

Some polynomial inequalities on real normed spaces

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ABSTRACT

We consider various inequalities for polynomials, with an emphasis on the most fundamental inequalities of approximation theory. In the sequel a key role is played by the generalized Minkowski functional $\alpha(K, x)$, already being used by Minkowski and contemporaries and having occurred in approximation theory in the work of Rivlin and Shapiro in the early sixties. We try to compare real, geometric methods and complex, pluripotential theoretical approaches, where possible, and formulate a number of questions to be decided in the future. An extensive bibliography is given to direct the reader even in topics we do not have space to cover in more detail.

Key words: convex body, generalized Minkowski functional, polynomials on normed spaces, Chebyshev problem, Bernstein-Szegő inequality, maximal chord, minimal width, Markov inequality, geometry of Banach spaces.

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1. Introduction

1.1.

In our present work, as well as throughout and all over approximation theory, a distinguished role is played by the (univariate) Chebyshev polynomials of the first kind. These can be defined as

$$\begin{aligned} T_n(x) &:= 2^{n-1} \prod_{j=1}^n \left(x - \cos \left(\frac{(2j-1)\pi}{2n} \right) \right) = \\ &= \frac{1}{2} \left\{ (x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n \right\}, \end{aligned} \tag{1}$$

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while the most used expression for them is the first part of the formula

$$T_n(x) = \begin{cases} \cos(n \arccos x), & |x| \leq 1 \\ \operatorname{sgn}(x)^n \cosh(n \cosh^{-1} |x|), & |x| \geq 1 \end{cases} \quad (2)$$

It is well-known that regarding the modulus of $p(x)/\|p\|_{[-1,1]}$, T_n is extremal simultaneously for all x with $|x| > 1$. For this we refer to [50, Theorem 1.2.2, Chapter 5] or [72, (2.37), p. 108]. The very same Chebyshev polynomial has many extremal properties, see for instance [72, §2.7]. In particular, it is also extremal concerning its “speed of growth towards infinity”, which can be precisely described by its leading coefficient. Since for all $p(x) = \sum_{k=1}^n a_k x^k \in \mathcal{P}_n(\mathbb{R})$ we have $a_n = \lim_{x \rightarrow +\infty} p(x)/x^n$, a polynomial is extremal concerning its growth towards infinity iff the leading coefficient is extremal. Thus, for

$$\max \left\{ a_n : p(x) = \sum_{k=0}^n a_k x^k \in \mathcal{P}_n(\mathbb{R}), \|p\|_{[-1,1]} \leq 1 \right\}, \quad (3)$$

Chebyshev’s polynomial is again the extremal case.

1.2.

In the present survey we focus on extensions of the most well-known and classical inequalities of approximation theory for algebraic polynomials on \mathbb{R} to the case of infinitely many variables, i.e. to normed spaces. In all what follows, X is a real normed space, $X^* = \mathcal{L}(X, \mathbb{R})$ is the usual dual space, and $S := S_X$, $S^* := S_{X^*}$, $B := B_X$ and $B^* := B_{X^*}$ are the unit spheres and (closed) unit balls of X and X^* , respectively. Moreover, $\mathcal{P} = \mathcal{P}(X)$ and $\mathcal{P}_n = \mathcal{P}_n(X)$ will denote the space of continuous (i.e., bounded) polynomials of free degree and of degree at most n , respectively, from X to \mathbb{R} .

There are several ways to introduce continuous polynomials over X , one being the linear algebraic way of writing

$$\mathcal{P}_n := \mathcal{P}_0^* + \mathcal{P}_1^* + \cdots + \mathcal{P}_n^*, \quad \text{and} \quad \mathcal{P} := \bigcup_{n=0}^{\infty} \mathcal{P}_n \quad (4)$$

with \mathcal{P}_k^* (or, in another notation, $\mathcal{P}^k(X; \mathbb{R})$) denoting the space of homogeneous (continuous) polynomials of degree (exactly) $k \in \mathbb{N}$. That is, one considers bounded k -linear forms

$$L \in \mathcal{L}(X^k \rightarrow \mathbb{R}) \quad (5)$$

together with their “diagonal functions”

$$\widehat{L} : X \rightarrow \mathbb{R}, \quad \widehat{L}(x) := L(x, x, \dots, x) \quad (6)$$

and defines \mathcal{P}_k^* as the set of all \widehat{L} for L running $\mathcal{L}^k(X) := \mathcal{L}(X^k \rightarrow \mathbb{R})$. In fact, it is sufficient to identify equivalent linear forms (that is, those having identical diagonal functions) by selecting the unique symmetric one among them: in other words, to let

L run over $\mathcal{L}^s({}^kX)$ denoting (real) *symmetric* k -linear forms. Building up the notion of polynomials that way is equivalent to the definition

$$\mathcal{P}_n := \{p : X \rightarrow \mathbb{R} : \|p\| < \infty, p|_{Y+y} \in \mathcal{P}_n(\mathbb{R}) \text{ for all } Y \leq X, \dim Y = 1, y \in X\} \quad (7)$$

or to the definition arising from combining (4) and

$$\mathcal{P}_k^* := \{p : X \rightarrow \mathbb{R} : \|p\| < \infty, p|_Y \in \mathcal{P}_n^*(\mathbb{R}^2) \text{ for all } Y \leq X, \dim Y = 2\} . \quad (8)$$

Here and throughout the paper for any set $K \subset X$ and function $f : X \rightarrow \mathbb{R}$ we denote, as usual,

$$\|f\|_K := \sup_K |f| \quad \text{and} \quad \|f\| := \|f\|_B .$$

For equivalent definitions of and introduction to polynomials over real normed spaces see [26, Chapter 1] and also [24, 37, 39, 82]. In particular, it is well-known that

$$\|\widehat{L}\| \leq \|L\| \leq C(n, X) \|\widehat{L}\| \quad \text{for all } L \in \mathcal{L}^s({}^nX) , \quad (9)$$

and that $C(n, X) \leq n^n/n!$, [26], while $C(n, X) = 1$ if X is a Hilbert space (Banach's Theorem, see [4, 26, 34]). Similarly to (9), one can consider special homogeneous polynomials which can be written as products of linear forms, i.e. $L(x_1, x_2, \dots, x_n) = \prod_{j=1}^n f_j(x_j)$ with $f_j \in \mathcal{L}(X \rightarrow \mathbb{R})$. Then $\|L\| = \prod_{j=1}^n \|f_j\|$, i.e. the product of the norms, and one compares to the norm of the corresponding homogeneous polynomial, i.e. to $\|\widehat{L}\| = \|\prod_{j=1}^n f_j\|$. Note that here L is far from being symmetric, and this yields to an essentially different question, with the similarly defined polarization constants now ranging up to n^n , see e.g [6, 16, 76].

These polarization problems are typical examples of genuinely multivariate inequalities, as in dimension 1 they simply degenerate. Since our focus is different, we direct the reader's attention to [3, 6, 7, 8, 16, 47, 27, 46, 56, 70] and also to [45] right in this volume. However, passing by we note that (9) already implies that a (symmetric) n -linear form L is bounded iff its diagonal function – i.e. the associated homogeneous polynomial defined by \widehat{L} – is bounded, and that a polynomial

$$p = p_0^* + p_1^* + \dots + p_n^* \quad \text{with} \quad p_k^* = \widehat{L}_k , \quad (10)$$

$L_k : X^k \rightarrow \mathbb{R}$ being a (symmetric) k -linear mapping, is bounded iff L_k are such for all $k = 1, \dots, n$. Hence in all what follows we are free to talk about boundedness or continuity of these polynomials without specifying in detail whether p , or p_k^* are assumed to be continuous or bounded.

1.3.

