A decomposition theorem for polymeasures✩

Fernando Bombal, David Pérez-García*, Ignacio Villanueva

Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad Complutense de Madrid, Madrid 28040, Spain

Received 29 November 2006
Available online 23 March 2007
Submitted by J. Diestel

Abstract

We prove that every countably additive polymeasure can be decomposed in a unique way as the sum of a “discrete” polymeasure plus a “continuous” polymeasure. This generalizes a previous result of Saeki.

© 2007 Elsevier Inc. All rights reserved.

Keywords: Polymeasures; Multilinear operators; Vector measures

1. Introduction

Since the appearance of the papers [1,18], the interplay between vector measure theory and linear operators defined on spaces of continuous functions have proven to be a very fruitful tool in mathematical analysis [10]. As in the linear case, multilinear operators on (or tensor products of) $C(K)$ spaces play a prominent role in the multilinear and holomorphy theory of Banach spaces [11] and have applications in Harmonic Analysis [16,17,26], Complex Analysis [3,8,9,12] and even Quantum Information Theory [22]. Thanks to the so-called polymeasures [6,13,27], one has at hand a multilinear measure theory that, as in the linear case, can help to get a deeper insight in the theory. Hence, the understanding of polymeasures can (and indeed does) improve our knowledge about multilinear operators on $C(K)$ spaces (as examples of that see [2,5,6,27,28]).

✩ Partially supported by MTM2005-00082, UCM-910346 and Programa Ramón y Cajal.
* Corresponding author.
E-mail address: david.perez.garcia@urjc.es (D. Pérez-García).

0022-247X/$ – see front matter © 2007 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2007.03.046
In this paper we work in this direction by providing a decomposition of every countably additive vector valued polymeasure (in a unique way) as the sum of a “discrete” polymeasure plus a “continuous” one. This simplifies the problem of studying general polymeasures to the study of the discrete and continuous case which, in many cases, is simpler [5]. This decomposition is similar in spirit to the decomposition of a countably additive vector measure in the sum of an atomic and a continuous measure, but it does not follow from it.

Our result provides (with a simple proof based on measure theory) a generalization of a result in [25], where the author proves a decomposition theorem for the elements of \((C(K_1) \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C(K_n) \hat{\otimes}_\pi X)^*\), where \(K_i\) are compact Hausdorff spaces, \(X\) is a Banach space such that \(X^*\) verifies certain geometrical decomposition (which turns out to be equivalent to \(X^*\) not containing isomorphic copies of \(c_0\)), and \(E \hat{\otimes}_\pi F\) stands for the projective tensor product of \(E\) and \(F\).

The notations and terminology used along the paper will be the standard ones in Banach Space Theory and Measure Theory, as for instance in [10]. However, before going any further we shall explain some terminology and state some known facts we shall use: \(X\) will always be a Banach space, and \(X^*\) its dual. \(K, K_i\) will always be compact Hausdorff spaces, and \(C(K)\) will denote the Banach space of the continuous scalar valued measures with the supremum norm.

Given a compact space \(K\) and its Borel \(\sigma\)-algebra \(\Sigma\), we write \(B(K)\) for the completion under the supremum norm of the space \(S(K)\) of the \(\Sigma\)-simple scalar valued functions. It is well known that \(C(K) \overset{1}{\hookrightarrow} B(K) \overset{1}{\hookrightarrow} C(K)^{*}\), where \(\hookrightarrow\) denotes isometric embedding. When we speak of a multilinear operator, we always assume it to be bounded.

We recall that, given \(\sigma\)-algebras \(\Sigma_1, \ldots, \Sigma_n\) and a Banach space \(X\), a (Banach space valued) polymeasure is a separately additive application \(\gamma : \Sigma_1 \times \cdots \times \Sigma_n \to X\).

We say that a polymeasure \(\gamma\) is countably additive (respectively regular) if it is separately countably additive (respectively regular).

Let \(T : C(K_1) \times \cdots \times C(K_n) \to X\) be a multilinear operator. Then, we know that \(T\) can be extended to a multilinear operator \(\overline{T} : B(K_1) \times \cdots \times B(K_n) \to X^{**}\), in a unique way if we ask \(\overline{T}\) to be separately weak*-weak* continuous, when the weak* topology we consider in \(B(K_i)\) is given by the isometric inclusion \(B(K_i) \hookrightarrow C(K_i)^{*}\) (see [6]). In fact, this extension is nothing but the restriction of the so called Aron–Berner extension \(AB(T) : C(K_1)^{*} \times \cdots \times C(K_n)^{*} \to X^{**}\) to the product \(B(K_1) \times \cdots \times B(K_n)\) (see, for instance [11]).

