Ergodicity of embedded singular laminations

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- Find measures associated to laminations.

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- First attempt: Invariant transversal measure ν (for the holonomy pseudogroup). Locally:

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$$\mu = vol_t \otimes \nu(t)$$

- Weaker concept: Harmonic measures (for the diffusion semigroup). Locally:

$$\mu = h(x,t) vol_t \otimes \nu(t)$$

Existence of harmonic measures

- Garnett: nonsingular real foliations[1,2]
- Berndtsson-Sibony: holomorphic foliations with singularities[3]
- Fornæss-Sibony: laminations by Riemann surfaces with singularities[4]

[1] L. Garnett. "Foliations, the ergodic theorem and Brownian motion". In: *J. Funct. Anal.* 51 (1983).

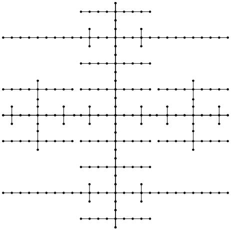
[2] A. Candel. "The harmonic measures of Lucy Garnett". In: *Adv. Math.* (2003).

[3] B. Berndtsson and N. Sibony. "The $\overline{\partial}$-equation on a positive current". In: *Invent. Math.* (2002).

[4] J. E. Fornæss and N. Sibony. "Harmonic currents of finite energy and laminations". In: *Geom. Funct. Anal.* (2005).

No Unicity

Furstenberg's example[5][6]
Lozano-Rojo repetitive graphs[7]



[5] J. E. Fornæss, N. Sibony, and E. F. Wold. "Examples of minimal laminations and associated currents". In: *Math. Z.* (2011).

[6] Bertrand Deroin. "Non unique-ergodicity of harmonic measures: smoothing Samuel Petite's examples". In: *Differential geometry*. 2009.

[7] A. Lozano-Rojo. "An example of a non-uniquely ergodic lamination". In: *Ergodic Theory Dynam. Systems* (2011).

Unicity

Who?	Where?	When?	How?
Deroin Kleptsyn[8]	C^1 foliations transversely conformal	NO	Brownian Motion
Bonatti Gómez-Mont[9]	Riccati foliations	INVARIANT	Fol. Geod. Flow
Fornæss Sibony[10]	Lam. Riem. Surf. in $\mathbb{P}^2$	MEASURES	Complex Var.

[8] B. Deroin and V. Kleptsyn. "Random conformal dynamical systems". In: *Geom. Funct. Anal.* (2007).

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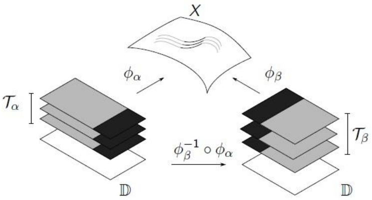
Lamination by Riemann surfaces transversely Lipschitz

Let X be a metric space, endowed with an atlas $\mathcal{U} = \{U_i, \phi_i\}$ with $\phi_i : \mathbb{D} \times \mathcal{T}_i \to U_i$ such that:

- $\phi_i^{-1} \circ \phi_j(z,t) = (f_{ij}(z,t), h_{ij}(t))$,
- f_{ij} holomorphic in the first variable and Lipschitz in the second one,
- h_{ij} is Lipschitz.



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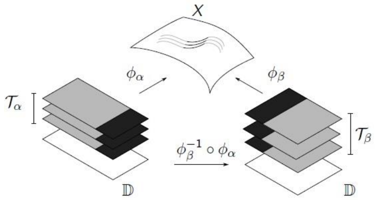
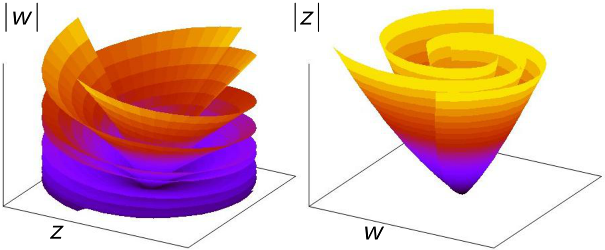


Lamination with singularities. $(X, \mathcal{L}, E)$

X compact metric space, $E \subset X$ and a lamination $\mathcal{L}$ on $X \setminus E$.



$zdw - (0.75 + 0.2i)wdz$

Hyperbolic Singularities

Let $(X, \mathcal{L}, E)$ in a compact complex surface M. We say that $p \in E$ is a hyperbolic singularity if we can find $U \subset M$ a neighborhood of p and some holomorphic coordinates (z, w) centered at p such that the leaves are invariant varieties for the holomorphic 1-form $\omega = zdw - \lambda wdz$, with $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

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Let M be a compact complex surface

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$$dT(\phi) = T(d\phi)$$

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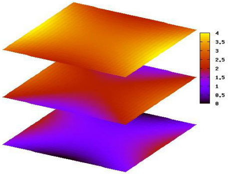
Positivity:

- A $(1,1)$ form ϕ is positive if it is a volume form on every 1 dimensional subvariety
- T is positive if $T(\phi) \geq 0$ for every $\phi \geq 0$

Harmonic Directed Positive $(1,1)$-Currents (HDPC). $\partial\overline{\partial}T = 0$

$$T = \int h_\alpha [V_\alpha] d\mu(\alpha)$$

$[V_\alpha]$ integration currents on plaques, h_α positive harmonic functions, μ a (not invariant) transversal measure.



