Laminations by Riemann surfaces in Kähler surfaces

Memoria presentada para obtener el grado de Doctor

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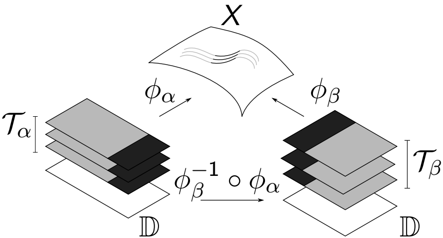

Lamination (by Riemann surfaces transversely $\mathcal{R}$-regular)

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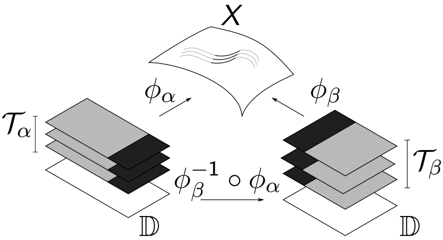

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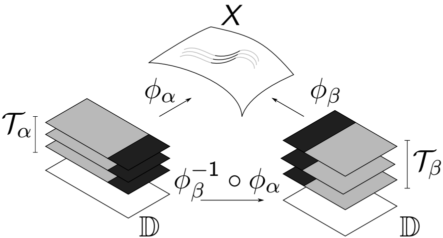

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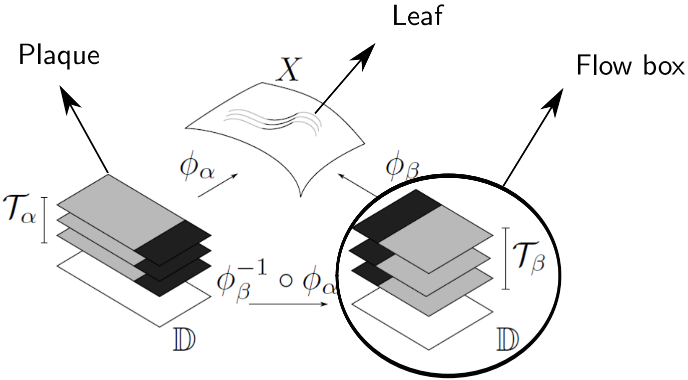
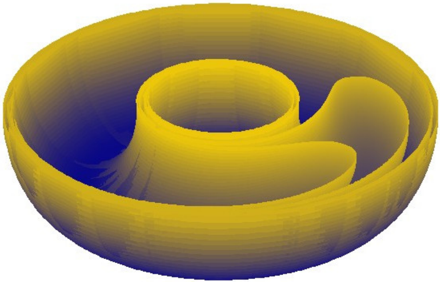
If X is a manifold we say it is a foliation


Leaf
Plaque
Flow box
X
ϕ_α
ϕ_β
$\mathcal{T}_\alpha$
$\mathcal{T}_\beta$
$\phi_\beta^{-1} \circ \phi_\alpha$
$\mathbb{D}$
$\mathbb{D}$

Example I. Reeb foliation



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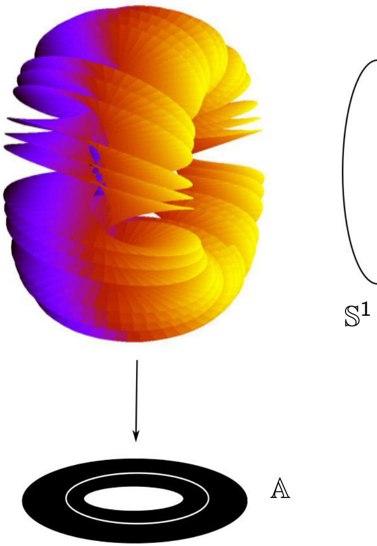
G is a Kleinian group hence Λ is its limit set (smallest invariant closed set), if it is a Schottky group Λ is a Cantor set.
We can extract a Λ fibration over S, carrying a lamination which is not a foliation.