Once we have defined \(\overline{T}\), we can define the set function \(\gamma : \Sigma_1 \times \cdots \times \Sigma_n \to X^{**}\) given by

\[
\gamma(A_1, \ldots, A_n) = \overline{T}(\chi_{A_1}, \ldots, \chi_{A_n}).
\]

Thus defined, \(\gamma\) is a weak* regular countably additive polymeasure, that is, for every \(x^* \in X^*\), \(x^* \circ \gamma\) is a regular countably additive polymeasure.

Given a polymeasure \(\gamma : \Sigma_1 \times \cdots \times \Sigma_n \to X\), as in the case \(n = 1\), its semivariation is defined as the set function

\[
\|\gamma\| : \Sigma_1 \times \cdots \times \Sigma_n \to [0, +\infty]
\]

given by

\[
\|\gamma\|(A_1, \ldots, A_n) = \sup \left\{ \left\| \sum_{m_1=1}^{r_1} \cdots \sum_{m_n=1}^{r_n} a_{m_1}^1 \cdots a_{m_n}^n \gamma(A_{m_1}^1, \ldots, A_{m_n}^n) \right\} \right\}.
\]
where the supremum is taken over all the finite $\Sigma_i$-partitions $(A_i^{m_i})_{m_i=1}^{r_i}$ of $A_i$ ($1 \leq i \leq n$) and all the collections $(a_i^{m_i})_{m_i=1}^{r_i}$ in the unit ball of the scalar field.

It is easy to see that, if for some $1 \leq i \leq n$ we have $A_i = B_i \cup C_i$ (disjointly), then
\[
\|\gamma\|(A_1, \ldots, A_n) \leq \|\gamma\|(A_1, \ldots, B_i, \ldots, A_n) + \|\gamma\|(A_1, \ldots, C_i, \ldots, A_n).
\]
\[
(1)
\]
We can also define the supremation of a polymeasure as the set function
\[
\overline{\gamma} : \Sigma_1 \times \cdots \times \Sigma_n \to [0, +\infty]
\]
given by
\[
\overline{\gamma}(A_1, \ldots, A_n) = \sup_{B_i \subset A_i, B_i \in \Sigma_i} \|\gamma(B_1, \ldots, B_n)\|.
\]

In [13] it is proved that $\overline{\gamma} \leq \|\gamma\| \leq 4^n \overline{\gamma}$.

Regarding this, we will also need the following lemma. Its simple proof, which uses just induction and the separate regularity of the polymeasure, can be seen in [23].

**Lemma 1.1.** Let $\gamma : \Sigma_1 \times \cdots \times \Sigma_n \to X$ be a regular polymeasure. Then, given $(A_1, \ldots, A_n) \in \Sigma_1 \times \cdots \times \Sigma_n$, for every $\epsilon > 0$ there exist compact sets $(C_1, \ldots, C_n) \in \Sigma_1 \times \cdots \times \Sigma_n$ with $C_i \subset A_i$ ($1 \leq i \leq n$) such that
\[
\|\gamma(A_1, \ldots, A_n) - \gamma(C_1, \ldots, C_n)\| < \epsilon.
\]

For the theory of polymeasures and its applications to the study of multilinear operators on $C(K)$ spaces see [6,23,27] and references therein.

If $E_1, \ldots, E_n, X$ are Banach spaces, a multilinear operator $T : E_1 \times \cdots \times E_n \to X$ is said to be completely continuous if, given weakly Cauchy sequences $(x_i^m)_{m \in \mathbb{N}} \subset E_i$ ($1 \leq i \leq n$), the sequence $(T(x_1^m, \ldots, x_n^m))$ is norm convergent, equivalently, if given weakly null sequences $(x_i^m)_{m \in \mathbb{N}} \subset E_i$ ($1 \leq i \leq n$), the sequence $(T(x_1^m, \ldots, x_n^m))$ is norm null. These operators are studied, among other places, in [20,21,24].

