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Theorems of Banach-Stone type for algebras of holomorphic functions on Banach spaces

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ABSTRACT

We prove that some topological algebras of holomorphic functions are topologically isomorphic if and only if their domains are biholomorphically equivalent.

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1. Introduction

In [3], S. Banach proved that two compact metric spaces X and Y are homeomorphic if and only if the Banach algebras $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ are isometrically isomorphic. M. H. Stone, in [18], generalized this result to arbitrary compact Hausdorff topological spaces, the well-known Banach-Stone theorem.

Let E and F be Banach spaces, $U \subseteq E$ and $V \subseteq F$ open subsets. Consider $\mathcal{H}(U)$ the algebra of all holomorphic functions $f : U \longrightarrow \mathbb{C}$, and likewise $\mathcal{H}(V)$, and let us endow $\mathcal{H}(U)$ and $\mathcal{H}(V)$ by one of its natural topologies $\tau = \tau_0, \tau_\omega, \tau_\delta$.

In Section 1 we prove that when E and F have the approximation property and $U \subseteq E$ and $V \subseteq F$ are absolutely convex open subsets, then U and V are biholomorphically equivalent if and only if the algebras $(\mathcal{H}(U), \tau)$ and $(\mathcal{H}(V), \tau)$ are topologically isomorphic, for $\tau = \tau_0$, τ_ω and τ_δ.

In Section 2, we have the same results for polynomially convex open subsets and in Section 3, it is possible to obtain the same results for pseudoconvex domains, but in this case, when the geometric properties of the open sets U and V are weaker, we need stronger properties on the Banach spaces E and F, namely, they need to be separable with the bounded approximation property. Finally we give a counterexample showing that the results are not valid for arbitrary pair U and V of open subsets.

In Section 4, if E and F are Tsirelson-like spaces and $U \subseteq E$ and $V \subseteq F$ are convex and balanced open subsets, we also prove that there is a special type of biholomorphic

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mapping between U and V if and only if the Fréchet algebras $\mathcal{H}_b(U)$ and $\mathcal{H}_b(V)$ are topologically isomorphic.

We refer to [6] or [13] for background information on infinite dimensional complex analysis.

2. Holomorphic functions on absolutely convex domains

Let E be a Banach space, and $U \subseteq E$ an open subset. Let $\mathcal{H}(U, F)$ denote the vector space of all holomorphic mappings $f : U \longrightarrow F$. When $F = \mathbb{C}$, we write $\mathcal{H}(U)$ instead of $\mathcal{H}(U, \mathbb{C})$, and in this case $\mathcal{H}(U)$ is an algebra. Let $z \in U$. The mapping $\delta_z : \mathcal{H}(U) \longrightarrow \mathbb{C}$ defined by $\delta_z(f) = f(z)$, for all $f \in \mathcal{H}(U)$, will denote the *evaluation at z*. It is clear that δ_z is a complex homomorphism of $\mathcal{H}(U)$. Let F be a Banach space, and V an open subset of F. If $\varphi \in \mathcal{H}(V, E)$ is such that $\varphi(V) \subseteq U$, then the mapping $C_\varphi : \mathcal{H}(U) \longrightarrow \mathcal{H}(V)$ given by $C_\varphi(f) = f \circ \varphi$, for all $f \in \mathcal{H}(U)$, is called *composition operator*. It is clear that C_φ is a homomorphism between the algebras $\mathcal{H}(U)$ and $\mathcal{H}(V)$.

Next we define some of the natural topologies for $\mathcal{H}(U)$.

Definition 2.1. Let E be a Banach space and $U \subseteq E$ be an open subset.

(i) We denote by τ_0 the topology in $\mathcal{H}(U)$ of the uniform convergence on the compact subsets of U.

(ii) A seminorm p in $\mathcal{H}(U)$ is ported by a compact subset $K \subset U$ if for each open set V, such that $K \subset V \subseteq U$, there exists a constant $C_V > 0$ such that

$$p(f) \leq C_V \|f\|_V, \text{ for all } f \in \mathcal{H}(U).$$

The topology τ_ω in $\mathcal{H}(U)$ is defined by the family of the seminorms which are ported by compact subsets of U.

