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The use of mathematics to read the book of nature. About Kepler and snowflakes

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Resum. «El coneixement esta escrit en aquest grandíssim llibre que tenim obert davant dels ulls, vull dir, l'Univers, però no es pot comprendre si abans no s'aprèn a entendre la llengua, a conèixer els caràcters en els quals esta escrit. Està escrit en llengua matemàtica i els seus caràcters son triangles, cercles i altres figures geomètriques, sense els quals es impossible entendre ni una paraula; sense ells es com girar vanament en un fosc laberint», va escriure Galileu (Il Saggiatore, capítol 6, pàg. 4). El 1611, el matemàtic Johannes Kepler, contemporani de Galileu i voraç lector del llibre del món, va escriure el seu llibre mes curt i sorprenent, *The Six-Cornered Snowflake: A New Year's Gift* («El floc de neu de sis puntes. Un regal d'Any Nou»). «Mentre escric això, ha començat a nevar, i molt més copiosament que fa una estona. He estat examinant amb deteniment els petits flocs. Bé, tots han estat caient amb un patró radial, però de dos tipus. Alguns de molt petits, i amb una quantitat indefinida de pues inserides pertot arreu, són de formes senzilles sense plomes i estries i molt fines, i tenen un glòbul una mica mes gros en el centre. Aquests formen la majoria de flocs. Però, esquitxats entre ells, apareixen els del segon tipus, les estrelles amb sis plomes.» (Kepler, 1611) Aquest text de Kepler–molt poc conegut fora de la comunitat matemàtica i física–va marcar una fita en l'ús de les matemàtiques per a entendre alguna cosa del món físic que ens envolta. Amb ell com a mapa, al llarg d'aquest article recorrerem part del terreny explorat per la geometria des del segle III aC fins avui.

Paraules clau: Johannes Kepler · floc de neu de sis puntes · geometria · conjectura de Kepler

Abstract. "Philosophy is written in that great book which ever lies before our eyes – I mean the universe – but we cannot understand it if we do not first learn the language and grasp the symbols, in which it is written. This book is written in the mathematical language, and the symbols are triangles, circles and other geometric figures, without whose help it is impossible to comprehend a single word of it; without which one wanders in vain through a dark labyrinth," wrote Galileo (*Il Saggiatore*, chapter 6, p. 4). In 1611, the mathematician Johannes Kepler, a contemporary of Galileo and voracious reader of the book of the world, wrote his shortest and most surprising book, *The Six-Cornered Snowflake: A New Year's Gift*. "Even as I write these things, it has begun to snow again, and more thickly than before. I have been attentively observing the tiny particles of snow, and although they were all falling with pointed radii, they were of two kinds. Some were exceedingly small, with varying numbers of radii that spread in every direction and were plain, without tufts or striations. These were most delicate, but at the same time joined together at the center in a somewhat larger droplet; and they were the majority. Sprinkled among them were the rarer, six-cornered snowflakes" (Kepler, 1611). This text by Kepler, little known outside the physics and mathematics community, marked a milestone in the use of mathematics to understand a part of the physical world that surrounds us. With this text as a map, this article covers part of the terrain explored by geometry, from the 3rd century AD until today.

Keywords: Johannes Kepler · six-cornered snowflakes · geometry · Kepler's Conjecture

Introduction

> "Philosophy is written in that great book which is constantly open in front of our eyes–I mean the universe–, but we cannot understand it if we do not learn first the language and grasp the symbols in which it is written. The book is written in mathematical language, and the symbols are triangles, circles and other geometric figures, without whose help it is impossible to understand a single word of it; without which one wanders in vain through a dark labyrinth."
>
> (Galileo Galiei, *Il saggitore,* chapter 6, p. 4)

When mathematicians study the book of the universe, in their attempts to read it they search for keys that will allow them to decodify its contents. The keys available to mathematicians are

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the structures that underlie the many different objects contained in the universe–what we, properly, call the *ideas* of things. *Idea* comes from the Greek words ειδω, which means to see, look, or observe, and ειδοζ which means figure, form, aspect, or vision. Behind the actual mountain lies the idea of *mountain*, an abstract sketch, a few lines that allow us to recognize the mountain behind the rocks, trees, or snow that cover it; the difference between *this* mountain and *mountain*, between a circle drawn on a blackboard and *circle*; the difference between the thing and the idea of the thing. Mathematicians seek the ideas of things. With the structures that they find in the universe, mathematicians construct maps that anyone who can read mathematics–because the maps constructed by mathematicians are written in mathematical language, a formal language with the wonderful advantage of being extremely precise and concise–to travel in the universe without getting lost.

One of the greatest seekers of ideas and one the most prominent cartographers of the universe was the mathematician Johannes Kepler (1571–1630), a contemporary of Galileo and a voracious reader of the book of the world. In 1611, Kepler wrote a surprisingly short and charming essay: *The Six-Cornered Snowflake. A New Year's Gift*. This text of Kepler's–barely known outside the community of mathematicians and phycisists–was a milestone in the use of mathematics to understand the universe around us.