In this note, [27, Theorem 5] will be specially relevant. Together with the results in [19], it states that, with the above notation, the following are equivalent:

(a) $T$ is completely continuous;
(b) $\gamma$ is countably additive;
(c) $\gamma$ is regular;
(d) $\overline{T}$ is $X$-valued;
(e) $A(B(T))$ is $X$-valued;
(f) $\gamma$ is $X$-valued;
(g) $T$ is unconditionally converging, in the sense defined in [14].

In the sequel we will always be interested in this type of multilinear operators/polymeasures.

The notation $[i]$ means that the $i$th coordinate is not involved.

**Definition 1.2.** We say that a polymeasure $\gamma : \Sigma_1 \times \cdots \times \Sigma_n \to X$, representing a multilinear operator $T : C(K_1) \times \cdots \times C(K_n) \to X$, is **continuous** if, given $(s_1, \ldots, s_n) \in K_1 \times \cdots \times K_n$, for any $\epsilon > 0$ there exist neighborhoods $A_i$ of $s_i$ ($1 \leq i \leq n$) such that
\[
\|\gamma\|(A_1, \ldots, A_n) < \epsilon.
\]
Definition 1.3. We say that a polymeasure \( \gamma : \Sigma_1 \times \cdots \times \Sigma_n \to X \), representing a multilinear operator \( T : C(K_1) \times \cdots \times C(K_n) \to X \), is discrete if there exist sequences \((s_i^{m_i})_{m_i \in \mathbb{N}} \subset K_i \) \((1 \leq i \leq n)\) and a family of scalars \((\alpha_{m_1 \cdots m_n})_{m_1, \ldots, m_n \in \mathbb{N}^n} \) such that
\[
\gamma(A_1, \ldots, A_n) = \sum_{m_1} \cdots \sum_{m_n} \alpha_{m_1 \cdots m_n} \delta_{s_i^{m_i}}(A_1) \cdots \delta_{s_i^{m_n}}(A_n),
\]
where
\[
\delta_{s_i^{m_i}}(A_i) = \begin{cases} 
1 & \text{if } s_i^{m_i} \in A_i, \\
0 & \text{if } s_i^{m_i} \notin A_i.
\end{cases}
\]

Remark 1.4. It is important to notice that, in this definition, the convergence of the series need not be unconditional. In fact, in the case of scalar measures, the convergence is unconditional if and only if \( v(\gamma) < \infty \), equivalently if and only if \( \gamma \) can be extended to a product measure [7].

2. The result

The following two lemmas contain most of the technical parts of the proof of the main result.

Lemma 2.1. Let \( T : C(K_1) \times \cdots \times C(K_n) \to X \) be a multilinear operator with associated countably additive polymeasure \( \gamma : \Sigma_1 \times \cdots \times \Sigma_n \to X \). Given any \((s_1, \ldots, s_n) \in K_1 \times \cdots \times K_n \) such that \( \gamma(\{s_1\}, \ldots, \{s_n\}) = 0 \), for every \( \epsilon > 0 \) there exist neighborhoods \( A_i \) of \( s_i \) \((1 \leq i \leq n)\) such that
\[
\|\gamma\|(A_1, \ldots, A_n) < \epsilon.
\]

Proof. First we will prove that given an operator \( T : C(K_1) \times \cdots \times C(K_n) \to X \) with a countably additive associated polymeasure \( \gamma : \Sigma_1 \times \cdots \times \Sigma_n \to X \), given \( s := (s_1, \ldots, s_n) \in K_1 \times \cdots \times K_n \) and given \( \epsilon > 0 \) there exist neighborhoods \( A_i \) of \( s_i \) \((1 \leq i \leq n)\) such that
\[
\|\gamma\|(A_1 \setminus \{s_1\}, \ldots, A_n \setminus \{s_n\}) < \epsilon. \tag{2}
\]

Assume this is not true. Then there exists \( \epsilon_0 > 0 \) such that for every neighborhood \( A_i \) of \( s_i \) \((1 \leq i \leq n)\)
\[
\|\gamma\|(A_1 \setminus \{s_1\}, \ldots, A_n \setminus \{s_n\}) > \epsilon_0. \tag{3}
\]