$\mathbb{S}^1$
$\mathbb{A}$

Minimal lamination

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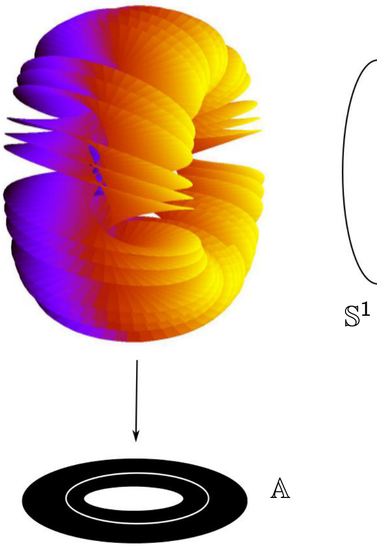
Lamination with singularities $(X, E, \mathcal{L})$

Let X be a compact topological space, $E \subset X$ and a lamination $\mathcal{L}$ on $X \setminus E$.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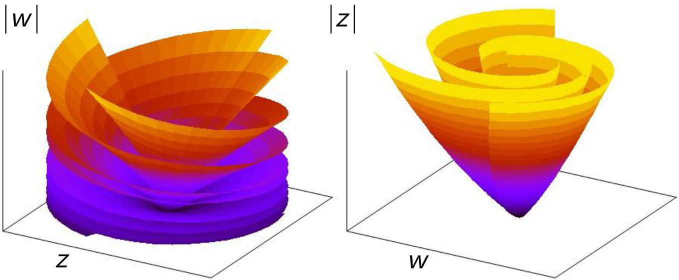

$zdw - (0.75 + 0.2i)wdz$

Hyperbolic Singularities

Let $(X, \mathcal{L}, E)$ in a compact complex surface M. We say that $p \in E$ is a hyperbolic singularity if we can find $U \subset M$ a neighborhood of p and some holomorphic coordinates (z, w) centered at p such that the leaves are invariant varieties for the holomorphic 1-form $\omega = zdw - \lambda wdz$, with $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Summary

We will study singular laminations:

- minimal
- transversely Lipschitz
- embedded in complex surfaces
- with at most hyperbolic singularities

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- An analytic subvariety Y of dimension m can be seen as a current $[Y]$ of bidimension (m, m) defined as $[Y](\phi) = \int_Y \phi$.

Closed Directed Positive $(1,1)$-Currents (CDPC). $dT = 0$

$$T = \int [V_\alpha] d\mu(\alpha)$$

$[V_\alpha]$ integration currents on plaques, μ an invariant transversal measure. Do not always exist.

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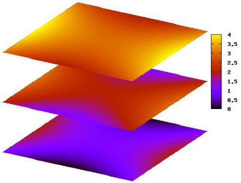
Harmonic Directed Positive $(1,1)$-Currents (HDPC). $\partial\bar{\partial} T = 0$

$$T = \int h_\alpha [V_\alpha] d\mu(\alpha)$$

$[V_\alpha]$ integration currents on plaques, h_α positive harmonic functions, μ a (not invariant) transversal measure. Do always exist.



Construction of Directed Currents

HDPC

$\phi : \mathbb{D} \to L$ with L leaf.

$$\tau_r := \frac{\phi_*(\log^+ \frac{r}{\xi}[(\Delta)])}{\|\phi_*(\log^+ \frac{r}{\xi}[(\Delta)]\|} \Rightarrow \tau_{r_n} \overset{\text{weak}*}{\to} T.$$

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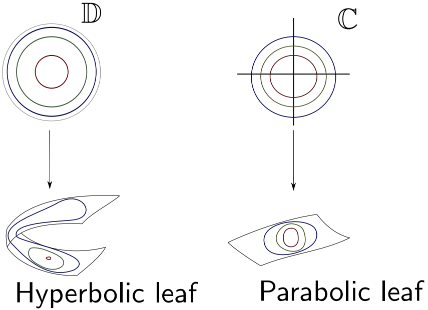

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Intersection Theory [FS05]