(iii) A seminorm p in $\mathcal{H}(U)$ is τ_δ-continuous if for each countable open cover $(U_n)_{n\in\mathbb{N}}$ of U there exists a constant $C > 0$ and an index $n_0 \in \mathbb{N}$ such that

$$p(f) \leq C \|f\|_{\cup_{n=1}^{n_0} U_n}, \text{ for all } f \in \mathcal{H}(U).$$

The topology τ_δ in $\mathcal{H}(U)$ is defined by the family of seminorms which are τ_δ-continuous.

It is clear that each evaluation $\delta_z : (\mathcal{H}(U), \tau) \longrightarrow \mathbb{C}$, $z \in U$, and each composition operator $C_\varphi : (\mathcal{H}(U), \tau) \longrightarrow (\mathcal{H}(V), \tau)$ are continuous τ_0, τ_ω or τ_δ.

We say that two open subsets $U \subseteq E$ and $V \subseteq F$ are *biholomorphically equivalent* if there exists a mapping $\varphi : V \longrightarrow U$ which is biholomorphic, that is, $\varphi : V \longrightarrow U$ is a bijection, and both φ and φ^{-1} are holomorphic.

Next theorem is the main result of this section.

Theorem 2.2. *Let E and F be Banach spaces with the approximation property and $U \subseteq E$ and $V \subseteq F$ be convex and balanced open subsets. Consider the following conditions.*

(i) *U and V are biholomorphically equivalent.*

(ii) *The algebras* $(\mathcal{H}(U), \tau_0)$ *and* $(\mathcal{H}(V), \tau_0)$ *are topologically isomorphic.*

(iii) *The algebras* $(\mathcal{H}(U), \tau_\omega)$ *and* $(\mathcal{H}(V), \tau_\omega)$ *are topologically isomorphic.*

(iv) *The algebras* $(\mathcal{H}(U), \tau_\delta)$ *and* $(\mathcal{H}(V), \tau_\delta)$ *are topologically isomorphic.*

Then (1), (2) *and* (3) *are equivalent and imply* (4). *If* E *and* F *are separable then* $(1) - (4)$ *are equivalent.*

Proof. (1) ⇒ (2) Let $\varphi : V \longrightarrow U$ be a biholomorphic mapping. Consider the composition operator $C_\varphi : \mathcal{H}(U) \longrightarrow \mathcal{H}(V)$. We have $(C_\varphi)^{-1} = C_{\varphi^{-1}}$ and then C_φ is a topological isomorphism between $\mathcal{H}(U)$ and $\mathcal{H}(V)$.
(2) ⇒ (1) Let $T : (\mathcal{H}(U), \tau_0) \longrightarrow (\mathcal{H}(V), \tau_0)$ be a topological isomorphism. To construct a biholomorphic mapping $\varphi : V \longrightarrow U$, let $w \in V$. Then $\delta_w \circ T : (\mathcal{H}(U), \tau_0) \longrightarrow \mathbb{C}$ is a continuous complex homomorphism of $(\mathcal{H}(U), \tau_0)$. By Proposition 4 from [8], there exists a unique $z \in U$ such that $\delta_w \circ T = \delta_z$. We define $\varphi : V \longrightarrow U$ by $\varphi(w) = z$ and show that φ is holomorphic. We have that

$$(\delta_w \circ T)(f) = f(z) = f(\varphi(w)), \text{ for all } w \in V \text{ and } f \in \mathcal{H}(U),$$