We pick first arbitrary neighborhoods \( A^1_i \) of \( s_i \) \((1 \leq i \leq n)\). Since (3) holds, we can apply Lemma 1.1 together with the comments above it to obtain that for every \( 1 \leq i \leq n \) there exists a compact set \( C^1_i \subset A^1_i \setminus \{s_i\} \) such that
\[
\|\gamma(C^1_i, \ldots, C^1_n)\| > \frac{\epsilon_0}{2 \cdot 4^n}. \tag{4}
\]

For every \( 1 \leq i \leq n \) we define now \( A^2_i := A^1_i \setminus C^1_i \). This is again a neighborhood of \( s_i \), so we can repeat the reasoning to find compact sets \( C^2_i \) \((1 \leq i \leq n)\) such that
\[
\|\gamma(C^2_i, \ldots, C^2_n)\| > \frac{\epsilon_0}{2 \cdot 4^n}. \tag{5}
\]

Proceeding inductively for every \( 1 \leq i \leq n \) we construct a sequence \((C^m_i)_{m \in \mathbb{N}}\) of compact mutually disjoint sets such that, for every \( m \in \mathbb{N} \),
\[
\|\gamma(C^m_1, \ldots, C^m_n)\| > \frac{\epsilon_0}{2 \cdot 4^n}. \tag{6}
\]
But this is not possible: first let us note that the operator \( \varphi : c_0 \to B(K_i) \) defined by
\[
\varphi(e_m) = \chi_{C_m^n}
\]
is clearly continuous, therefore the sequence \((\chi_{C_m^n})_{m \in \mathbb{N}} \subset B(\Sigma_i)\) weakly converges to 0. On the other hand, according to [19, Corollary 4.8] or [27, Proposition 8] the multilinear operator \( \overline{T} \) defined in the introduction is completely continuous, therefore the sequence \((\overline{T}(\chi_{C_m^n}, \ldots, \chi_{C_m^n}))_m = (\gamma(C_1^n, \ldots, C_n^n))_m \) must converge to 0 in norm, a contradiction with (4). This proves (2).

To finish the proof of the lemma, we use induction. The result is known to be true for \( n = 1 \); we suppose it true for \( n - 1 \) and we prove the case \( n \). We first observe that for every \( 1 \leq i \leq n \), the restricted polymeasures \( \gamma_i : \Sigma_1 \times \cdots \times \Sigma_n \to X \) defined by
\[
\gamma_i(A_1, [i], A_n) = \gamma(A_1, \ldots, A_{i-1}, \{s_i\}, A_{i+1}, \ldots, A_n)
\]
are all of them as in the hypothesis of the lemma. Therefore, given \( \epsilon > 0 \), by taking finite intersections if necessary, we can consider, for every \( 1 \leq i \leq n \) a neighborhood \( \hat{A}_i \) of \( s_i \) verifying (2) and the result of the lemma for all of the restricted polymeasures mentioned above.

Finally, using (1), we have that
\[
\|\gamma\|(A_1, \ldots, A_n) \leq \|\gamma_i\|(A_2, \ldots, A_n) + \|\gamma\|(A_1 \setminus \{s_i\}, A_2, \ldots, A_n),
\]
and we can use again (1) in \( \gamma(A_1 \setminus \{s_i\}, A_2, \ldots, A_n) \) to obtain that
\[
\|\gamma\|(A_1, \ldots, A_n) \leq \|\gamma_i\|(A_2, \ldots, A_n) + \|\gamma_{s_2}\|(A_1 \setminus \{s_1\}, A_3, \ldots, A_n) + \|\gamma\|(A_1 \setminus \{s_1\}, A_2 \setminus \{s_2\}, A_3, \ldots, A_n).
\]
Continuing this procedure we can conclude that
\[
\|\gamma\|(A_1, \ldots, A_n) \leq \|\gamma_i\|(A_2, \ldots, A_n) + \|\gamma_{s_2}\|(A_1 \setminus \{s_1\}, A_3, \ldots, A_n) + \|\gamma\|(A_1 \setminus \{s_1\}, \ldots, A_{n-1} \setminus \{s_{n-1}\}) + \|\gamma\|(A_1 \setminus \{s_1\}, \ldots, A_n \setminus \{s_n\}) \leq (n + 1)\epsilon.
\]

**Remark 2.2.** What this lemma essentially says is that a regular countably additive polymeasure \( \gamma \) is continuous if and only if \( \gamma(\{s_1\}, \ldots, \{s_n\}) = 0 \) for any point \( (s_1, \ldots, s_n) \in K_1 \times \cdots \times K_n \).