Fornæss and Sibony defined a geometrical self-intersection $\wedge_g$.

$$\lim_{\epsilon \to 0} \int \sum_{p \in J^{\epsilon}_{\alpha,\beta}} h_\alpha(p) h^{\epsilon}_{\beta}(p) \psi(p) d\mu(\alpha) d\mu(\beta)$$

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Theorem. Fornæss-Sibony[FS05]

For a lamination $(X, \mathcal{L}, E)$ transversely Lipschitz with E finite and no compact leaves in a Kähler surface (M, ω), if $T \wedge_g T = 0$ for every HDPC,

- either there are CDPC
- or there is only one HDPC of mass one ($T \wedge \omega = 1$).

Sufficient condition for the geometric self-intersection to vanish. Condition 1

If there exist:

- a family of automorphisms close to the identity Φ_ϵ,
- a covering by flow boxes $\mathcal{U} = (U_i, \varphi_i)$,
- a positive number $\epsilon_0 > 0$ and
- a positive integer $N \in \mathbb{N}$

such that if $|\epsilon| < \epsilon_0$, for every pair of plaques Γ_α, Γ_β in the same flow box U_i, the number of intersection points between Γ_α and $\Phi_\epsilon(\Gamma_\beta)$ (i.e. $\sharp J^\epsilon_{\alpha,\beta}$) is smaller than N. Then $T \wedge_g T = 0$ for every HDPC.

Unicity [PG13a],[PG13b]

Theorem

Let X be a minimal transversely Lipschitz lamination by Riemann surfaces with only hyperbolic singularities in a homogeneous compact Kähler surface M satisfying the hypothesis below. Then $T \wedge_g T = 0$ for every HDPC.

Hypothesis on the lamination depending on the surface

- $\mathbb{P}^2$: no compact leaves (Fornæss- Sibony),
- $\mathbb{P}^1 \times \mathbb{P}^1$: no compact leaves,
- $\mathbb{P}^1 \times \mathbb{T}^1$: no compact leaves,
- $\mathbb{T}^2$: no compact leaves nor invariant complex segments.

Sketch of the proof on $\mathbb{P}^1 \times \mathbb{P}^1$

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We will consider a family

$$\Phi_\epsilon([z_0:z_1],[w_0:w_1]) = ([z_0+\epsilon v_1 z_1 : z_1],[w_0+\epsilon v_2 w_1 : w_1]).$$

Expressions of $\varphi_i^{-1} \circ \Phi_\epsilon \circ \varphi_i$

$i = 1$

$$(z, w) \mapsto \left(\frac{z}{1 + \epsilon v_1 z}, \frac{w}{1 + \epsilon v_2 w} \right)$$

$i = 2$

$$(z, w) \mapsto \left(z + \epsilon v_1, \frac{w}{1 + \epsilon v_2 w} \right)$$

$i = 3$

$$(z, w) \mapsto \left(\frac{z}{1 + \epsilon v_1 z}, w + \epsilon v_2 \right)$$

$i = 4$

$$(z, w) \mapsto (z + \epsilon v_1, w + \epsilon v_2)$$

Behavior at $([1:0],[1:0])$

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$$\mathrm{dist}_z(\Gamma_p, \Gamma_p^\epsilon) = \left| f_p(z) - \frac{f_p(\frac{z}{1-\epsilon v_1 z})}{1+\epsilon v_2 \frac{z}{1-\epsilon v_1 z}} \right|$$

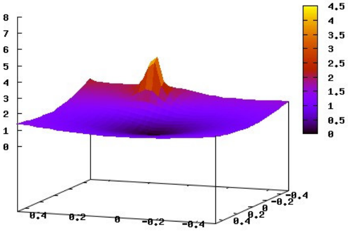

Figure : $\mathrm{dist}_z(\Gamma_p, \Gamma_p^\epsilon)/\epsilon$ for $\epsilon = 1$