Kepler's brief tract, now considered one of the pioneering works in glaceology and theoretical mineralogy, elegantly illustrates how to place our reflections about the world around us in a scientific framework and how to mathematically phrase our questions about the nature of the universe. In the process of analyzing what he calls "the formative faculty" at work in the universe, specifically, in the shaping of snowflakes, Kepler casually takes us through some of the major classical works examining the nature or shape of the universe as perceived during his time: the atomism of Epicurus, *Timeaus* of Plato, *Elements* of Euclid, *The Sand Reckoner* of Archimedes [1], and the *Collection* of Pappus [7].

In mathematics, good solutions to problems tend to produce new questions. Thus, the answers to some of the questions about the universe that Kepler found in the works and ideas of his mathematical ancestors led him to pose new questions, both to himself and to the mathematical community in *The Six-Cornered Snowflake*. Using the book as a map, we will explore part of the tiny speck of the universe that, from Antiquity to our times, geometers have explored in search of ideas. The discovery of new semi-regular solids and the statements of what became known as *Kepler's Conjecture* and *The Beehive Conjecture* are some of the highlights that we will encounter in our exploration.

The Six-Cornered Snowflake of Kepler

To the honorable Counselor at the Court of his Imperial Majesty, Lord Matthäus Wachker von Wackenfels, a Decorated Knight and Patron of Writers and Philosophers, my Lord and Benefactor.

> "I am well aware of how fond you are of Nothing, not so much on account of its inexpensive price as for the charming and subtle *jeu d'esprit* of playful Passereau. Thus, I can easily tell that a gift will be the more pleasing and welcome to you the closer it comes to Nothing.
>
> Whatever it is, then, that will please you by its evocation of nothing, ought to be both small and insignificant, inexpensive and fleeting–that is to say, almost nothing. And since there are many such things in the realm of nature, a choice is to be made among them." [4]

Kepler's search for something close to Nothing while at the same time worthy of thought, which he hoped to give his benefactor as a gift, leads Kepler to consider but then to reject the atoms and the four elements as described in the Greek texts. In doing so, he wittily jokes about some of the best known classical theories describing the universe.

After calling "truly Nothing" the atoms of Epicurus (341–270 BCE, of whom we know about through Lucretius' *De Rerum Natura* [5]), he goes through each of the four elements, starting with earth. But, as Kepler reflects in a clear reference to one of the jewels of antiquity, *The Sand Reckoner*, how could one pick a grain of sand knowing that this would confuse Archimedes' calculations? In this short text, barely a dozen pages, Archimedes (287–212 BCE) proves that the number of grains of sand that would fill the universe (defined as the sphere of the fixed stars) is finite, and he determines an upper boundary for it. This may seem to us, readers of the 21st century, not terribly impressive, but, in fact, it was a significant achievement, if only because the Greeks of 3rd century BCE still assigned a letter to each number and, so, Archimedes had to devise a new system of numerical notation capable of naming very large numbers (Archimedes bases his calculations on the assumption that 10,000 grains of sand fit within a poppy seed, which in turn measures 1/40th of a finger breadth).

The next element considered by Kepler is fire, whose particles–which he describes as pyramidal shapes, following Plato's *Timaeus*–when burnt, produce too much ash to be considered Nothing. Smoke (air) is too expensive, a drop of water too big.

> "While anxiously considering these matters, I crossed over the bridge, mortified by my incivility in having appeared before you without a New Year's gift, except perhaps (to keep playing the same chord) the one that I always bring you, namely, Nothing. Nor was I able to think of something that, while being next to Nothing, would yet allow for subtle reflection. Just then, by a happy occurrence, some of the vapor in the air was gathered into snow by the force of the cold, and a few scattered flakes fell on my coat, all six-cornered, with tufted radii. By Hercules! Here was something smaller than a drop, yet endowed with a shape. Here, indeed, was a most desirable New Year's gift for the lover of Nothing, and one worthy as well of a mathematician (who has Nothing, and receives Nothing), since it descends from the sky and bears a resemblance to the stars."

> "Our question is why snowflakes, when they first fall, before they are entangled into larger clumps, always come down with six-corners and with six radii, tufted like feathers."

> "To see how these questions should be decided, let us make use of some well-known examples. But let us present them in a geometric manner, for a digresssion of this kind will contribute greatly to our inquiry." ([4], p. 31)

The first of the examples analyzed by Kepler is that of the beehives, made out of two stacks of cells with hexagonal openings attached by their backs (Fig. 1). After examining carefully the entrance and bottoms of each of the stacks of cells, Kepler gives a clear description of the geometric shape of each small prism. He then focuses on the rhombic shape of their peculiar end walls (Fig. 2).

> "Intrigued by these rhombi, I began looking into geometry to see whether a body resembling the five regular solids and the fourteen Archimedean solids could be constructed exclusively from rhombi. I discovered two." ([4], p. 133)

The geometry known to Kepler and his European contemporaries was that of the Greeks as explained in *Elements* of Euclid and *Mathematical Collection* of Papus. The origin of Greek geometry (measurement of land) is explained by Herodotus.