In the following classical inequalities of approximation theory the usual condition of normalization is that $\|p\|_I \leq 1$, where $p \in \mathcal{P}_n(\mathbb{R})$ and $I = [-1, 1]$ (or, sometimes, some other interval $[a, b]$). In \mathbb{R} all the convex bodies are just intervals, and linear substitution allows to restrict ourselves to I , but in higher dimensions there is a great variety of convex bodies to deal with. Recall that a set $K \subset X$ is called *convex body* in a normed space (or in a topological vector space) X if it is a bounded, closed convex set that has a non-empty interior.

The convex body K is *symmetric*, iff there exists a center of symmetry x so that reflection of K at x leaves the set invariant, that is, $K = -(K - x) + x = -K + 2x$. In the following we will term K to be *centrally symmetric* if it is symmetric with respect to the origin, i.e. if $K = -K$. This occurs iff K can be considered the unit ball with respect to a norm $\|\cdot\|_{(K)}$, which is then equivalent to the original norm $\|\cdot\|$ of the space X in view of $B_{X, \|\cdot\|}(0, r) \subset K \subset B_{X, \|\cdot\|}(0, R)$.

The *central symmetrization* or *half difference body* (cf. [33], p. 135 and 362, respectively) of a set K in a normed space X is

$$C := C(K) := \frac{1}{2}(K - K) := \left\{ \frac{1}{2}(x - y) : x, y \in K \right\}. \quad (11)$$

The central symmetrization of K is centrally symmetric with respect to the origin. In case K is a convex body, we also have $0 \in \text{int } C^1$. On the other hand, even though K is assumed to be closed, C is not necessarily closed (c.f. [68, Section 6]), hence C is not a convex body in general. Nevertheless, the closure \overline{C} of C is a symmetric convex body, which is also *fat*, and $\text{int } C \subset C \subset \text{int } \overline{C} = \overline{C}$.

The “*maximal chord*” of K in direction of $v \neq 0$ is

$$\begin{aligned} \tau(K, v) &:= \sup\{\lambda \geq 0 : \exists y, z \in K \text{ s.t. } z = y + \lambda v\} = \\ &= \sup\{\lambda \geq 0 : K \cap (K + \lambda v) \neq \emptyset\} = \\ &= \sup\{\lambda \geq 0 : \lambda v \in K - K\} = 2 \sup\{\lambda > 0 : \lambda v \in C\} = \\ &= 2 \max\{\lambda \geq 0 : \lambda v \in \overline{C}\} = \tau(C, v). \end{aligned} \quad (12)$$

Usually $\tau(K, v)$ is not a “maximal” chord length, but only a supremum, however we shall use the familiar finite dimensional terminology (see for example [83]).

The *support function* to K , where K can be an arbitrary set, is defined for all $v^* \in X^*$ (sometimes only for $v^* \in S^*$) as

$$h(K, v^*) := \sup_K v^* = \sup\{\langle v^*, x \rangle : x \in K\}, \quad (13)$$

and the *width of K in direction $v^* \in X^*$* (or $v^* \in S^*$) is

$$\begin{aligned} w(K, v^*) &:= h(K, v^*) + h(K, -v^*) = \sup_K v^* + \sup_K (-v^*) = \\ &= \sup\{\langle v^*, x - y \rangle : x, y \in K\} = 2h(C, v^*) = w(C, v^*). \end{aligned} \quad (14)$$

Let us introduce the notations

$$X_t(v^*) := \{x \in X : \langle v^*, x \rangle \leq t\}, \quad X(K, v^*) := X_{h(K, v^*)}(v^*). \quad (15)$$

Clearly the closed halfspace $X(K, v^*)$ contains K , and the hyperplane

$$H(K, v^*) := H_{h(K, v^*)}(v^*), \quad H_t(v^*) := \{x \in X : \langle v^*, x \rangle = t\} = \partial X_t(v^*) \quad (16)$$

is a supporting hyperplane² to K .

¹Throughout the paper we denote when convenient $C(K)$, $\tau(K, v)$, $\alpha(K, x)$, $w(K, v^*)$, etc. by C , τ , α , w , etc., respectively.

²Note that throughout the paper we mean “supporting” in the weak sense, that is, we do not require $K \cap H(K, v^*) \neq \emptyset$, but only $\text{dist}(K, H(K, v^*)) = 0$. The same convention is in effect for other supporting objects as halfspaces, layers etc.

A *layer* (sometimes also called *slab*, *plank* or *strip*) is the region of X enclosed by two parallel hyperplanes, i.e.

$$L_{r,s}(v^*) := \{x \in X : r \leq \langle v^*, x \rangle \leq s\} = X_s(v^*) \cap X_{-r}(-v^*), \quad (17)$$

while the *supporting layer* or *fitting layer* of K with normal v^* is

$$\begin{aligned} L(K, v^*) &:= X(K, v^*) \cap X(K, -v^*) = L_{-h(K, -v^*), h(K, v^*)}(v^*) = \\ &= \{x \in X : -h(K, -v^*) \leq \langle v^*, x \rangle \leq h(K, v^*)\}. \end{aligned} \quad (18)$$

1.4.

In \mathbb{R} the position of a point $x \in \mathbb{R}$ with respect to the "convex body" I can be expressed simply by $|x|$ (as $\pm x$ occupy symmetric positions). However, to quantify the position of $x \in X$ with respect to the convex body $K \subset X$ is a problem of several possible answers. In this regard the most frequent tool is the Minkowski functional. For any $x \in X$ the *Minkowski functional* or (*Minkowski distance function* [33, p. 57] or *gauge* [74, p. 28] or *Minkowski gauge functional* [59, §1.1(d)] is defined as

$$\varphi_K(x) := \inf\{\lambda > 0 : x \in \lambda K\}. \quad (19)$$

Clearly (19) is a norm on X if and only if the convex body K is centrally symmetric with respect to the origin. If $K \subset X$ is a centrally symmetric convex body, then the norm $\|\cdot\|_{(K)} := \varphi_K$ can be used successfully in approximation theoretic questions as well. As said above, for $\|\cdot\|_{(K)}$ the unit ball of X will be K itself, $B_{X, \|\cdot\|_{(K)}}(0, 1) = K$.

In case K is nonsymmetric, (19) still can be used. But then even the choice of the homothetic centre is questionable since the use of any alternative gauge functional

$$\varphi_{K, x_0}(x) := \inf\{\lambda > 0 : x \in \lambda(K - x_0)\}$$

is equally well justified. Moreover, neither is good enough for the applications.

One of the key points of these notes is to highlight the role of the so called *generalized Minkowski functional* in the above quantification problem. It seems that the most appropriate means to apply in the inequalities of our interest are provided by this notion. This generalized Minkowski functional $\alpha(K, x)$ also goes back to Minkowski [51] and Radon [61], see also [31], [68]. There are several ways to introduce it, but perhaps the most appealing is the following construction.

By convexity, K is the intersection of its "supporting halfspaces" $X(K, v^*)$, and grouping opposite normals we get

$$K = \bigcap_{v^* \in S^*} X(K, v^*) = \bigcap_{v^* \in S^*} L(K, v^*). \quad (20)$$

Any layer (17) can be homothetically dilated with quotient $\lambda \geq 0$ at any of its symmetry centers lying on the symmetry hyperplane $H_{r+\frac{s}{2}}(v^*)$ to obtain

$$L_{r,s}^\lambda(v^*) := \left\{x \in X : \frac{\lambda+1}{2}r - \frac{\lambda-1}{2}s \leq \langle v^*, x \rangle \leq \frac{\lambda+1}{2}s - \frac{\lambda-1}{2}r\right\}. \quad (21)$$

In particular, we have also defined

$$L^\lambda(K, v^*) = L_{-h(K, -v^*), h(K, v^*)}^\lambda(v^*) \quad (22)$$

and by using (22) one can even define

$$K^\lambda := \bigcap_{v^* \in S^*} L^\lambda(K, v^*). \quad (23)$$

Note that K^λ can be empty for small values of λ . Although not needed here, it is worth mentioning that a nice formula, due to R. Schneider and E. Makai (for $\lambda \geq 1$ and for $\lambda \leq 1$, respectively) states that

$$K^\lambda = \begin{cases} \overline{K + (\lambda - 1)C(K)} = \overline{\frac{\lambda+1}{2}K - \frac{\lambda-1}{2}K} \\ K \sim (1 - \lambda)\overline{C} = \frac{1+\lambda}{2}K \sim \frac{1-\lambda}{2}(-K), \end{cases} \quad (24)$$

see [68, Propositions 7.1 and 7.3], with \sim denoting Minkowski difference: $A \sim B := \{x \in X : x + B \subset A\}$. The sets (24) were first extensively studied by Hammer [32].