**Lemma 2.3.** Let \( T : C(K_1) \times \cdots \times C(K_n) \to X \) be a multilinear operator with associated countably additive polymeasure \( \gamma : \Sigma_1 \times \cdots \times \Sigma_n \to X \). Then there exist at most countably many \( s = (s_1, \ldots, s_n) \in K_1 \times \cdots \times K_n \) such that
\[
\gamma(\{s_1\}, \ldots, \{s_n\}) \neq 0.
\]

**Proof.** We reason by induction on \( n \). For \( n = 1 \) (in that case \( \gamma \) is just a regular countably additive measure) the result is well known. We suppose the result true for \( n - 1 \) and we set to prove it for \( n \). Suppose the result is not true in this case. Then there exists \( \epsilon > 0 \) such that the set
\[
C := \{ (s_1, \ldots, s_n) \text{ such that } \|\gamma(\{s_1\}, \ldots, \{s_n\})\| > \epsilon \}
\]
is not countable. We will inductively choose a sequence \((x_m)_{m \in \mathbb{N}} \subset C\). We start by choosing \( x_1 := (s_1, \ldots, s_n) \) any element of \( C \). We show now how to choose \( x_m \) once \( x_1 =
(s_1^1, \ldots, s_n^1), \ldots, x_{m-1} = (s_1^{m-1}, \ldots, s_n^{m-1}) have been chosen. For every 1 \leq j \leq m - 1 and for every 1 \leq i \leq n we consider the (regular, countably additive) (n - 1)-polymeasure
\[
\gamma_{s_i}^j : \Sigma_1 \times \cdots \times \Sigma_n \to X
\]
given by
\[
\gamma_{s_i}^j (A_1, [i], A_n) = \gamma (A_1, \ldots, A_{i-1}, \{s_i^j\}, A_{i+1}, \ldots, A_n)
\]
for every \((A_1, [i], A_n) \in \Sigma_1 \times \cdots \times \Sigma_n\).

By the inductive step (on \(n\)) we know that the set
\[
C_i^j := \{ (s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) \text{ such that } (s_1, [i], s_n) \in K_1 \times \cdots \times K_n \text{ and } \gamma (s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) = \gamma_{s_i}^j (s_1, [i], s_n) \neq 0 \}
\]
is at most countable.

Hence the set
\[
C' := \bigcup_{i=1}^n \bigcup_{j=1}^{m-1} C_i^j
\]
is at most countable, and therefore \(C \setminus C'\) is not empty. So, we choose \(x_m\) any element in \(C \setminus C'\).

Once we have chosen \((x_m = (s_1^m, \ldots, s_n^m))_{m \in \mathbb{N}} \subset C\) as above, for every 1 \(\leq i \leq n\) we consider the set
\[
S_i := \bigcup_{m \in \mathbb{N}} \{ s_i^m \}.
\]

Since \(S_i \in \Sigma_i\) and since \(\gamma\) is countably additive, we have that \(\gamma (S_1, \ldots, S_n)\) is well defined and
\[
\gamma (S_1, \ldots, S_n) = \sum_{m_1} \cdots \sum_{m_n} \gamma (\{s_1^{m_1}\}, \ldots, \{s_n^{m_n}\}).
\]

But the construction of the sequence \((x_m)\) implies that whenever there exist \(i, k \in \{1, \ldots, n\}\) such that \(m_i \neq m_k\), we have that \(\gamma (\{s_1^{m_1}\}, \ldots, \{s_n^{m_n}\}) = 0\), hence
\[
\gamma (S_1, \ldots, S_n) = \sum_m \gamma (\{s_1^m\}, \ldots, \{s_n^m\})
\]
but this series cannot converge because \(\|\gamma (\{s_1^m\}, \ldots, \{s_n^m\})\| > \epsilon\), thus we reached a contradiction. \(\square\)