> "This king also (they said) divided the country among all the Egyptians by giving each an equal parcel of land, and made this his source of revenue, assessing the payment of a yearly tax. And any man who was robbed by the river of part of his land could come to Sesostris and declare what had happened; then the king would send men to look into it and calculate the part by which the land was diminished, so that thereafter it should pay in proportion to the tax originally imposed. From this, in my opinion, the Greeks learned the art of measuring land." (Herodotus, Book 2, p.109 [3])

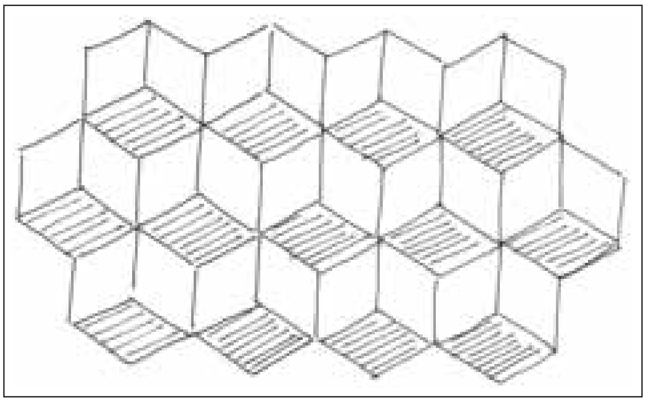

Fig. 1. Stack of beehive cells as seen from behind.

Measuring the effect of the tides of the river on the parcels of land led the Egyptians to the study of the area of polygons, as the problems that appear in the Rhind Mathematical Papyrus (1800 BCE) illustrate (Fig. 3).

The Greeks took from the Egyptians the study of geometry and then carried on their own research. In the 7th century BCE, Thales studied how to determine the distance from the coast to a ship, developing the famous Thales Theorem that we all have learned in school (Fig. 4).

But, what happens if the ship is moving along a straight line and then changes its course? To the observer located on top of the tower on land, the position triangle turns into the vertex of a solid body. Problems like this led geometers to the study of regular solids. Regular solids are convex bodies having similar regular polygons as faces and equal angles at all their vertices. While there are infinitely many regular polygons (the equilateral triangle, the square, the regular pentagon, the regular hexagon, etc.), there are only five regular solids: the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron. They are also known as the Platonic solids, because they are the figures used by Plato in the model of the universe that he describes in *Timaeus* (53–63 AD). Four of them give shape

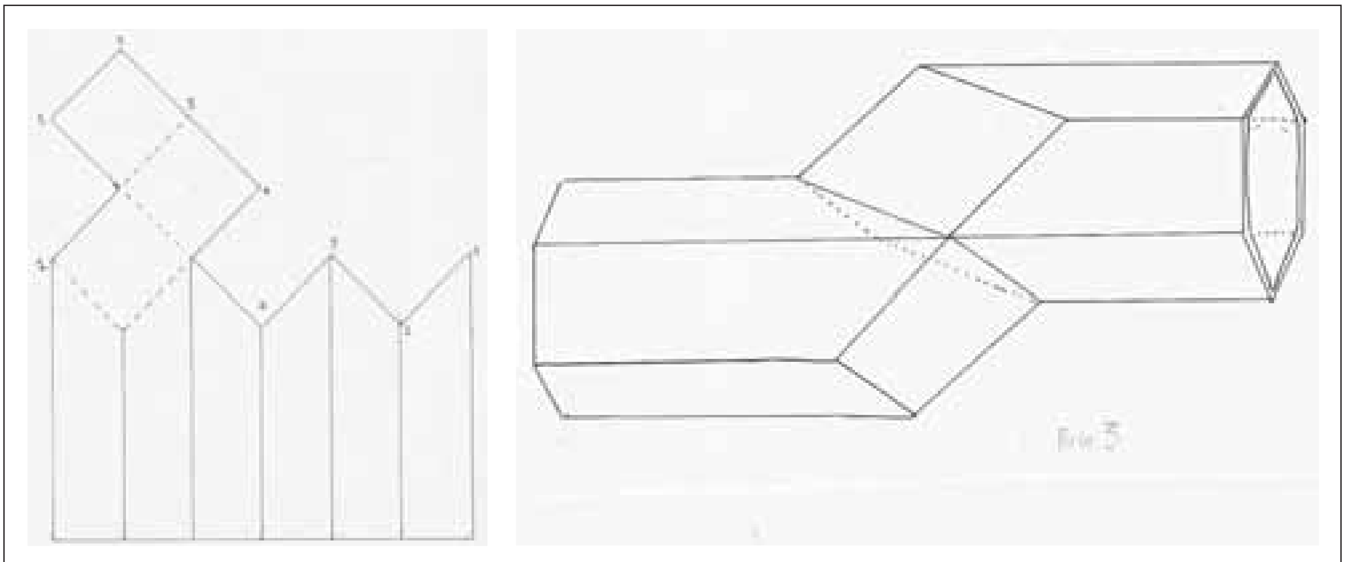

Fig. 2. Rhombic shape of the end walls of beehive cells.