Using the convex, closed, bounded, increasing and (as easily seen, c.f. [68, Proposition 3.3]) even absorbing set system $\{K^\lambda\}_{\lambda \geq 0}$, the *generalized Minkowski functional* or *gauge functional* is defined as

$$\alpha(K, x) := \inf\{\lambda \geq 0 : x \in K^\lambda\}. \quad (25)$$

There are other possibilities to define $\alpha(K, x)$ equivalently. First let

$$\gamma(K, x) := \inf \left\{ 2 \frac{\sqrt{\|x - a\| \|x - b\|}}{\|a - b\|} : a, b \in \partial K, \text{ such that } x \in [a, b] \right\}. \quad (26)$$

Then we have

$$\alpha(K, x) = \sqrt{1 - \gamma^2(K, x)}. \quad (27)$$

Also, one can consider the original definition of Minkowski [51], as presented in Grünbaum's article, [31, p. 246]. Denote

$$t := t(K, v^*, x) := \frac{2\langle v^*, x \rangle - h(K, v^*) + h(K, -v^*)}{w(K, v^*)}. \quad (28)$$

For fixed v^* this function is an affine linear functional in $x \in X$, while for fixed x it is a norm-continuous mapping from S^* (or $X^* \setminus \{0\}$) to \mathbb{R} . In fact, for fixed $v^* \in S^*$, t maps the layer $L(K, v^*)$ to $[-1, 1]$, and $L^\eta(K, v^*)$ to $[-\eta, \eta]$. Therefore, the two forms of the following definition are really equivalent;

$$\begin{aligned} \lambda := \lambda(K, x) &:= \sup \{ \eta > 0 : \exists v^* \in S^*, x \in \partial L^\eta(K, v^*) \} \\ &= \sup \{ |t(K, v^*, x)| : v^* \in S^* \} = \sup \{ t(K, v^*, x) : v^* \in S^* \}. \end{aligned} \quad (29)$$

Note that $t(K, v^*, x) = -t(K, -v^*, x)$ and therefore we don't have to use the absolute value.

In fact, $\lambda(K, x)$ expresses the supremum of the ratios of the distances between the point x and the symmetry hyperplane $\frac{1}{2}(H(K, v^*) + H(K, -v^*))$ of any layer $L(K, v^*)$ and the half-width $w(K, v^*)/2$. Now Minkowski's definition was

$$\varphi(K, x) := \inf \left\{ \frac{\min\{\text{dist}(x, H(K, v^*)), \text{dist}(x, H(K, -v^*))\}}{\max\{\text{dist}(x, H(K, v^*)), \text{dist}(x, H(K, -v^*))\}} : v^* \in S^* \right\},$$

which clearly implies the relation

$$\varphi(K, x) = \frac{1 - \lambda(K, x)}{1 + \lambda(K, x)} \quad (x \in K).$$

Although this $\varphi(K, x)$ seems to be used traditionally only for $x \in K$, extending the definition to arbitrary $x \in X$ yields the similar relation

$$\varphi(K, x) = \frac{|1 - \lambda(K, x)|}{1 + \lambda(K, x)} \quad (x \in X).$$

Now the above definitions are connected simply by

$$\lambda(K, x) = \alpha(K, x) . \quad (30)$$

In fact, usefulness of (25) and the possibility of the wide ranging applications stems from the fact that this geometric quantity incorporates quite nicely the geometric aspects of the configuration of x with respect to K , which is mirrored by about a dozen (!), sometimes strikingly different-looking, equivalent formulations of it. For the above and many other equivalent formulations with full proofs, further geometric properties and some notes on the applications in approximation theory see [68] and the references therein.

2. Chebyshev type problems of polynomial growth

Chebyshev problems are, in fact, a large class of problems. We select from these only Chebyshev-type extremal problems concerning growth of real polynomials. There are further questions we do not address here, one important class being the problem of approximating a prescribed "main term", i.e. some homogeneous term or polynomial of given degree n , by the collection of lower degree or lower rank (in lexicographical order) terms. To these questions we refer to [18, 20, 21, 28, 55, 58, 63, 64], and the references therein.

The general question we will be dealing with can be formulated as follows: "How large can a polynomial be at a point $x \in X$, or when $x \rightarrow \infty$?" More precisely, we are interested in determining for arbitrary fixed $x \in X$

$$C_n(K, x) := \sup \{p(x) : p \in \mathcal{P}_n, \|p\|_K \leq 1\}, \quad (31)$$

or for some $v \in X$ (say, with $\|v\| = 1$)

$$A_n(K, v) := \sup \{p_n^*(v) : p \in \mathcal{P}_n \text{ satisfying (10), } \|p\|_K \leq 1\}. \quad (32)$$

Clearly C_n specifies the possible size of a polynomial at a given point, while A_n is its order of growth towards infinity in a given direction. Note the appearance of the n -homogeneous part p_n^* in (32). Hence it is apparent that A_n is a kind of a formulation of the limiting case of C_n . Indeed, it is easy to see by lower order homogeneity of all the other terms, that for p represented as in (10) we have

$$p_n^*(v) = \lim_{\lambda \rightarrow +\infty} \frac{p(\lambda v)}{\lambda^n}, \quad (33)$$

hence a precise knowledge of $p(\lambda v)$ suffices. Both problems are classical and fundamental in the theory of approximation, see e.g. [50] or [72] for the one and a half century old single variable result and its many consequences, variations and extensions.

2.1.

As we have mentioned in the introduction, even the above formulation (23) and (25) of the definition of $\alpha(K, x)$ was applied first in work on these questions, particularly on (31), where a quantification of the position of x with respect to K is needed. To the best of our knowledge, application of the generalized Minkowski functional penetrated into approximation theory and polynomial inequalities first in the fundamental work [73] by Rivlin–Shapiro. There they proved the following.

Theorem A. (Rivlin–Shapiro, 1961). *Let $K \subset \mathbb{R}^d$ be a strictly convex body and $x \in \mathbb{R}^d \setminus K$. Then we have*

$$C_n(K, x) = T_n(\alpha(K, x)). \quad (34)$$

Moreover, $C_n(K, x)$ is actually a maximum, attained by

$$P(x) := T_n(t(K, v^*, x)). \quad (35)$$

Here T_n is the classical Chebyshev polynomial (1), while $t(K, v^*, x)$ is the linear expression defined in (28) and v^* is some appropriately chosen linear functional from S^* .

Note that actually the restriction $x \notin K$ is natural, as $p \equiv 1 \in \mathcal{P}_n$, and thus for $x \in K$ we always have $C_n(K, x) = 1$.

Apart from involving the generalized Minkowski functional, Rivlin and Shapiro naturally used the following helpful auxiliary proposition from the geometry of \mathbb{R}^d .

Lemma A. (Parallel supporting hyperplanes lemma). *Let $K \subset \mathbb{R}^d$ be a convex body, and $x \in K$ arbitrary. Then there exists at least one straight line ℓ through the point x so that $K \cap \ell = [a, b]$ with some $a \neq b$ and $a, b \in \partial K$ and K has parallel supporting hyperplanes at a and b .*

This standard fact was well-known to geometers for long, and many authors used it without reference or proof, see eg. [5, p. 990], [32], nevertheless, in approximation theory some reproving occurred later on. It is useful both in proving the result and to find the extremal polynomial exhibiting exactness of the upper estimate.