**Theorem 2.4.** Let \(T : C(K_1) \times \cdots \times C(K_n) \to X\) be a multilinear operator with associated countably additive polymeasure \(\gamma : \Sigma_1 \times \cdots \times \Sigma_n \to X\). Then \(\gamma\) can be written in an unique way as
\[
\gamma = \gamma_c + \gamma_d,
\]
where \(\gamma_c\) is continuous and \(\gamma_d\) is discrete.
Proof. Let γ be as in the hypothesis. Let \((s_m)_{m \in \mathbb{N}} := (s_1^m, \ldots, s_n^m)_{m \in \mathbb{N}} \subset K_1 \times \cdots \times K_n\) be the sequence (possibly finite or even empty) of elements in \(K_1 \times \cdots \times K_n\) such that \(\gamma(\{s_1\}, \ldots, \{s_n\}) \neq 0\) (see Lemma 2.3). For every \(1 \leq i \leq n\) we define the set \(S_i \subset K_i\) by

\[
S_i = \bigcup_{m \in \mathbb{N}} \{s_i^m\}.
\]

\(S_i\), being the countable union of closed sets, is a Borel set of \(K_i\). Hence we can define \(\gamma_d\) by

\[
\gamma_d(A_1, \ldots, A_n) = \gamma(A_1 \cap S_1, \ldots, A_n \cap S_n)
\]

for every \((A_1, \ldots, A_n) \in \Sigma_1 \times \cdots \times \Sigma_n\). Since \(\gamma\) is countably additive, so is \(\gamma_d\), therefore

\[
\gamma_d(A_1, \ldots, A_n) = \sum_{m_1} \cdots \sum_{m_n} \gamma(\{s_1^{m_1}\}, \ldots, \{s_n^{m_n}\}) \delta_{s_1^{m_1}}(A_1) \cdots \delta_{s_n^{m_n}}(A_n)
\]

and \(\gamma_d\) is discrete. As remarked above, the series is known to be convergent in the given order, but in general it is not unconditionally convergent.

We define now \(\gamma_c = \gamma - \gamma_d\). Then \(\gamma_c\) is regular and countably additive and clearly

\[
\gamma_c(\{s_1\}, \ldots, \{s_n\}) = 0
\]

for every \((s_1, \ldots, s_n) \in K_1 \times \cdots \times K_n\). By Lemma 2.1, this makes \(\gamma_c\) continuous.

The uniqueness of the decomposition is clear. □

To see the relation with Saeki’s result [25, Theorem 1], note that condition \(\mathcal{P}\) in [25, p. 34] is equivalent to \(X^*\) not containing isomorphic copies of \(c_0\). It follows now from [4, Proposition 11] that in that case every multilinear operator \(T : C(K_1) \times \cdots \times C(K_n) \to X^*\) (with representing polymeasure \(\gamma : \Sigma_1 \times \cdots \times \Sigma_n \to X^*\) is unconditionally converging (as for the definition, see [14]), and now it follows from [19, Theorem, Corollary 4.8] and [27, Theorem 5] that \(\gamma\) is countably additive.

However, the converse is not at all as easy, that is, it is not trivial (if possible) to derive our result directly from Saeki’s. The reason is that this would be essentially the same as giving a positive answer to the following question.

Let \(X\) be a Banach space, let \(T : C(K_1) \times \cdots \times C(K_n) \to X\) be a multilinear operator with representing polymeasure \(\gamma\). Suppose that \(T\) is unconditionally converging (equivalently, suppose that \(\gamma\) is countably additive). Does \(T\) factor through a Banach space not containing \(c_0\)?

If this were the case, there would be no complemented copies of \(c_0\) in \(\ell_{\infty} \hat{\otimes}_\pi \ell_{\infty}\), a question long time (and still) open. The reasoning for this is the following: Every bilinear operator \(T : \ell_{\infty} \times \ell_{\infty} \to c_0\) is unconditionally converging (this follows from [19]). Suppose that \(c_0\) is complemented in \(\ell_{\infty} \hat{\otimes}_\pi \ell_{\infty}\), let \(\hat{P} : \ell_{\infty} \hat{\otimes}_\pi \ell_{\infty} \to c_0\) be the projection and let \(P : \ell_{\infty} \times \ell_{\infty} \to c_0\) be the bilinear operator associated to \(\hat{P}\). If \(P\) factors through a space not containing \(c_0\), so does \(\hat{P}\), and this forces \(\hat{P}\) to be unconditionally converging, which is clearly impossible.

We note that in [15], the authors show an example of a Banach lattice \(X\) and an unconditionally converging operator \(T : X \to c_0\) such that \(T\) does not factor through any space not containing \(c_0\).

Acknowledgment

We want to thank Joe Diestel for bringing our attention to this problem and for several useful talks concerning it.
References