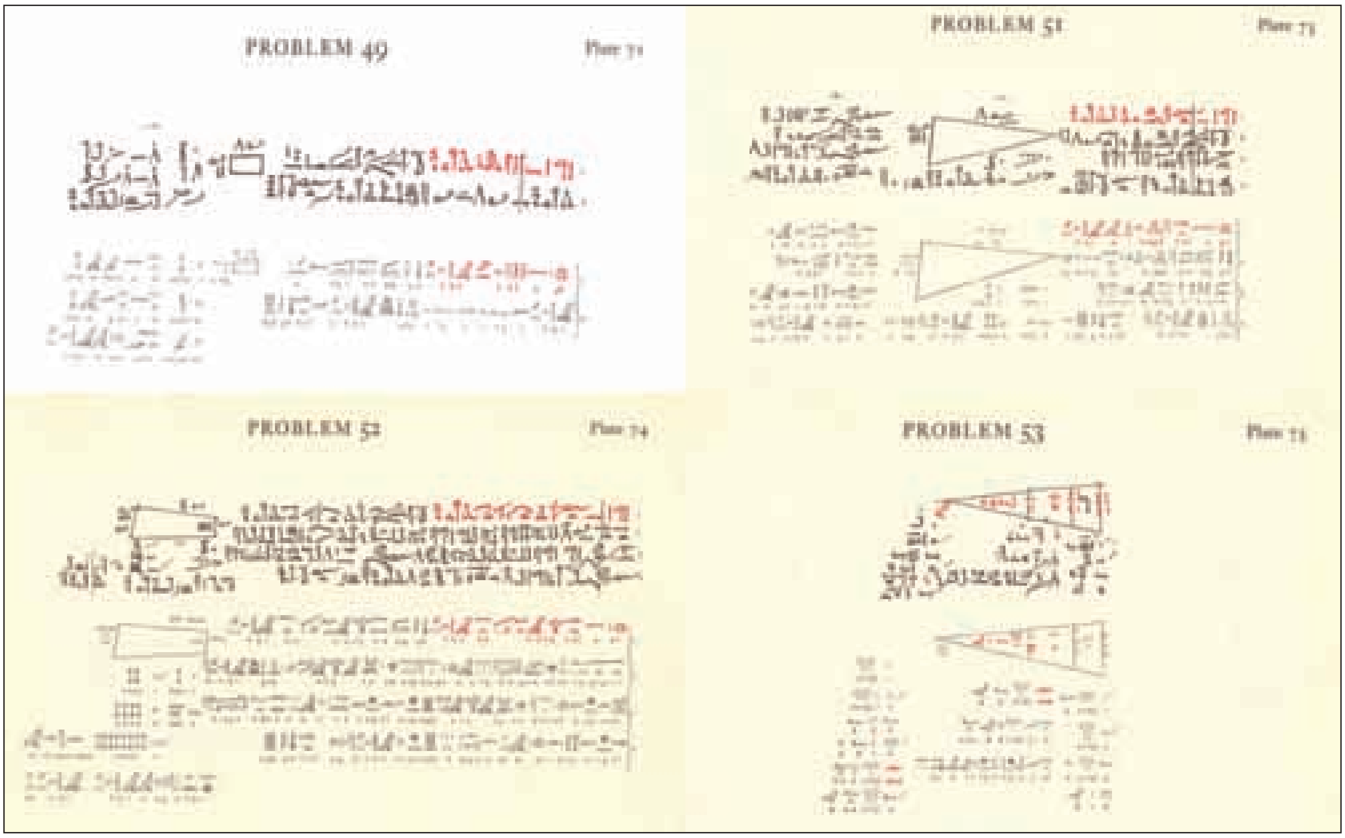

Fig. 3. Rhind Mathematical Papyrus (1800 BC).

to the particles of the elements that compose the universe: the tetrahedron (fire), the cube (earth), the octahedron (air), and the icosahedron (water), while the fifth regular solid, the dodecahedron, contains all that there is. The study of the five regular solids can be traced to the *Elements* (Alexandria, 300 BCE), the text in which Euclid collected all of the geometry that had been developed by the Greeks up to his time.

The next great Greek Geometer was Archimedes. All that we know of his work has come to us through Pappus. Archimedes studied what are known as "semi-regular" solids, i.e., those formed by equilateral and equiangular polygons of more than one kind. He found thirteen of them –although Kepler, mistakenly, cites fourteen. These are now known as the Archimidean solids and they have come down to us in Book V of Pappus' *Collection* [7]. Kepler added two new semi-regular solids to the Platonic and Archimedean polyhedra, the rhombic dodecahedron and the rhombic triacontahedron, formed, respectively, by twelve and thirty identical rhombi as faces (Fig. 5).