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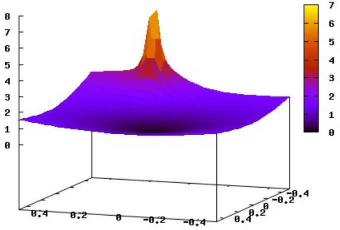

Figure : $\mathrm{dist}_z(\Gamma_p, \Gamma_p^\epsilon)/\epsilon$ for $\epsilon = 0.8$

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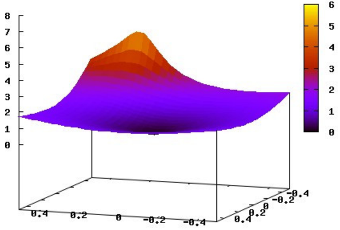

Figure : $\mathrm{dist}_z(\Gamma_p, \Gamma_p^\epsilon)/\epsilon$ for $\epsilon = 0.6$

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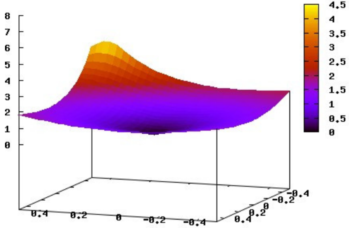

Figure : $\mathrm{dist}_z(\Gamma_p, \Gamma_p^\epsilon)/\epsilon$ for $\epsilon = 0.5$

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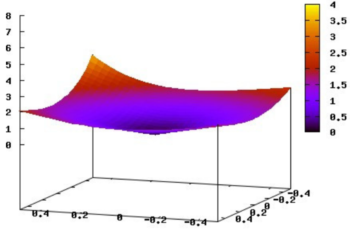

Figure : $\mathrm{dist}_z(\Gamma_p, \Gamma_p^\epsilon)/\epsilon$ for $\epsilon = 0.3$

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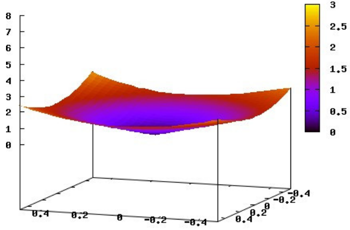

Figure : $\mathrm{dist}_z(\Gamma_p, \Gamma_p^\epsilon)/\epsilon$ for $\epsilon = 0.1$