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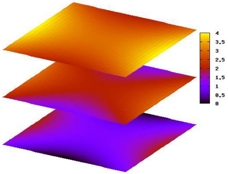

Closed Directed Positive $(1,1)$-Currents (CDPC). $dT = 0$

$$T = \int [V_\alpha] d\mu(\alpha)$$

$[V_\alpha]$ integration currents on plaques, μ an invariant transversal measure.

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Cohomological intersection

- Define a Hilbert space of equivalence classes among real harmonic currents $T_1 \sim T_2$ then $T_1 = T_2 + \partial\overline{\partial}u$ for $u \in L^1_{\mathrm{loc}}$
- Define $[\cdot] \wedge [\cdot]$ cohomological intersection

Theorem

$(X, \mathcal{L}, E)$ in M homogeneous compact Kähler manifold. If $[T] \wedge [T] = 0$ for every T HDPC, then there exists only one equivalence class (nonzero and normalized).
If there are no CDPC, there is only one HDPC of mass one.

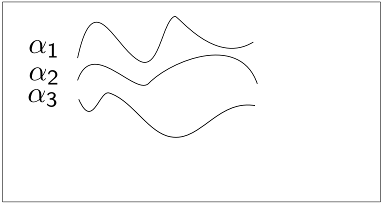
Geometric self-intersection

If $T \wedge_g T = 0$ for every T HDPC, then $[T] \wedge [T] = 0$.
Consider $\Phi_\epsilon \in Aut(M)$ a perturbation of the identity.
Define

$$T \wedge_g T(\psi) = \lim_{\epsilon \to 0} T \wedge_g \Phi_\epsilon^*(T)(\psi) =$$

$$= \lim_{\epsilon \to 0} \int \sum_{p \in J^\epsilon_{\alpha,\beta}} h_\alpha(p) h^\epsilon_\beta(p) \psi(p) d\mu(\alpha) d\mu(\beta)$$

Where $J^\epsilon_{\alpha,\beta}$ denotes the intersection points between Γ_α and Γ^ϵ_β.
If $\sharp J^\epsilon_{\alpha,\beta} \leq N$ independently of ϵ, then $T \wedge_g T = 0$.



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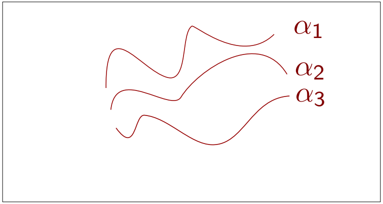

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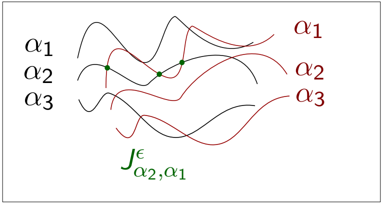

Summary

Summing up

For $(X, \mathcal{L}, E)$ a lamination with singularities

- in M a compact homogeneous Kähler surface,
- with a discrete set singularities,
- no CDPC,

if we find a perturbation of the identity Φ_ϵ such that $\sharp J^\epsilon_{\alpha,\beta} \leq N$ for ϵ small enough then

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Remark: Tits' theorem says $M = \mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^1 \times \mathbb{T}^1, \mathbb{T}^2$

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Theorem. Fornæss-Sibony, P-G

[11][12] Let $(X, \mathcal{L}, E) \subset M$ a minimal transversely Lipschitz lamination with at most hyperbolic singularities embedded in a homogeneous compact Kähler surface.
If there are no CDPC there exists only one HDPC of mass 1.

[11] John Erik Fornæss and Nessim Sibony. "Unique ergodicity of harmonic currents on singular foliations of $\mathbb{P}^2$". In: *Geom. Funct. Anal.* (2010).

[12] P-G. "Ergodicity of laminations with singularities in Kähler surfaces". In: *Math. Z.* (2013).

Corollary

If $(X, \mathcal{L}) \subset \mathbb{P}^1 \times \mathbb{P}^1$ transversely Lipschitz and with no invariant compact curves, then it does not admit CDPC.

Remark: Vanishing of self-intersection of closed currents (invariant measures) was already considered by Hurder and Mitsumatsu[13]

Corollary

If $(X, \mathcal{L}, E) \subset \mathbb{P}^1 \times \mathbb{P}^1$ transversely Lipschitz with no compact invariant curves and hyperbolic singularities, then there exists only one HDPC current of mass one, in particular a unique minimal set.

[13] S. Hurder and Y. Mitsumatsu. "The intersection product of transverse invariant measures". In: *Indiana Univ. Math. J.* 40 (1991).

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- Exceptional minimal set problem: Does there exists any non trivial non singular lamination by Riemann surfaces in $\mathbb{P}^2$? Same question for M hom. comp. Kähler surface. Answer:

[14] S. C. Coutinho and J. V. Pereira. "On the density of algebraic foliations without algebraic invariant sets". In: *J. Reine Angew. Math.* 594 (2006).

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  Is there any lamination by Riemann surfaces in M hom. comp. Kähler surface with no CDPC?
- Genericity of holomorphic foliations: Coutinho and Pereira[14] proved that a generic foliation on algebraic surfaces of high degree has no invariant curves. Therefore the result applies generically for foliations in $\mathbb{P}^1 \times \mathbb{P}^1$.

[14] S. C. Coutinho and J. V. Pereira. "On the density of algebraic foliations without algebraic invariant sets". In: *J. Reine Angew. Math.* 594 (2006).

Thanks for your attention