Rivlin and Shapiro assumed strict convexity – which means that no straight line segment can lie on the boundary ∂K – for they needed it in order to apply their basic method, that of the extremal signatures. In fact they needed this condition in proving Lemma A *by use of extremal signatures*, which was their goal in illustrating the diverse applications of the method. Indeed, their method proved to be very successful in multivariate polynomial problems, but they themselves remarked in [73] that regarding the Chebyshev problem, a direct, more geometrical argument can give more³. The – from here quite straightforward – proof for the case of a not necessarily strictly convex $K \subset \mathbb{R}^d$ was then presented in [43].

³In fact, they present this as Problem 3 on pages 694–696 of the paper, and start by explicitly

2.2.

However, there is no need for any new proof until we keep working in \mathbb{R}^d , as Theorem A of Rivlin and Shapiro for strictly convex bodies directly implies the general case once we take into account the next standard fact.

Lemma B. (Convex bodies approximation lemma). *Any convex body $K \subset \mathbb{R}^d$ can be approximated arbitrarily closely by strictly convex bodies of \mathbb{R}^d .*

Here, naturally, the approximation is meant in the Hausdorff distance sense, that is in

$$\delta(K, M) := \max \left\{ \sup_{x \in K} \inf_{y \in M} \|x - y\|, \sup_{y \in M} \inf_{x \in K} \|x - y\| \right\}. \quad (36)$$

This can be a kind of folklore among geometers, but to page out a proof was difficult. Nevertheless, several ideas of proofs were suggested by colleagues working in geometry, so it can certainly not be considered an unknown fact. In fact, e.g. in multivariate complex analysis this is used even with the stronger requirement that the approximating convex bodies monotonically decrease to K and have even real analytic boundaries, see e.g. [23, Proposition 2.2]. Anyway, we sketch two proofs in the sequel.

First proof of Lemma B. Working in \mathbb{R}^d one may fix any positive ϵ , approximate the given convex body K within $\epsilon/3$ by some polyhedra P , then P by a special polyhedra S with only 1-codimensional simplices as sides, and then finally change very slightly the (then finitely many) halfspaces, giving as their intersection S , so that the resulting body be strictly convex. A way for this last change is to substitute each halfspace Q by a large ball B , exhibiting a very small Hausdorff distance $\delta(Q \cap B(0, R), B \cap B(0, R))$, where R is taken so large (but fixed) that a given large neighbourhood of K is already contained in it. In fact, it is easier to see that we get what we want if we construct these balls the following way. We pick up one point from the relative interior of each of the sides of S , and move it slightly outward in normal direction: then the corresponding balls B are defined as the balls drawn around those simplices of full dimension, which arise from the original sides and the corresponding, slightly moved points outside. Clearly if the points to be moved are fixed, and the length of the move is fixed for all sides equally as, say, δ , then in function of $\delta \rightarrow 0$, the intersection of these balls, (which always contain S for small enough δ), will finally shrink to S . That concludes the proof of the lemma⁴.

Having this approximation lemma the proof of the general case of Theorem A is done by referring to the continuity of T_n and also of α , the latter understood as a function on $\mathcal{K} \times X$, where \mathcal{K} denotes the set of all convex bodies, and is equipped with the metric of Hausdorff distance (36). Even the extremal polynomial (35) obtains us-

writing "... this problem ... may also be solved without the methods of this paper." Then after proving the assertion of Lemma A, they remark once again: "It is, of course, possible to obtain this result more geometrically but with (42) at hand we prefer to utilize it." And ending the application to Problem 3, they state once again: "To sum up: To solve Problem 3 we need only a pair of parallel supporting hyperplanes to K such that the points of tangency, P_1 and P_2 , are collinear with P_0 ." (P_0 in their notation corresponds to the point x in ours.) Then they describe once again how the corresponding extremal value and the extremal ridge Chebyshev polynomial is found once these hyperplanes are given.

⁴This nice constructive proof was communicated to us by Balázs Csikós.

ing the corresponding extremal polynomials of the strict convex case and compactness in \mathbb{R}^d .

But is continuity clear? Well, continuity of α can be checked explicitly, but it may be rather tedious, compared to our expectations that it should be such anyway. So the best is to get around any tedious calculations, and prove something even better, that of *convexity* in x and admitting a Lipschitz bound even as a two-variable function on $K \times X$, see [68, Theorem 5.5]. (Actually, here [68, Lemma 5.4] suffices.)

In fact, continuity of α holds even in the normed space setting, which is something we could not get through compactness or direct calculations, but by combining convexity, Lipschitz bounds, and, in view of infinite dimension, even the fact that α is bounded on bounded sets, which is also necessary, see [59, 74]. For the whole assertion see [68, Corollary 6.1].

2.3.

All that raise the question whether we can go further, to achieve a similar result even in normed spaces of infinite dimension. Provided we have a result of the Rivlin-Shapiro type, this is possible for the approximation lemma extends to infinite dimensional spaces, too.

Analogously to the above first proof, one may want to represent an arbitrary convex body $K \subset X$ as intersection of balls; but that does not always go through. The property of a normed space X that all convex bodies are intersections of closed balls is called the Mazur Intersection Property, and this property fails in some spaces: see e.g. [15, 29, 30]. Nevertheless, an even nicer proof of Lemma B can be presented if we apply the general fact, also well-known to geometers, that strict convexity, in fact, is the dual property to smoothness⁵ (cf. [59], e.g.), which gives way for the proof of the general version.

Second proof of Lemma B. Consider the dual convex body $K^* \subset X^*$ of K , approximate it closely by a smooth convex body $Q^* \subset X^*$, and then take the pre-dual $Q \subset X$ of Q^* , which is then a strictly convex body in X and comes arbitrarily close to K as Q^* approximates K^* . To make everything explicit, one may argue by adding a small ball to K^* , that is consider say $Q^* := \delta B^* + K^*$, and then let $\delta \rightarrow 0$ while taking the predual Q of Q^* .

But to start with (to get a Rivlin-Shapiro type result, say), do we also have a parallel supporting hyperplane lemma? Standard proofs use compactness, which is no longer available in X . Interestingly, even Lemma A continues to hold in some Banach spaces, in particular in reflexive Banach spaces, see [69, Proposition 2]. That gives a way to recover the finite dimensional, geometric proofs even in these normed spaces. However, examples show⁶ (see [69, Example 2]), that the same assertion *fails* in some other Banach spaces. Hence to settle the general case in a satisfactory way one should combine our knowledge of $\alpha(K, x)$ more effectively. We have the following result.

Theorem 2.1 (Révész–Sarantopoulos, 2001, [68]). *If $K \subset X$ is an arbitrary convex body and if $x \in X \setminus K$ is arbitrary, then we have (34). Moreover, $C_n(K, x)$ is actually a maximum, attained by (35), where the notation is as in Theorem A.*

⁵We thank Károly Böröczky Jr. reminding us to this idea, quite relevant in the present context.

⁶As written in [69], this example was constructed by P. Wojtaszczyk.

Here we can observe that our results on the linear speed of growth of $\alpha(K, x)$ (see [68, Corollary 5.8., (5.17)]), together with Theorem 2.1 give strong indications even for the other Chebyshev problem, as we know that the Chebyshev polynomial itself has leading coefficient 2^{n-1} . Indeed, we have the following result.

Theorem 2.2 (Révész–Sarantopoulos, 2001 [68]). *Let $K \subset X$ be an arbitrary convex body and let $v \in X$. Then we have*

$$A_n(K, v) = \frac{2^{2n-1}}{\tau(K, v)^n} ,$$

and the supremum is actually a maximum attained by a polynomial of the form (35) with some appropriately chosen $v^* \in S^*$.