It is clear that rhombic faces appear in nature as a product of the struggle of round organic shapes, such as the seeds of pomegranates and peas, for space, pressing agaisnt each other, as Kepler observes (Fig. 6). But, what if these spherical shapes struggling for space are made out of solid material? How should we pack them in order to optimize the use of space?

> "Now, if you should go on to the tighest possible solid arrangement, and pile layer upon layer of those that have been gathered first on a plane, their arrangement will be either square or triangular" (Fig. 7) ([4], p. 55)

This statement of Kepler–which, after all, is simply telling us that the best way to stack oranges on a table is the way or-

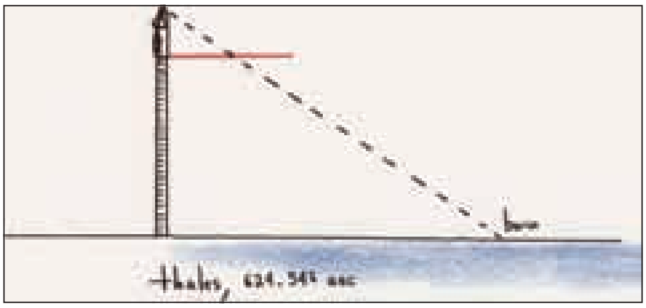

Fig. 4. Thales' determination of the distance from the coast to a ship, which led to the development of the famous Thales Theorem.

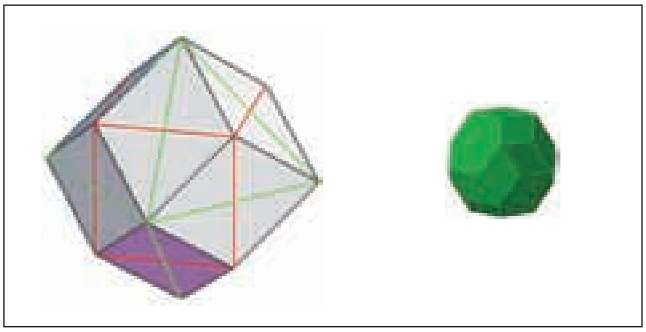

Fig. 5. The rhombic dodecahedron and the rhombic triacontahedron.

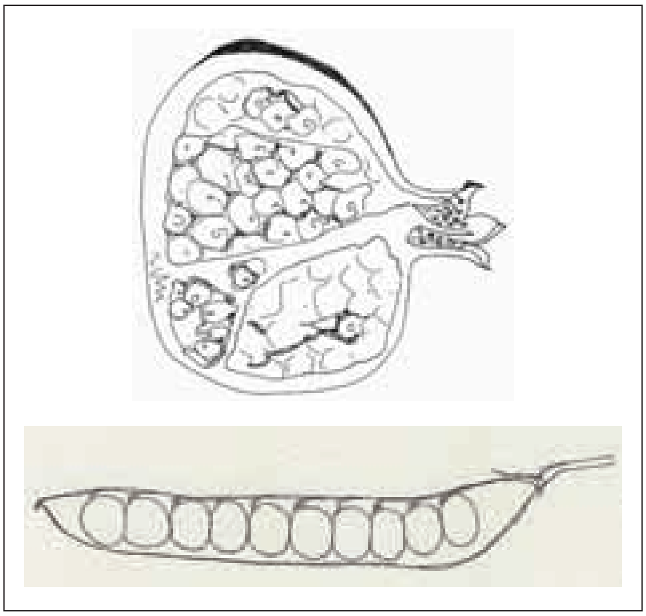

Fig. 6. Rhombic faces in pomegranates and peas.

anges are stacked on any market table!–became known within the mathematical community as Kepler's Conjecture. Over time, this conjecture became so important as to form one of the 23 problems (the 18th) that, acording to the the famous list produced by Hilbert in 1900, would guide mathematical research during the coming century.

In 1998, Thomas Hales (USA) produced a proof of Kepler's Conjecture that, as mathematical proofs go, stands as an exceptional case. Hales' proof has two parts, one is mathematical and the other computational. The mathematical part was published in *Annals* [2], one of the most prominent mathematical journals (http://annals.princeton.edu/annals/2005/162-3/p01.xhtml), and there is no doubt about its being correct. On the other hand, the computational part is so technical that, after years of studying it, the editors of *Annals* could not decide upon its veracity. While it seemingly has no mistakes, the techniques it uses are too hard to evaluate. Thus, the editors of the journal reached a decision, unique in the history of jounals of mathematics, to publish the mathematical part and include in their web page a link to the computational part. The complete proof was finally obtained in 2006 and was published in the computation journal *Discrete and Computational Geometry* and "it is generally considered sound."

From the packing of spheres, Kepler goes back to the packing of beehive cells, stating the second of the two famous conjectures that we find in his essay.