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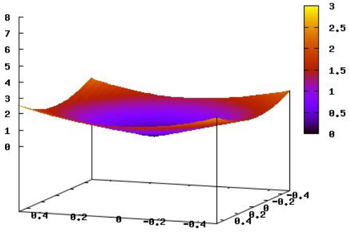
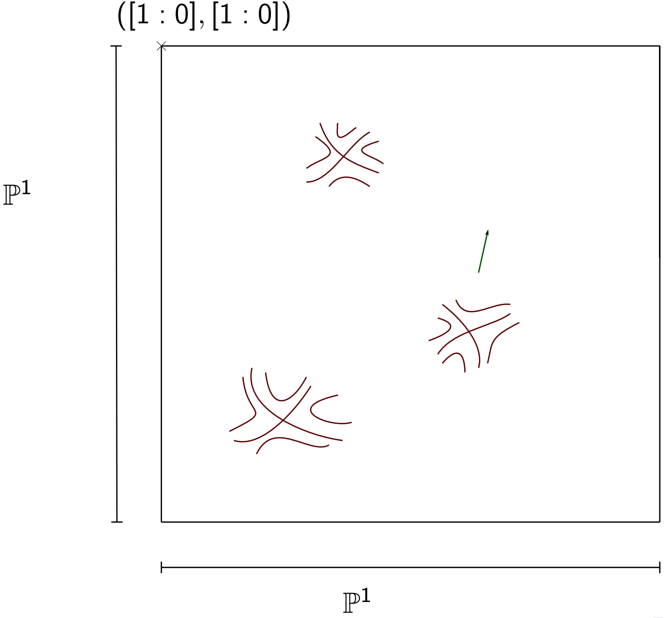
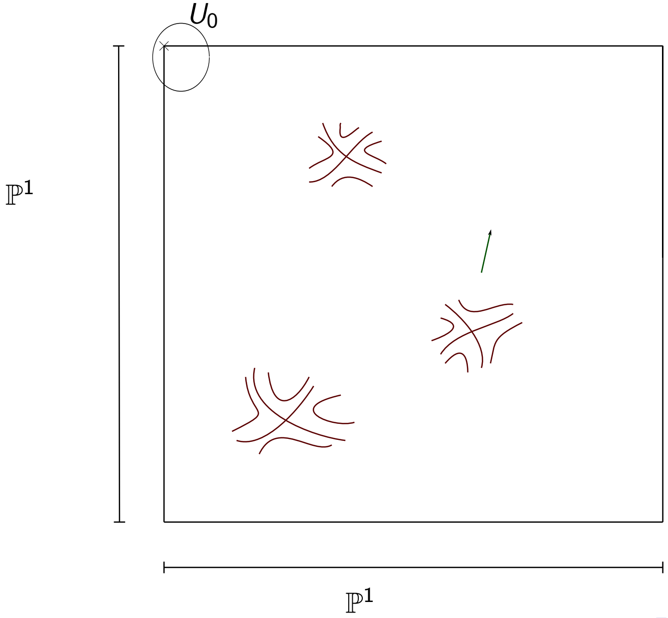
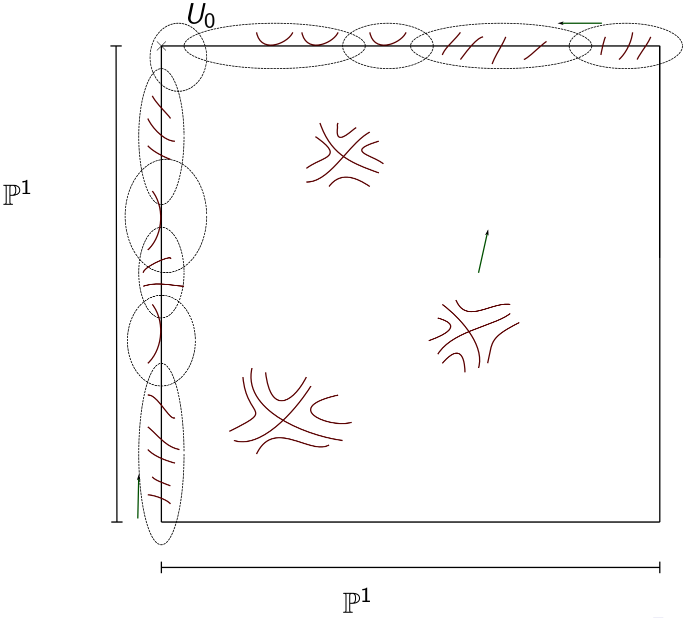
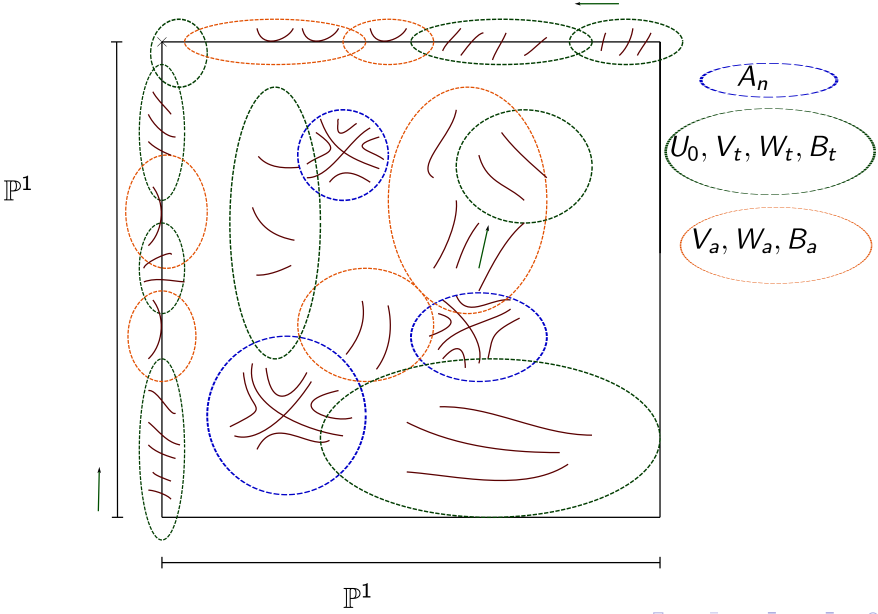