Based on the determination of these extremal quantities, other related questions were already addressed in approximation theory, such as the uniqueness of the extremal polynomials, or the existence of the so-called universal majorant polynomials. These, in turn, have consequences e.g. concerning the approximation of convex bodies by convex hulls of algebraic surfaces. For further details we refer to [40] and [67].

3. Bernstein's Inequality

If a univariate algebraic polynomial p is given with degree at most n , then by the classical Bernstein-Szegő inequality ([81], [25], [19]) we have

$$|p'(x)| \leq \frac{n\sqrt{\|p\|_{C[a,b]}^2 - p^2(x)}}{\sqrt{(b-x)(x-a)}} \quad (a < x < b). \quad (37)$$

This inequality is sharp for every n and every point $x \in (a, b)$, as

$$\sup \left\{ \frac{|p'(x)|}{\sqrt{\|p\|_{C[a,b]}^2 - p^2(x)}} : \deg p \leq n, |p(x)| < \|p\|_{C[a,b]} \right\} = \frac{n}{\sqrt{(b-x)(x-a)}} .$$

We may say that the upper estimate (37) is exact, and the right hand side is just the "true Bernstein factor" of the problem.

In the multivariate setting a number of extensions were proved for this classical result. However, due to the geometric variety of possible convex sets replacing intervals of \mathbb{R} , our present knowledge is still not final. The exact Bernstein inequality is known only for symmetric convex bodies, and we are within a bound of some constant factor in the general, nonsymmetric case.

For more precise notation we may define formally for any topological vector space X , a subset $K \subset X$, and a point $x \in K$ the n^{th} "Bernstein factor" as

$$B_n(K, x) := \frac{1}{n} \sup \left\{ \frac{\|Dp(x)\|}{\sqrt{\|p\|_{C(K)}^2 - p^2(x)}} : \deg p \leq n, |p(x)| < \|p\|_{C(K)} \right\} , \quad (38)$$

where $Dp(x)$ is the derivative of p at x , and even for an arbitrary unit vector $y \in X$

$$B_n(K, x, y) := \frac{1}{n} \sup \left\{ \frac{\langle Dp(x), y \rangle}{\sqrt{\|p\|_{C(K)}^2 - p^2(x)}} : \deg p \leq n, |p(x)| < \|p\|_{C(K)} \right\}. \quad (39)$$

The perhaps nicest available method – and, anyway, our favorite – is the *method of inscribed ellipses*, introduced into the subject by Y. Sarantopoulos [77]. This works for arbitrary interior points of any, possibly nonsymmetric convex body. However, other methods are in use and there is a striking connection, only recently revealed, and still not fully understood, between the method of inscribed ellipses and the general approach through *pluripotential theory*. In this survey we explain the method of inscribed ellipses, list the known results, consider an instructive analysis of the case of the simplex, and shortly comment on the intriguing questions still open.

3.1.

Although for the reader's convenience we include some short proofs, let us emphasize that, unless otherwise stated, results in this section are due to Sarantopoulos [77]. The key of all of the method is the next

Lemma C. (Inscribed Ellipse Lemma, Sarantopoulos, 1991). *Let K be any subset in a vector space X . Suppose that $x \in K$ and the ellipse*

$$\mathbf{r}(t) = \cos t \, a + b \sin t \, y + x - a \quad (t \in [-\pi, \pi]). \quad (40)$$

lies inside K . Then we have for any polynomial p of degree at most n the Bernstein type inequality

$$|\langle Dp(x), y \rangle| \leq \frac{n}{b} \sqrt{\|p\|_{C(K)}^2 - p^2(x)}. \quad (41)$$

Proof. Consider the trigonometric polynomial $T(t) := p(\mathbf{r}(t))$ of degree at most n . Since $\mathbf{r}(t) \subset K$ we clearly have $\|T\| \leq \|p\|_{C(K)}$. According to the Bernstein-Szegő inequality [81] (see also [25]) for trigonometric polynomials,

$$|T'(t)| \leq n \sqrt{\|T\|^2 - T(t)^2} \leq n \sqrt{\|p\|_{C(K)}^2 - p(\mathbf{r}(t))^2} \quad (\forall t \in \mathbb{R}).$$

In particular, for $t = 0$, we get

$$|T'(0)| \leq n \sqrt{\|p_n\|_{C(K)}^2 - p_n^2(\mathbf{x})}.$$

By the chain rule

$$T'(0) = \langle Dp_n(\mathbf{r}(0)), \mathbf{r}'(0) \rangle = \langle Dp_n(x), by \rangle,$$

which completes the proof.

Lemma D. (Sarantopoulos, 1991). *Let K be a centrally symmetric convex body in a vector space X and $x \in K$. The ellipse $\mathbf{r}(t) = \cos t \, x + b \sin t \, y$ ($t \in [-\pi, \pi]$) lies in K whenever*

$$\|y\|_{(K)} = 1 \text{ and } b = \sqrt{1 - \|x\|_{(K)}^2}.$$

Proof. The assertion is equivalent to $\|\mathbf{r}(t)\|_{(K)} \leq 1$ for every t . By the triangle and Cauchy inequalities

$$\|\mathbf{r}(t)\|_{(K)} \leq |\cos t| \|x\|_{(K)} + b |\sin t| \|y\|_{(K)} \leq \sqrt{\cos^2 t + \sin^2 t} \sqrt{\|x\|_{(K)}^2 + b^2 \|y\|_{(K)}^2} = 1.$$

Lemma D is proved.

Mutatis mutandis to the previous lemma we can deduce also the following variant.

Lemma E. (Sarantopoulos, 1991). *Let K be a centrally symmetric convex body in X , where $(X, \|\cdot\|)$ is a normed space. Let $\varphi_K = \|\cdot\|_{(K)}$ be the Minkowski functional (norm) generated by K . Then for every nonzero vector $y \in X$ the ellipse $\mathbf{r}(t) = \cos t \, x + b \sin t \, y$ ($t \in [-\pi, \pi)$) lies in K with*

$$b := \frac{\sqrt{1 - \varphi^2(K, x)}}{\varphi(K, y)}.$$

Theorem B. (Sarantopoulos, 1991). *Let p be any polynomial of degree at most n over the normed space X . Then we have for any unit vector $y \in X$ the Bernstein type inequality*

$$|\langle Dp(x), y \rangle| \leq \frac{n \sqrt{\|p\|_{C(K)}^2 - p^2(x)}}{\sqrt{1 - \|x\|_{(K)}^2}}. \quad (42)$$

Proof. The proof follows from combining Lemmas C and D.

Theorem C. (Sarantopoulos, 1991). *Let K be a symmetric convex body and y a unit vector in the normed space X . Let p_n be any polynomial of degree at most n . We have*

$$|\langle Dp_n(x), y \rangle| \leq \frac{2n \sqrt{\|p_n\|_{C(K)}^2 - p_n^2(x)}}{\tau(K, y) \sqrt{1 - \varphi^2(K, x)}}.$$

In particular, with $w(K)$ standing for the width of K we have

$$\|Dp_n(x)\| \leq \frac{2n \sqrt{\|p_n\|_{C(K)}^2 - p_n^2(x)}}{w(K) \sqrt{1 - \varphi^2(K, x)}}.$$

Proof. Here we need to combine Lemmas C and E to obtain Theorem C.

It can be rather difficult to determine, or even to estimate the b -parameter of the "best ellipse", what can be inscribed into a convex body K through $x \in K$ and tangential to direction of y . Still, we can formalize what we are after.