that is, $T(f) = f \circ \varphi$, for all $f \in \mathcal{H}(U)$. In particular we have that $T(f) = f \circ \varphi \in \mathcal{H}(U)$, for all $f \in E'$, that is, φ is w-holomorphic and consequently holomorphic, by Theorem 8.12.b from [13]. Therefore $T = C_\varphi$. The same argument yields a mapping $\psi \in \mathcal{H}(U, F)$ with $\psi(U) \subseteq V$, such that $T^{-1} = C_\psi$. Then $Id = C_\varphi \circ C_\psi = C_{\psi \circ \varphi}$, that is, $g = g \circ \psi \circ \varphi$, for all $g \in \mathcal{H}(V)$ and in particular for all $g \in F'$. By the Hahn-Banach Theorem, we conclude that $\psi \circ \varphi = id$ and by the same arguments we have that $\varphi \circ \psi = id$. Then φ is bijective and $\varphi^{-1} = \psi \in \mathcal{H}(U, F)$, and therefore φ is biholomorphic.
(1) ⇒ (3) Apply the same arguments used in (1) ⇒ (2).
(3) ⇒ (1) By Proposition 4 from [8], $\mathcal{S}(\mathcal{H}(U), \tau_0) = \mathcal{S}(\mathcal{H}(U), \tau_\omega) = U$. Then apply the same arguments used in (2) ⇒ (1).
(1) ⇒ (4) Apply the same arguments used in (1) ⇒ (2).
(4) ⇒ (1) Use Theorem 3.1. from [11], and the same arguments of (2) ⇒ (1). □

Remark 2.3. Theorem 2.2 is true if we suppose that only one of the Banach spaces has the approximation property. To prove this we are based on ideas from [4]. In Theorem 2.2, if we suppose that E has the approximation property, then we have a holomorphic mapping $\varphi : V \longrightarrow U$ such that $T = C_\varphi$. For each $z \in U$ we have that $\delta_z \circ T^{-1}$ is τ_0-continuous and then by Mackey-Arens's Theorem there exists $w \in F$ such that $T^{-1}(g) = g(w)$, for all $g \in F'$. If we denote $w = \psi(z)$, we actually have a holomorphic mapping $\psi : U \longrightarrow F$, such that $\psi \circ \varphi : V \longrightarrow F$ is the identity mapping. Taking derivatives at the origin, we have that $d\psi(\varphi(0)) \circ d\varphi(0) : F \longrightarrow F$ is the identity operator, and hence F is a complemented subspace of E, and then F has the approximation property. The rest of the arguments to show that $\psi(U) \subseteq V$ and that $\psi = \varphi^{-1}$ are the same of Theorem 2.2. The same arguments above are also true for the bounded approximation property, and so this remark is true for Theorems 3.1 and 4.1 of this paper.

For open balls, we have a stronger result (next corollary), which is a consequence of the main theorem of [9] and Theorem 2.2.

Corollary 2.4. *Let E and F be Banach spaces with the approximation property. Let us consider the following conditions.*

(i) *E and F are isometrically isomorphic.*

(ii) *B_E and B_F are biholomorphically equivalent.*

(iii) *The algebras $(\mathcal{H}(B_E), \tau_0)$ and $(\mathcal{H}(B_F), \tau_0)$ are topologically isomorphic.*

(iv) *The algebras $(\mathcal{H}(B_E), \tau_\omega)$ and $(\mathcal{H}(B_F), \tau_\omega)$ are topologically isomorphic.*

(v) *The algebras $(\mathcal{H}(B_E), \tau_\delta)$ and $(\mathcal{H}(B_F), \tau_\delta)$ are topologically isomorphic.*

Then (1), (2), (3) *and* (4) *are equivalent and imply* (5). *If E and F are separable, then* $(1)-(5)$ *are equivalent.*

3. Holomorphic functions on polynomially convex domains

In Section 1 we saw that once we have characterized the spectra of the topological algebra $(\mathcal{H}(U), \tau)$, then we can prove Theorem 2.2. In this and in the next section, we will be able to generalize Theorem 2.2 to more general open sets, as soon as we have results on the spectra of the respective algebras of holomorphic functions. We say that an open subset U of a Banach space E is *polynomially convex* if $\widehat{K}_{\mathcal{P}(E)} \cap U$ is compact for each compact set $K \subset U$. Here $\widehat{K}_{\mathcal{P}(E)}$ denotes the set

$$\Big\{x \in E \,:\, |P(x)| \leq \sup_K |P|, \text{ for all } P \in \mathcal{P}(E)\Big\}.$$

Let $P \in \mathcal{P}(E)$. Then the open set $U = \{x \in E \,:\, |P(x)| < 1\}$ is polynomially convex but not convex.