Figure : $\mathrm{dist}_z(\Gamma_p, \Gamma_p^\epsilon)/\epsilon$ for $\epsilon = 0.01$


([1 : 0], [1 : 0])
$\mathbb{P}^1$
$\mathbb{P}^1$


U_0
$\mathbb{P}^1$
$\mathbb{P}^1$


U_0
$\mathbb{P}^1$
$\mathbb{P}^1$


A_n
U_0, V_t, W_t, B_t
V_a, W_a, B_a
$\mathbb{P}^1$
$\mathbb{P}^1$

Control on the local behavior

Singular flow boxes

The motions were chosen such that we can apply the local study carried out in Fornaess-Sibony [FS10].

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If Γ_α and Γ_β are plaques on the flow box, there exist $C_1, C_2 > 0$ such that

$$\max d(\Gamma_\alpha, \Gamma_\beta^\epsilon) < C_1|\epsilon| \Rightarrow \min d(\Gamma_\alpha, \Gamma_\beta) > C_2|\epsilon|$$

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Flow boxes along the motions

If Γ_β is a plaque on the flow box, then, for $|\epsilon|$ small enough,

$$\Gamma_\beta \cap \Gamma_\beta^\epsilon \neq \emptyset.$$

End of the proof

- Start with a f.b.t. to the motion and $\#(\Gamma_\alpha \cap \Gamma_\beta^\epsilon) > N$.

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- Their continuations satisfy
  - $\min d(\Gamma'_\alpha, \Gamma'_\beta) > \frac{C_2|\epsilon|}{M_2}$
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- Applying the triangular inequality,

$$\min d(\Gamma'_\beta, \Gamma'^\epsilon_\beta) \geq \min d(\Gamma'_\alpha, \Gamma'^\epsilon_\beta) - \max d(\Gamma'_\alpha, \Gamma'^\epsilon_\beta) \geq$$

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if N big enough.

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Table of Contents

Corollary

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Reason: No CDPC $\Rightarrow$ no compact leaves.
If $M = \mathbb{T}^2$, No CDPC $\Rightarrow$ no complex segments.

Corollary

A minimal non singular transversely Lipschitz lamination which is not a single leaf in $\mathbb{P}^1 \times \mathbb{P}^1$ does not admit CDPC.

Remark: the same is true on $\mathbb{P}^2$ (Hurder-Mitsumatsu [HM91])

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Corollary

Let X be a transversely Lipschitz lamination by Riemann surfaces in $\mathbb{P}^1 \times \mathbb{P}^1$ with at most hyperbolic singularities and without invariant compact curves. Then it has only one minimal set.

Proof: X, X' two minimal sets with T, T' HDPC and $X'' = X' \cup X$.
X'' admits a unique HDPC T''. But it already had two: X'' and X'.

Applications

Non singular case

- Every non singular foliation on $\mathbb{T}^2$ carries a CDPC.
- The associated foliation to every Levi-flat on $\mathbb{T}^2$ carries a CDPC. [Ohs06]

Generalization of the Exceptional Minimal Set Problem:
Is there any lamination in a homogeneous compact Kähler surface without CDPC?

Singular case

Jouanoulou's Theorem has been generalized by Coutinho and Pereira [CP06] for every algebraic surface. A modification of an easier proof of $\mathbb{P}^2$ can be made for $\mathbb{P}^1 \times \mathbb{P}^1$.

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Future work

- Providing examples
- Harmonic flow
- Higher dimension