Definition 3.1. For arbitrary $K \subset X$ and $x \in K$, $y \in X$ the corresponding "best ellipse constants" are the extremal quantities

$$E(K, x, y) := \sup\{b : \mathbf{r} \subset K \text{ with } \mathbf{r} \text{ as given in (40)}\} \quad (43)$$

and

$$E(K, x) := \inf\{E(K, x, y) : y \in X, \|y\| = 1\}. \quad (44)$$

Clearly, the inscribed ellipse method yields Bernstein type estimates whenever we can derive some estimate of the ellipse constants. In case of symmetric convex bodies, Sarantopoulos's Theorems B and C are sharp; for the nonsymmetric case we know only the following result.

Theorem D. (Kroó–Révész, [42]). *Let K be an arbitrary convex body, $x \in \text{int}K$ and $\|y\| = 1$, where X can be an arbitrary normed space. Then we have*

$$|\langle Dp(x), y \rangle| \leq \frac{2n\sqrt{\|p\|_{C(K)}^2 - p^2(x)}}{\tau(K, y)\sqrt{1 - \alpha(K, x)}}, \quad (45)$$

for any polynomial p of degree at most n . Moreover, we also have

$$\|Dp(x)\| \leq \frac{2n\sqrt{\|p\|_{C(K)}^2 - p^2(x)}}{w(K)\sqrt{1 - \alpha(K, x)}} \leq \frac{2\sqrt{2}n\sqrt{\|p\|_{C(K)}^2 - p^2(x)}}{w(K)\sqrt{1 - \alpha^2(K, x)}}. \quad (46)$$

Note that in [42] the best ellipse is not found; the construction there gives only a good estimate, but not an exact value of (43) or (44). In fact, here we quoted [42] in a strengthened form: the original paper contains a somewhat weaker formulation only.

One of the most intriguing questions of the topic is the following conjecture, formulated first in [68].

Conjecture A. (Révész–Sarantopoulos). *Let X be a topological vector space, and K be a convex body in X . For every point $x \in \text{int}K$ and every (bounded) polynomial p of degree at most n over X we have*

$$\|Dp(x)\| \leq \frac{2n\sqrt{\|p\|_{C(K)}^2 - p^2(x)}}{w(K)\sqrt{1 - \alpha^2(K, x)}},$$

where $w(K)$ stands for the width of K .

3.2.

We denote $|x|_2 := (\sum_{i=1}^d x_i^2)^{1/2}$ the Euclidean norm of $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. Let

$$\Delta := \Delta_d := \{(x_1, \dots, x_d) : x_i \geq 0, i = 1, \dots, d, \sum_{i=1}^d x_i \leq 1\}$$

be the standard simplex in \mathbb{R}^d . For fixed $x \in \text{int}\Delta$, and $y = (y_1, \dots, y_d)$, $|y|_2 = 1$ the best ellipse constant of Δ is, by Definition 3.1, $E(\Delta, x, y)$. By a tedious calculation via the Kuhn-Tucker theorem and some geometry, the following was obtained in [49].

Proposition 3.2 (Milev–Révész, 2003). *We have*

$$E(\Delta, x, y) = \left\{ \frac{y_1^2}{x_1} + \dots + \frac{y_d^2}{x_d} + \frac{(y_1 + \dots + y_d)^2}{1 - x_1 - \dots - x_d} \right\}^{-1/2}. \quad (47)$$

Theorem 3.3 (Milev-Révész, 2003). *Let $p_n \in \mathcal{P}_n^d$. Then for every $x \in \text{int}\Delta$ and $y \in \mathbb{S}^{d-1}$ we have*

$$|D_y p_n(x)| \leq \frac{n \sqrt{\|p_n\|_{C(\Delta)}^2 - p_n^2(x)}}{E(\Delta, x, y)}, \quad (48)$$

where $E(\Delta, x, y)$ is as given in (47).

From now on let us restrict ourselves to the case $d = 2$. We denote the vertices of Δ by $O = (0, 0)$, $A = (1, 0)$, $B = (0, 1)$ and the centroid (i.e. mass point) of Δ by $M = (1/3, 1/3)$. A calculation shows that $1 - \alpha(\Delta, x) = 2r(x)$, with

$$r := r(x) := \min\{x_1, x_2, 1 - x_1 - x_2\} = \begin{cases} x_1, & x \in \triangle OMB \\ x_2, & x \in \triangle OMA \\ 1 - x_1 - x_2, & x \in \triangle AMB \end{cases} \quad (49)$$

and if $y = (\cos \varphi, \sin \varphi)$ ($0 \leq \varphi \leq \pi$) then

$$\tau(\Delta, y) = \begin{cases} 1/(y_1 + y_2), & \varphi \in [0, \pi/2] \\ 1/y_2, & \varphi \in (\pi/2, 3\pi/4] \\ -1/y_1, & \varphi \in (3\pi/4, \pi]. \end{cases}$$

Note that the inequality

$$\frac{1}{E(\Delta, x, y)} \leq \frac{2}{\tau(\Delta, y) \sqrt{1 - \alpha(\Delta, x)}} \quad (50)$$

holds true for every $x \in \text{int}\Delta$ and $y \in \mathbb{S}^1$, i.e. estimate (48) is better than (45) when $K = \Delta$. Accordingly, we can derive a new estimation for $Dp_n(x)$.

Proposition 3.4 (Milev-Révész, 2003). *Let $p_n \in \mathcal{P}_n^2$. Then for every $x \in \text{int}\Delta$ we have*

$$|Dp_n(x)|_2 \leq nE(x) \sqrt{\|p_n\|_{C(\Delta)}^2 - p_n^2(x)}, \quad (51)$$

where

$$E(x) = \sqrt{\frac{x_1(1 - x_1) + x_2(1 - x_2) + D(x)}{2x_1x_2(1 - x_1 - x_2)}} \quad (52)$$

with

$$D(x) = \sqrt{[x_1(1 - x_1) + x_2(1 - x_2)]^2 - 4x_1x_2(1 - x_1 - x_2)}.$$

Note that the inequality

$$[x_1(1 - x_1) + x_2(1 - x_2)]^2 - 4x_1x_2(1 - x_1 - x_2) > [x_1(1 - x_1) - x_2(1 - x_2)]^2$$

holds true for $x \in \text{int}\Delta$, hence $D(x) > 0$.

Using this estimate, an improvement of the constant 2 to $\sqrt{3}$ was achieved in Theorem D for the special case of $K = \Delta$, c.f. [49]. Of more interest is the next estimate comparing to the conjectured quantity with $1 - \alpha^2(\Delta, x)$.

Theorem 3.5 (Milev-Révész, 2003). *Let $p_n \in \mathcal{P}_n^2$ and $\|p_n\|_{C(\Delta)} = 1$. Then for every $x \in \text{int } \Delta$ we have*

$$|Dp_n(x)|_2 \leq \frac{\sqrt{3+\sqrt{5}} n \sqrt{\|p_n\|_{C(\Delta)}^2 - p_n^2(x)}}{w(\Delta) \sqrt{1 - \alpha^2(\Delta, x)}}. \quad (53)$$

It was checked that this is the most what follows from the inscribed ellipse method, interpreted as considering $E(\Delta, x)$ the exact yield it gives.

This improves the constant in Theorem D but falls short of Conjecture A, since $2\sqrt{2} = 2.8284\dots > \sqrt{3+\sqrt{5}} = 2.2882\dots > 2$.

3.3.

Let us consider the following question. All known lower estimates for the Bernstein factors used some kind of ridge polynomials, i.e. polynomials composed from a linear form and some (in fact, a Chebyshev) polynomial. Can one sharpen these lower estimates to the extent that Conjecture A will be disproved?

Recall that ridge polynomials are defined as

$$\mathcal{R}_n := \{p \in \mathcal{P} : p(x) = P(L(x)), L \in X^*, P \in \mathcal{P}_n(\mathbb{R})\}, \quad \mathcal{R} := \bigcup_{n=1}^{\infty} \mathcal{R}_n.$$

By easy linear substitution we may assume that ridge polynomials are expressed by using some $L(x) = t(K, v^*, x)$, as defined in (28).