> "The only figures that can fill up a plane without leaving empty spaces are the triangle, the square, and the hexagon. Of these, the hexagon is the most capacious, and bees avail themselves of this capacity to store their honey.
>
> This reasoning can be extended to a consideration of solid figures, as follows: the only way to divide a solid space without leaving any gaps is by means of cubes and rhombic figures, and rhombic figures are more capacious than cubes." ([4], p. 61)

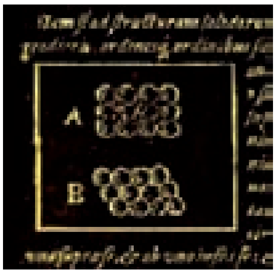

Fig. 7. (A) Square and **(B)** triangular arrangements of spherical shapes.

In this statement, Kepler provides an answer, albeit without proof, to the question already posed in Antiquity, known as *The Honeycomb Conjecture*: Which geometric figure has a minimum perimeter that divides a plane in equal regions with a given area? Since the begining of Geometry, the answer was believed to be the hexagon.

> "There being then three figures capable by themselves of exactly filling up the space about the same point, the bees by reason of their instinctive wisdom chose for the construction of the honeycomb the figure which has the most angles, because they conceived that it would contain more honey than either of the two others."
>
> Bees, then, are aware of the very fact, through its practical application, that the hexagon is greater than the square and the triangle and will hold more honey for the same expenditure of material used in constructing the different geometric figures. We, however, claiming as we do a greater share of wisdom than bees, will investigate a problem of even broader application, namely: "of all equilateral and equiangular plane figures having an equal perimeter, that which has the greater number of angles is always greater, and the greatest plane figure of all those which have a perimeter equal to that of the polygons is the circle." ([7], Introduction to Book V)

The next mathematician, after Kepler, to address this question, was Colin MacLaurin ([6]). He was, in fact, the first to assert that not only the hexagonal opening of the beehive cell is the best possible, but that also the rhombic keel has the property of being *isoperimetric*; that is, the cell as a whole is optimal, in the sense of being the most capacious among those prisms with the same surface area, thus minimizing the amount of wax required to build it (Fig. 8).

To everyone's surprise, 200 years later, in 1943, the Hungarian mathematician László Fejes Tóth proved the Honeycomb Conjecture, with its restriction to figures with straight sides. He also proved that the bases of the cells are not optimal shapes. Toth found an alternative solid figure that would allow the bees to save 0.35% more wax [8]. Thomas C. Hales solved, finally, the Honeycomb Conjecture in 1990, in the process of solving Kepler's Conjecture.

The Beehive Conjecture is the two-dimensional version of what is known as Kelvin's problem, named after Lord Kelvin: what surface of minimum area distributes space in equal regions of a given volume?

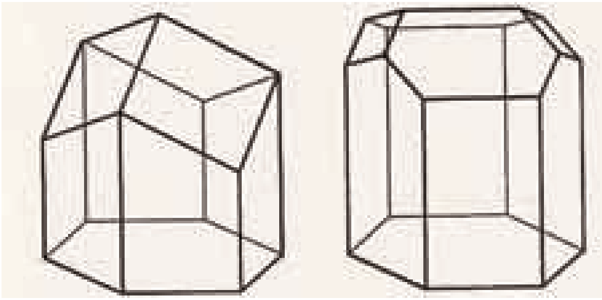

Fig. 8. Beehive cells withan isoperimetric rhombic keel.

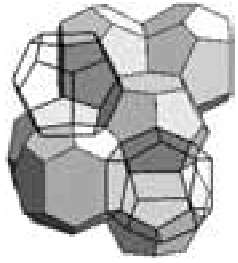

Fig. 9. Tetrakaidecaedra, as well as dodecaedra: arrangement of the molecules of water in chlorine hydrate crystal.

Kelvin himself proposed an answer to the question in 1887: the *tetrakaidecahedron*, a truncated octahedron with 14 faces, 6 of them square and 8 of them hexagonal. It was generally believed that the tetrakaidecahedron solved the problem until, in 1993, in the course of their research the Irish chemists D. Weaire and R. Phelan found that the arrangement of water molecules in chlorine hydrate crystals in tetrakaidecaedra as well as dodecaedra was 0.3% better than in the structure proposed by Kelvin (Fig. 9). Weaire and Phelan's structure is, to date, still the best possible solution of Kelvin's problem [9].

In the final part of his essay, Kepler addresses the star-like shape of snowflakes.

"Even as I write these things, it has begun to snow again, and more thickly than before. I have been attentively observing the tiny particles of snow, and although they were all falling with pointed radii, they were of two kinds. Some were exceedingly small, with varying numbers of radii that spread in every direction and were plain, without tufts or striations. These were most delicate, but at the same time joined together at the center in a somewhat larger droplet; and they were the majority. Sprinkled among them were the rarer, six-cornered snowflakes, all of which were flat, not just as they floated about but also as they fell, and even the little tufts were on the same plane as their stalk." ([4], p. 572) (Fig. 10)

Kepler suggests that cold may be the cause for the six-cornered patterns of snowflakes, and his analysis of why water vapor freezes in six-fold shapes–a question later addressed and answered practically by the Japanese phycisist Ukiro Nakaya (1900–1962)– led him to research concave polyhedra. He thus discovered two figures that he was to describe a few years later in *Harmonice Mundi*.