For polynomially convex open subsets, J. Mujica proved in [12], Theorem 5.5, and in [11], Theorem 4.1, that the spectrum $(\mathcal{H}(U), \tau)$ is identified with U, for the three topologies, under certain hypotheses on E and U. Then we have the following theorem, whose proof is similar to the proof of Theorem 2.2.

Theorem 3.1. *Let E and F be Banach spaces with the approximation property and $U \subseteq E$ and $V \subseteq F$ be polynomially convex open subsets. Let us consider the following conditions.*

(i) *U and V are biholomorphically equivalent.*

(ii) *The algebras $(\mathcal{H}(U), \tau_0)$ and $(\mathcal{H}(V), \tau_0)$ are topologically isomorphic.*

(iii) *The algebras $(\mathcal{H}(U), \tau_\omega)$ and $(\mathcal{H}(V), \tau_\omega)$ are topologically isomorphic.*

(iv) *The algebras $(\mathcal{H}(U), \tau_\delta)$ and $(\mathcal{H}(V), \tau_\delta)$ are topologically isomorphic.*

Then (1), (2) *and* (3) *are equivalent and imply* (4). *If E and F are separable and have the bounded approximation property, and U and V are connected, then* $(1)-(4)$ *are equivalent.*

4. Holomorphic functions on pseudoconvex domains

Finally, we can also obtain similar results for more general domains, the pseudoconvex domains. We say that an open subset U of a Banach space E is *pseudoconvex* if $\widehat{K}_{\mathcal{P}sc(U)}$ is compact for each compact set $K \subset U$. Here, $\mathcal{P}sc(U)$ denotes the family of all continuous plurisubharmonic functions on U, and $\widehat{K}_{\mathcal{P}sc(U)}$ denotes the set

$$\Big\{x \in U \,:\, f(x) \leq \sup_K f, \text{ for all } f \in \mathcal{P}sc(U)\Big\}.$$

Let $f \in \mathcal{P}sc(E)$. Then the open set $U = \{x \in E \,:\, f(x) < 0\}$ is pseudoconvex.

M. Schottenloher, in [17] (main theorem), shows that the spectrum of $(\mathcal{H}(U), \tau)$ is identified with U, for τ_0 and τ_ω, when E is separable with the bounded approximation property and U is connected and pseudoconvex. J. Mujica, in [14], Theorem 2.1, shows the same for τ_δ. So, finally, for pseudoconvex domains, we have the following equivalence.

Theorem 4.1. *Let E and F be separable Banach spaces with the bounded approximation property and $U \subseteq E$ and $V \subseteq F$ be connected pseudoconvex open subsets. Then the following conditions are equivalent.*

(i) *U and V are biholomorphically equivalent.*

(ii) *The algebras $(\mathcal{H}(U), \tau_0)$ and $(\mathcal{H}(V), \tau_0)$ are topologically isomorphic.*

(iii) *The algebras $(\mathcal{H}(U), \tau_\omega)$ and $(\mathcal{H}(V), \tau_\omega)$ are topologically isomorphic.*

(iv) *The algebras $(\mathcal{H}(U), \tau_\delta)$ and $(\mathcal{H}(V), \tau_\delta)$ are topologically isomorphic.*

Under the hypotheses of Theorem 4.1, U and V are domains of holomorphy, by Theorem 54.12 from [13]. In the next example, we show that in every Banach space it is possible to construct a pair of open subsets H and D, one of them not a domain of holomorphy, such that the algebras $(\mathcal{H}(H), \tau)$ and $(\mathcal{H}(D), \tau)$ are topologically isomorphic, for $\tau = \tau_0$, τ_ω, τ_δ, but H and D are not biholomorphically equivalent.

Example 4.2. Let E be a Banach space such that $dim(E) \geq 2$, and let us write $E = \mathbb{C}^2 \oplus N$, where N is a Banach space. Let $D = \{z = (z_1, z_2, w) \in E \,:\, |z_1| < R_1,\, |z_2| < R_2 \text{ and } \|w\| < R\}$ and $H = \{z = (z_1, z_2, w) \in D \,:\, |z_1| > r_1 \text{ or } |z_2| < r_2\}$, where $0 < R \leq \infty$ and $0 < r_j < R_j \leq \infty$, for $j = 1, 2$. Then the algebras $(\mathcal{H}(H), \tau)$ and $(\mathcal{H}(D), \tau)$ are topologically isomorphic, for $\tau = \tau_0$, τ_ω, τ_δ, but H and D are not biholomorphically equivalent.