Definition 3.6. For any $n \in \mathbb{N}$ the corresponding "ridge Bernstein constant" is

$$C_n(K, x, y) := \frac{1}{n} \sup_{R \in \mathcal{R}_n, |R(x)| < \|R\|_{C(K)}} \frac{|\langle DR(x), y \rangle|}{\sqrt{\|R\|_{C(K)}^2 - R^2(x)}}.$$

Proposition 3.7 (Milev-Révész, 2003). *For every convex body K and $x \in \text{int} K$, $y \in S^*$ we have*

$$C_n(K, x, y) \leq \frac{2}{\tau(K, y)} \frac{1}{\sqrt{1 - \alpha^2(K, x)}}.$$

Proof. By the chain rule we have for any $R \in \mathcal{R}_n$, $R = P(t(x))$ the formula

$$|\langle DR(x), y \rangle| = \left| P'(t(x)) \frac{2}{w(K, v^*)} \langle v^*, y \rangle \right| \leq \frac{2}{\tau(K, y)} |P'(t(x))|.$$

Applying the Bernstein-Szegő inequality for $s \in (-1, 1)$ we get

$$\frac{|P'(s)|}{\sqrt{\|P\|_{C[-1,1]}^2 - P^2(s)}} \leq \frac{n}{\sqrt{1 - s^2}}.$$

Note that for T_n , the classical Chebyshev polynomial of degree n , (and only for that) this last inequality is sharp. Putting $s := t(x)$ and combining the previous two inequalities we are led to

$$\frac{1}{n} \frac{|\langle DR(x), y \rangle|}{\sqrt{\|R\|_{C(K)}^2 - R(x)}} \leq \frac{2}{\tau(K, y)} \frac{1}{\sqrt{1 - t^2(K, v^*, x)}}.$$

Taking supremum with respect to $v^* \in S^*$ on the right hand side, we obtain a bound independent of v^* . In fact, according to (29) and (30) (see also [68, Proposition 4.1]), the supremum is a maximum and is equal to $\frac{2}{\tau(K, y)} \frac{1}{\sqrt{1 - \alpha^2(K, x)}}$. Thus taking supremum also on the left hand side, Theorem 3.7 obtains.

Whence ridge polynomials satisfy Conjecture A, always.

It follows from the definitions and Lemma C that

$$C_n(K, x, y) \leq B_n(K, x, y) \leq \frac{1}{E(K, x, y)}.$$

For the case of the standard simplex we have a converse inequality.

Proposition 3.8 (Milev-Révész, 2003). *For every $x \in \text{int}\Delta$ and $y \in \mathbb{S}^{d-1}$ we have the inequality*

$$\frac{1}{E(\Delta, x, y)} \leq \sqrt{d} C_n(\Delta, x, y).$$

Corollary 3.9 (Milev-Révész, 2003). *For every $x \in \text{int}\Delta$ and $y \in \mathbb{S}^{d-1}$ we have*

$$1 \leq \frac{B_n(\Delta, x, y)}{C_n(\Delta, x, y)} \leq \sqrt{d}, \quad 1 \leq \frac{B_n(\Delta, x)}{C_n(\Delta, x)} \leq \sqrt{d}.$$

Note that in the paper [42] it is proved that for every $x_0 \in \text{int}K$ there is a direction y_0 and a ridge polynomial $T_n(K, v_0^*, x)$ such that

$$\frac{1}{n} \frac{D_{y_0} T_n(K, v_0^*, x_0)}{\sqrt{1 - T_n(K, v_0^*, x_0)^2}} = \frac{2}{\tau(K, y_0) \sqrt{1 - \alpha^2(K, x_0)}}.$$

Consequently,

$$C_n(K, x_0, y_0) \geq \frac{2}{\tau(K, y_0) \sqrt{1 - \alpha^2(K, x_0)}}.$$

Hence, for every $x_0 \in \text{int}K$ there is a y_0 such that

$$\frac{B_n(K, x_0, y_0)}{C_n(K, x_0, y_0)} \leq \sqrt{2}.$$

Comparing this to Corollary 3.9, we see that (for the case of the simplex) the latter ratio remains uniformly bounded for all x and y .

3.4.

Another method of considerable success in proving Bernstein (and Markov) type inequalities is the pluripotential theoretical approach. Classically, all that was considered only in the finite dimensional case, but nowadays even the normed spaces setting is cultivated. To explain these, one needs an understanding of *complexifications of real normed spaces*, see e.g. [52, 13], as well as the *Siciak-Zaharjuta extremal function* $V(z)$. In fact, the latter, by the celebrated Siciak-Zaharjuta Theorem, can be expressed both by *plurisubharmonic functions* from the *Lelong class*, and also just by logarithms of the absolute values of polynomials. We spare the reader from the first,

referring to [38] as a general, nice introduction to pluripotential theory, and restrict ourselves to the latter, perhaps easier to digest formulation. That is very much like the Chebyshev problem (31) in §2, except that we consider it all over the complexification $Y := X + iX$ of X , take logarithms, and after normalization by the degree, merge the information derived by all polynomials of any degree into one clustered quantity. Namely, for any bounded $E \subset Y$ V_E vanishes on E , while outside E we have the definition

$$V_E(z) := \sup \left\{ \frac{1}{n} \log |p(z)| : 0 \neq p \in \mathcal{P}_n(Y), \|p\|_E \leq 1, n \in \mathbb{N} \right\} \quad (z \notin E) \quad (54)$$

For $E \subset X$ one can easily restrict even to $p \in \mathcal{P}(X)$. For the theory related to this function and some recent developments concerning Bernstein and also Markov type inequalities for convex bodies or even more general sets, we refer to [10, 11, 12, 13, 38, 48, 44, 57, 60].

Now consider $E = K \subset X$, where K is now a convex body. Our more precise result in Theorem 2.1 yields $V_K(x) = \sup_{n \in \mathbb{N}} \log |C_n(K, x)|/n = \lim_{n \in \mathbb{N}} \log |C_n(K, x)|/n = \alpha(K, x) + \sqrt{\alpha(K, x)^2 - 1}$, as an easy calculation with the last expression in (1) shows together with the fact that $\log |C_n(K, x)|/n$ increases. However, in the Bernstein problem the values of V_K are much more of interest for complex points $z = x + iy$, in particular for $x \in K$ and y small and nonzero. More precisely, the important quantity is the normal (sub)derivative

$$D_y^+ V_E(x) := \liminf_{\epsilon \rightarrow 0} \frac{V_E(x + i\epsilon y)}{\epsilon}, \quad (55)$$

as this quantity occurs in the next estimation of the directional derivative.

Theorem 3.10 (Baran, 1994 & 2004). *Let $E \subset X$ be a bounded, closed set, $x \in \text{int } E$ and $0 \neq y \in X$. Then for all $p \in \mathcal{P}_n(X)$ we have*

$$|\langle Dp(x), y \rangle| \leq n D_y^+ V_E(x) \sqrt{\|p\|_E^2 - p(x)^2}. \quad (56)$$

In fact, [10] contains this only for \mathbb{R}^d and partial derivatives, but by applying rotations of E , all directional derivatives follow; the case of infinite dimensional spaces are considered in [13]. See also [22, 66].

It is not obvious, how such estimates can be applied to concrete cases. First, one has to find the precise value of V_E , in such a precision, that even the derivative can be computed: then the derivatives must be obtained and only then do we really have something. However, even that is addressed by considering the Bedford-Taylor theory of the Monge-Ampere equation and the equilibrium measure [14], as the density of the equilibrium measure gives the extremal function. In some concrete applications all that may be calculated. A particular example (see [9], [38, Example 5.4.7], [12, Example 4.8]) is the following.