Both the Platonic and the Archimidean solids, as all solids considered in Antiquity, were convex bodies. But what happens if we allow our figures to be concave, that is, star-shaped? Kepler re-discovered two of them, the small stellate dodecahedron (which we can think of as a dodecahedron with a pentagonal pyramid built on each face, and which can be seen in Paolo Uccello's mosaic on the floor of San Marco's Cathedral in Venice, ca. 1430), and the great stellate dodecahedron (which we can think of as an icosahedron with a triangular pyramid on each face, and which had been already published by Wenzel Jamnitzer in 1568).

The discovery of these two stellate polyhedra posed a new question: how many regular polyhedra, either convex or stellate, are there? In 1809, the French mathematician Louis Poinsot rediscovered two more, the great dodecahedron and the great icosahedron. In 1813, Augustin-Louis Cauchy proved them to be sophisticated stellate versions of the dodecahedron and the icosahedron. He also proved that the four stellate solids rediscovered by Kepler and Poinsot, known today as the Kepler-Poisont solids, are the only possibilities for regular star polyhedra and, consequently, that there are only nine regular polyhedra: five convex (the Platonic solids) and four stellate (the Kepler-Poisont solids).

Kepler offers us one more mathematical surprise. In comparing the six-fold and five-fold patterns that we find in the univers, he writes:

"There are two regular solids, the dodecahedron and the icosahedron, the former of which is composed expressly of pentagons, the latter of triangles, but joined together in a pentagonal arrangement. The structure of each of these solids, as well in fact as that of the pentagon, cannot be produced without the proportion that modern geometers call "divine." It is ordered in such a way that the two smaller terms of a continuous proportion taken together produce the third; and subsequently, any two adjacent terms add up to the one following, to infinity, since the same proportion holds forever. It is impossible to give a perfect example in numbers. The further we proceed from the unit, the more perfect our example becomes. Let the smallest terms be 1 and 1, which you must think of as unequal. Add them together and they make 2. Add to this the larger 1 to make 3. Add 2 and they make 5. Add 3 and they make 8. Add 5 and they make 13. Add 8 and they make 21. As 5 is to 8, so is 8 approximately to 13; as 8 is to 13, so is 13 approximately to 21; and so on, forever." ([4], p. 67)

The divine proportion! Known to the Greek geometers as the mean and extreme ratio, and described with great precision by Euclid in 300 BCE (Book VI of the *Elements*: "a straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less"), the numerial value of this proportion is: $a/b=(1+\sqrt{5})/2$. As an example, the ratio between the side of a regular pentagon and its diagonal is $1:(1+\sqrt{5})/2$. The numerical value of this proportion is usually denoted by the Greek letter φ and it is an irrational number. Kepler seems to have been among the

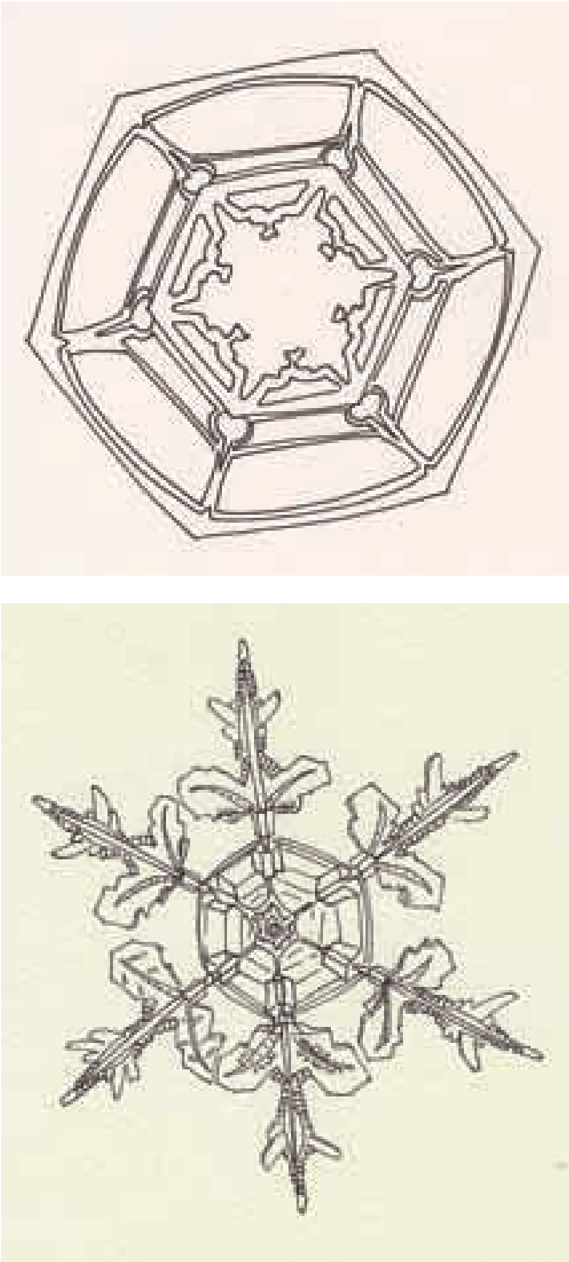

Fig. 10. Illustrations of Kepler's six-cornered snowflakes.