Proof. First we will prove that each $f \in \mathcal{H}(H)$ has a unique extension $\tilde{f} \in \mathcal{H}(D)$. With that purpose, let $\rho_1 > 0$ such that $r_1 < \rho_1 < R_1$ and let $D' = \{z = (z_1, z_2, w) \in D \,:\, |z_1| < \rho_1\}$. Observe that $D = H \cup D'$. Given $f \in \mathcal{H}(H)$, let us define

$$g(z) = \frac{1}{2\pi i} \int_{|\zeta_1| = \rho_1} \frac{f(\zeta_1, z_2, w)}{\zeta_1 - z_1} d\zeta_1, \text{ for all } z \in D'. \tag{1}$$

Since

$$(\zeta_1 - z_1)^{-1} = \sum_{m=0}^{\infty} {z_1}^m {\zeta_1}^{-(m+1)},$$

it follows that for z_2 and w fixed, g is a holomorphic function of z_1, for all $|z_1| < \rho_1$. For z_1 and w fixed, the function $\frac{f(\zeta_1, z_2, w)}{\zeta_1 - z_1}$ is a differentiable function of z_2, and therefore g is differentiable function of z_2 (Proposition 13.14 from [13]). By the same argument we have that, for z_1 and z_2 fixed, g is differentiable in w. Hence g is separately holomorphic by Theorem 14.7 from [13], and therefore holomorphic by Theorem 36.8 from [13]. In other words $g \in \mathcal{H}(D')$. By the Cauchy Integral Formula for holomorphic functions of one variable, we have that $g(z) = f(z)$ for every $z = (z_1, z_2, w)$ such that $|z_1| < \rho_1$ and $|z_2| < r_2$, and therefore for every $z \in D' \cap H$. Then the function $\tilde{f}$ defined by $\tilde{f} = f$ on H and $\tilde{f} = g$ on D' is the desired extension. The uniqueness of the extension is clear. Hence H cannot be a domain of holomorphy. Now, let us define $T : \mathcal{H}(H) \longrightarrow \mathcal{H}(D)$ by $T(f) = \tilde{f}$, for every $f \in \mathcal{H}(H)$. Then T is well defined and is an isomorphism between algebras because $\tilde{f}$ is unique. It is easy to see that $T^{-1} : \mathcal{H}(D) \longrightarrow \mathcal{H}(H)$ is the restriction operator. Let us show that the algebras $(\mathcal{H}(H), \tau)$ and $(\mathcal{H}(D), \tau)$ are topologically isomorphic, for $\tau = \tau_0$ and τ_ω. First we will show that $D \subset \mathcal{S}(\mathcal{H}(H), \tau_0)$. With that purpose, let $z \in D$ and let us define

$$h_z : \mathcal{H}(H) \longrightarrow \mathbb{C} \text{ by } h_z(f) = \tilde{f}(z), \text{ for all } f \in \mathcal{H}(H).$$

If $z \in H$, it is clear that h_z is continuous. If $z \in D'$, then $h_z(f) = g(z)$, where g is defined in (1). We see that there exists $C > 0$ such that

$$|g(z)| \leq C \sup_{|\zeta_1| = \rho_1} |f(\zeta_1, z_2, w)| = C \sup_{z \in K} |f(z)|, \text{ for every } f \in \mathcal{H}(H),$$

where $K = \{\zeta \in \mathbb{C} : |\zeta| = \rho_1\} \times \{z_2\} \times \{w\}$, which is a compact subset. Hence h_z is continuous and therefore $D \subset \mathcal{S}(\mathcal{H}(H), \tau_0) = \mathcal{S}(\mathcal{H}(H), \tau_\omega) = \Sigma$. Let us consider the mapping $G : \mathcal{H}(H) \longrightarrow \mathcal{H}(\Sigma)$ given by $G(f) = \hat{f}$, where $\hat{f}(h) = h(f)$, for every $h \in \Sigma$. We have that G is continuous for τ_0 (see H. Alexander [2], Sections 2 and 4) and for τ_ω (see M. Matos [10]). Since $H \subset D \subset \Sigma$ it follows that the following mappings are continuous

$$(\mathcal{H}(\Sigma), \tau) \hookrightarrow (\mathcal{H}(D), \tau) \xrightarrow{T^{-1}} (\mathcal{H}(H), \tau) \xrightarrow{G} (\mathcal{H}(\Sigma), \tau), \quad \text{for } \tau = \tau_0, \tau_\omega.$$

