1$.

Let us suppose that the map ϕ has degree $a \geq 2$. In this case, when we take a point $p \in M$, there exist at least two planes Π_1 and Π_2 of C such that the corresponding pencils X_1 and X_2 have the same center p. The lines of these two pencils give rise to a 2-dimensional family of bad planes such that their union is the whole $\mathbb{P}^3 = \langle \Pi_1, \Pi_2 \rangle$. Again, if we let the point p move on M, we find a 4-fold covered by bad planes, which means that the component $S''X$ of the secant variety is the whole $G(1,5)$.

Then it must be $a = 1$, and C is a rational curve of degree d, for some $d \geq 3$ (if $d < 3$, then X is degenerate). Therefore $C \cong \mathbb{P}^1$, and the morphism $\mathbb{P}^1 \to G(2,5)$ defining $\mathcal{G}$ is given by 6 sections of the rank-3 vector bundle $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d)$. Hence, as we have seen in Example 18, X is the cone on the rational curve defined by the rank-2 vector bundle $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(d)$, with vertex m. □

As a corollary of Theorem 2.2, we have that the only smooth surface of $G(1,5)$ with small secant variety is the Veronese surface, and hence this is the only smooth surface that can be projected from $G(1,5)$ to $G(1,3)$ (this result has been proved in [**AS92**] using strongly several classification results).

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