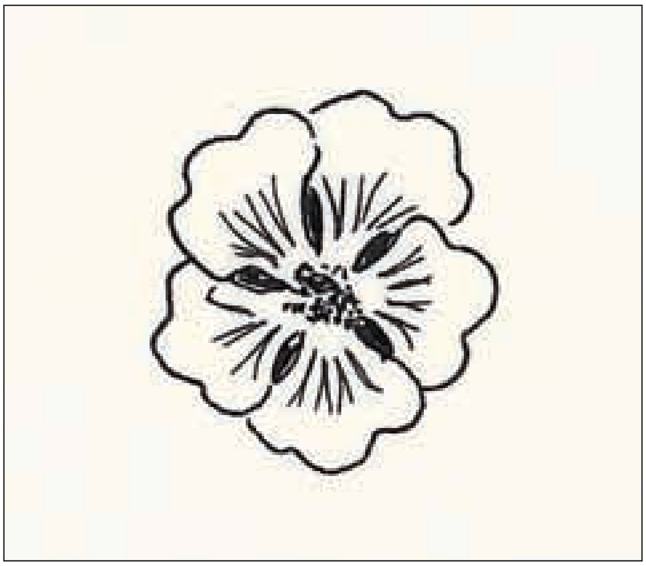

Fig. 11. The five-cornered shape of flowers.

first to observe that in the numerical sequence 1,1, 2, 3, 5, 8, 13, 21..., introduced by Leonardo Fibonacci in his *Liber Abaci* (1202), the ratio between successive terms gradually approaches φ, also known nowadays as the golden number (a name coined by the Rumanian engineer and diplomat Matila Ghyka in his 1931 book, *Le nombre d'or*).

Epilogue

I would like to close our text with Kepler's words.

"By contrast, if one inquires why it is that all trees and shrubs (or certainly the majority) produce flowers in a five-cornered shape, which is to say with five petals; and why, in apple and pear trees, the fruit follows the same five-fold pattern of the flower, or a related one of ten (for inside there are five little chambers to contain the seeds, and ten veins, as can also be seen in cucumbers and other plants of this kind); *here*, I say, it becomes appropriate to consider the beauty and the peculiar nature of the figure that characterizes the soul of these plants." (p. 69) (Fig. 11)

"When means are directed to an established end, there is no accident, but only order, pure reason, and a clear design. No such purpose can be observed in the shaping of the snowflake, since the six-cornered arrangement does not make it last longer, or produce a fixed, natural body of definite and lasting shape. My response is that the formative principle does not act only for the sake of ends, but also for the sake of adornment." (p. 91)

"But the formative faculty of the earth does not embrace one figure, but is practised and well-versed in the whole of geometry." (p. 111)

And, finally, with his last sentence:

"Let the chemists, then, tell us whether there is any salt in snow, and what kind, and what shape it takes. I have knocked on the doors of chemistry, and seeing how much remains to be said on this subject before we know the cause, I would rather hear what you think, my most ingenious man, than wear myself out with further discussion."

References

[1] Archimedes (c250 BCE) The Sand Reckoner. In: TL Heath (ed) The Works of Archimedes. Cambridge University Press 221–232. Available online at [http://www.ar-

chive.org/stream/worksofarchimede029517mbp#page/n3/mode/2up]
[2] Hales TC (2005) A Proof of the Kepler Conjecture. Ann Math 162:1065-1185
[3] Herodotus (440 BCE) The History of Heterodotus. Translated by George Rawlinson. Available online at [http://classics.mit.edu/Herodotus/history.html]
[4] Kepler J (2010) The Six-Cornered Snowflake. A New Year's Gift (Kepler, 1610). Bilingual edition Latin/English, Paul Dry Books, Philadelphia
[5] Lucretius (c 50 BCE) De Rerum Natura. Available online at [http://classics.mit.edu/Carus/nature_things.html]
[6] MacLaurin C (1743) Of the bases of the cells wherein the bees deposit their honey. Transactions of the Royal Society of London
[7] Pappus (1981) Mathematical Collection. In: Heath TL (ed) A History of Greek Mathemathics, vol. 2. Dover Publications, London
[8] Toth LF (1964) What the bees know and what they do not know. Bull Amer Math Soc 70, 4:468-481. Also available online at [http://projecteuclid.org]
[9] Weaire D, Phelan R (1994) A counter-example to Kelvin's conjecture on minimal surfaces. Philosophical Magazine Letters 69, 2:107-110

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