And consequently we have that $(\mathcal{H}(H), \tau)$ and $(\mathcal{H}(D), \tau)$ are topologically isomorphic, for $\tau = \tau_0$, τ_ω. It remains to prove that $T : (\mathcal{H}(H), \tau_\delta) \longrightarrow (\mathcal{H}(D), \tau_\delta)$ is a topological isomorphism, but this is showed in [5] by G. Coeuré, in [7] by A. Hirschowitz and in [16] by M. Schottenloher. Finally we will show that H and D are not biholomorphically equivalent. In fact, since D is convex, we have that D is domain of holomorphy, so if there exists a biholomorphic mapping between H and D, we have that H is a domain of holomorphy, by Theorem 1.8 from [16] or Theorem 2.15 from [7], but this is a contradiction. □

When $E = \mathbb{C}^2$, the pair (H, D) is called a *Hartogs figure* in $\mathbb{C}^2$.

5. Holomorphic functions of bounded type on Tsirelson's space

As we saw in the previous sections, to have theorems of Banach-Stone type we need results on the spectra and also be able do define composition operators between those algebras that are under consideration. In this section we look for Banach-Stone theorems for algebras of holomorphic functions of bounded type, and investigate conditions for a mapping to define a continuous composition operator between two algebras of holomorphic functions of bounded type.

Let E and F be Banach spaces. Let $\mathcal{P}(^mE, F)$ denote the Banach space of all continuous m-homogeneous polynomials from E into F. Let $\mathcal{P}_f(^mE, F)$ denote the subspace of $\mathcal{P}(^mE, F)$ generated by all polynomials of the form $P(x) = \varphi(x)^m b$, with $\varphi \in E'$ and $b \in F$. Finally, let $\mathcal{P}(E, F)$ denote the vector space of all continuous polynomials from E into F. When $F = \mathbb{C}$, we write $\mathcal{P}(^mE)$, $\mathcal{P}_f(^mE)$ and $\mathcal{P}(E)$ instead $\mathcal{P}(^mE, \mathbb{C})$, $\mathcal{P}_f(^mE, \mathbb{C})$ and $\mathcal{P}(E, \mathbb{C})$, respectively.

Let U be an open subset of E. We say that an increasing sequence of open subsets $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ of E is a *regular cover of* U if

$$U = \bigcup_{n=1}^{\infty} U_n \text{ and } d_{U_{n+1}}(U_n) > 0, \text{ for all } n \in \mathbb{N},$$

where $d_{U_{n+1}}(U_n)$ denotes the distance from U_n to the boundary of U_{n+1}. Let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ be a regular cover of U. We denote by $\mathcal{H}^\infty(\mathcal{U}, F)$ the vector space formed by all holomorphic mappings $f : U \longrightarrow F$ which are bounded on each U_n, for $n \in \mathbb{N}$. $\mathcal{H}^\infty(\mathcal{U}, F)$ is a Fréchet space for the topology of the uniform convergence on the sets U_n. When $F = \mathbb{C}$, we write $\mathcal{H}^\infty(\mathcal{U})$ instead of $\mathcal{H}^\infty(\mathcal{U}, F)$. In this case we have that $\mathcal{H}^\infty(\mathcal{U})$ is a Fréchet algebra.

Suppose that U is balanced and each U_n is bounded and circular, that is, $e^{i\theta}U_n \subset U_n$, for all $\theta \in \mathbb{R}$. If follows from the Cauchy inequalities that the Taylor series of each $f \in \mathcal{H}^\infty(\mathcal{U}, F)$ at the origin converges uniformly on each U_n. In particular, $\mathcal{P}(E, F)$ is dense in $\mathcal{H}^\infty(\mathcal{U}, F)$.