Proposition 3.11 (Baran, 1988). *The extremal function of the standard simplex in \mathbb{R}^d is $V_\Delta(z) = \log h(|z_1| + |z_2| + \dots + |z_n| + |1 - (z_1 + z_2 + \dots + z_n)|)$.*

From this and the calculation with the rotated directions etc, we calculated in [66] the following surprising corollary.

Proposition 3.12. *The above pluripotential theoretical estimate of Baran gives for the standard triangle of \mathbb{R}^2 the result exactly identical to (48).*

Much remains to explain in this striking coincidence, the first being the next.

Hypothesis A. *Let $K \subset X$ be a convex body. Then for all points $x \in \text{int } K$ the inscribed ellipse method and the pluripotential theoretical method of Baran results in exactly the same estimate, i.e. for all $y \in S^*$ we have*

$$D_y^+ V_K(x) = \frac{1}{E(K, x, y)} . \quad (57)$$

All people like to believe that his method(s) are the ultimate ones. However, it is quite unclear which one is the right one in the Bernstein problem. If any of the inscribed ellipse method or the pluripotential theoretical method of Baran is right - or, in case of validity of Hypothesis A, if both are precise - then Conjecture A would fail. Still, it seems worthy to formulate these contradictory assumptions.

Hypothesis B. *Let $K \subset X$ be convex body. Then for all points $x \in \text{int } K$ the exact Bernstein factor is just what results from the pluripotential theoretical method of Baran:*

$$B_n(K, x) = \sup_{y \in S^*} D_y^+ V_K(x) . \quad (58)$$

Hypothesis C. *Let $K \subset X$ be convex body. Then for all points $x \in \text{int } K$ the exact Bernstein factor is just what results from the inscribed ellipse method of Sarantopoulos:*

$$B_n(K, x) = \frac{1}{E(K, x)} . \quad (59)$$

Note that we already know that these hypothesis are certainly not true for the directional derivatives of *all* directions $y \in S^*$, where both methods can be improved upon for some y , see [65]. Care has to be exercised in formulating conjectures and hypothesis in these matters: the situation is more complex than one might like to have, and the simple heuristics of extending the results of the symmetric case do fail sometimes. In this respect see also [23, 22, 44, 48]

For some other interesting assertions and conjectures, (sometimes more addressed to the pluripotential theoretical aspects than the Bernstein inequality itself), and an analysis of them we refer to [13, 65].

Also, another real, geometric method, of obtaining Bernstein type inequalities, due to Skalyga [78, 80], is to be mentioned here: the difficulty with that is that to the best of our knowledge, no one has ever been able to compute, neither for the seemingly least complicated case of the standard triangle of \mathbb{R}^2 , nor in any other particular non-symmetric case the yield of that abstract method. Hence in spite of some remarks that the method is sharp in some sense, it is unclear how close these estimates are to the right answer and what use of them we can obtain in any concrete cases.

4. Further inequalities and problems for solution

4.1.

Finally let us touch upon a few other questions and problems generally in the center of interest for approximation theorists. One is the so-called Markov problem, which is the question of obtaining *uniform* estimates, (as opposed to pointwise ones in the Bernstein problem), to the size of the gradient vector of a polynomial all over the convex body $K \subset X$. Note that while the Bernstein-Szegő type estimates are quite good for a given point x , their use is less and less towards the boundary: in fact, at the boundary the estimate tends to infinity. This is so even in the one dimensional case of \mathbb{R} , and is inherent in the problem, due to the improvement, generally valid only inside the body, with respect to dependence on the degree of the polynomial. Indeed, $B_n(x)$ was normalized just by n , the degree, while the classical Andrei Markov inequality

$$\|p'\|_{[a,b]} \leq \frac{2n^2}{b-a} \|p\|_{[a,b]} \quad (60)$$

is sharp, excluding a "uniform Bernstein inequality" even in \mathbb{R} .

For symmetric convex bodies, also by the above described method of inscribed ellipses, Sarantopoulos was able to obtain that

$$\|Dp\|_K \leq \frac{2n^2 \|p\|_K}{w(K)}, \quad (61)$$

or, in case $\|\cdot\| = \|\cdot\|_{(K)}$, i.e. when K is the unit ball of the normed space X , then

$$\|Dp\| \leq n^2 \|p\|, \quad (62)$$

a fully satisfactory answer for the symmetric case. A nice, elementary argument of Wilhelmsen [84] presented (61) in the full generality of convex, not necessarily symmetric bodies, but with a factor 4 in place of 2. It was shown in [54] that (61) does not remain true for all convex bodies. Finally, Skalyga [79, 80] found

$$\|Dp\|_K \leq \frac{2n \cot\left(\frac{\pi}{4n}\right) \|p\|_K}{w(K)}. \quad (63)$$

Note that

$$2n \cot\left(\frac{\pi}{4n}\right) \sim \frac{8}{\pi} n^2 \quad \text{as } n \rightarrow \infty.$$

The author of [79] also remarks that the estimate (63) is sharp in the sense of being subject to no improvement in the full generality of all convex bodies and in all normed spaces. For some further information see [2, 13, 17, 19, 41, 57, 60].

4.2.

It is important, in particular for doing analysis on infinite dimensional spaces, to have a control over the size of derivatives of any order. In the classical case of a real interval it was done by Vladimir Markov, and the answer e.g. for $I = [-1, 1]$ is

$$\|p^{(k)}\|_I \leq T_n^{(k)}(1) \|p\|_I = \frac{(n^2(n^2-1) \dots (n^2 - (k-1)^2))}{(2k-1)!} \|p\|_I \quad (k = 1, 2, \dots, n), \quad (64)$$

which is sharp again for the Chebyshev polynomial T_n .

At present we are far from having a nearly as precise estimate as (64) for the general case of normed spaces. These can not be obtained, not even for dimension one, by simple iterations of the estimates for the first derivative, what gives only substantially weaker results. However, for the important special case of a Hilbert space an exact extension of this inequality is known, see [53].

Theorem E. (Munoz–Sarantopoulos, 2002). *Let H be any Hilbert space and p be an arbitrary polynomial of degree n , that is $p \in \mathcal{P}_n(H)$. Then we have*

$$\|p^{(k)}\| \leq T_n^{(k)}(1)\|p\| \quad (k = 1, 2, \dots, n) . \quad (65)$$

Harris has a number of results on Bernstein and Markov inequalities related also to higher order derivatives, see [36], and the extremely readable survey [35] in particular. One would like to decide the following.

Conjecture B. (Harris). *Let X be any Banach space and $p \in \mathcal{P}_n(X)$. Then (65) holds true.*

It seems that neither the inscribed ellipse, nor the pluripotential theoretic methods above can be applied to higher derivatives, at least not directly. Hence even in the symmetric (i.e., norm unit ball) case there is no obvious way to get close to the conjecture (or the truth).

4.3.

As mentioned above, combining or iterating known estimates does not necessarily give best results even if the parts put together are exact in their kind. Another example is the following classical question, which can be considered a composition of the Bernstein problem and the Chebyshev problem (although simple and basic in itself). The question is that how large can the *derivative* of a polynomial be at point x not inside, but outside (may be distant) of the set of normalization. The classical version for dimension 1 was already known to Chebyshev and reads

$$\max\{p^{(k)}(x) : p \in \mathcal{P}_n(\mathbb{R}), \|p\|_{[-1,1]} \leq 1\} = T^{(k)}(x) , \quad (66)$$

another extremal property of the classical Chebyshev polynomials, see e.g. [72, p. 93]. This classical inequality can easily be obtained by considering Lagrange interpolation of p on the nodes of maxima of the Chebyshev polynomials, that is at the point system $\{\cos(k\pi/n)\}_{k=0}^n$. But what is the answer for the similar question in the multivariate case? Since now directional derivatives $D_y p(x) = \langle Dp(x), y \rangle$ of all directions $y \in S$ occur, the problem does not reduce to a one dimensional question. It would be interesting, but non-trivial, to settle this question even in case $k = 1$.

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