Defining composition operators between the algebras $\mathcal{H}^\infty(\mathcal{U})$ and $\mathcal{H}^\infty(\mathcal{V})$ is not as trivial as it is between the algebras $\mathcal{H}(U)$ and $\mathcal{H}(V)$. To illustrate this, we have the following theorem.

Theorem 5.1. *Let E and F be Banach spaces, $U \subseteq E$ and $V \subseteq F$ be open subsets, $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ and $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ be regular covers of U and V, respectively. Consider $\varphi \in \mathcal{H}(V, E)$, with $\varphi(V) \subseteq U$. Then the operator $C_\varphi : \mathcal{H}^\infty(\mathcal{U}) \longrightarrow \mathcal{H}^\infty(\mathcal{V})$ given by $C_\varphi(f) = f \circ \varphi$, for all $f \in \mathcal{H}^\infty(\mathcal{U})$, is well defined and is a continuous homomorphism if and only if for each $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $\varphi(V_k) \subseteq \widehat{(U_{n_k})}_{\mathcal{H}^\infty(\mathcal{U})}$.*

Proof. Let us suppose that C_φ is well defined and is a continuous homomorphism; and let $k \in \mathbb{N}$. Since C_φ is continuous, there exists $c > 0$ and $n_k \in \mathbb{N}$ such that

$$\sup_{V_k} |C_\varphi(f)| \le c \sup_{U_{n_k}} |f|, \text{ for all } f \in \mathcal{H}^\infty(\mathcal{U}).$$

Replacing f by f^n, taking nth roots and letting $n \longrightarrow \infty$, we have that

$$\sup_{V_k} |C_\varphi(f)| \le \sup_{U_{n_k}} |f|, \text{ for all } f \in \mathcal{H}^\infty(\mathcal{U}).$$

Let $y \in V_k$. Then $|f(\varphi(y))| = |C_\varphi(f)(y)| \leq \sup_{U_{n_k}} |f|$, for all $f \in \mathcal{H}^\infty(\mathcal{U})$, that is, $\varphi(y) \in (\widehat{U_{n_k}})_{\mathcal{H}^\infty(\mathcal{U})}$. Since y is an arbitrary element of V_k, it follows that $\varphi(V_k) \subseteq (\widehat{U_{n_k}})_{\mathcal{H}^\infty(\mathcal{U})}$.

Conversely, to show that C_φ is well defined, we have to guarantee that $f \circ \varphi \in \mathcal{H}^\infty(\mathcal{V})$, for all $f \in \mathcal{H}^\infty(\mathcal{U})$. To see this, let $k \in \mathbb{N}$. By hypothesis there exists $n_k \in \mathbb{N}$ such that $\varphi(V_k) \subseteq (\widehat{U_{n_k}})_{\mathcal{H}^\infty(\mathcal{U})}$. Let $y \in V_k$. Then $\varphi(y) \in (\widehat{U_{n_k}})_{\mathcal{H}^\infty(\mathcal{U})}$, that is,

$$|f(\varphi(y))| \leq \sup_{U_{n_k}} |f| < \infty, \text{ for all } f \in \mathcal{H}^\infty(\mathcal{U}).$$

Since y is an arbitrary element of V_k, it follows that $f \circ \varphi$ is bounded on V_k, for all $k \in \mathbb{N}$ and $f \in \mathcal{H}^\infty(\mathcal{U})$, proving that $f \circ \varphi \in \mathcal{H}^\infty(\mathcal{V})$, for all $f \in \mathcal{H}^\infty(\mathcal{U})$. Finally, now it is easy to check that C_φ is a continuous homomorphism between $\mathcal{H}^\infty(\mathcal{U})$ and $\mathcal{H}^\infty(\mathcal{V})$. □

Now we can precisely define composition operators between two algebras $\mathcal{H}^\infty(\mathcal{U})$ and $\mathcal{H}^\infty(\mathcal{V})$.

Definition 5.2. Let $U \subseteq E$ and $V \subseteq F$ be open subsets and $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ and $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ be regular covers of U and V, respectively.

(i) We denote by $\mathcal{H}^\infty(\mathcal{V},\mathcal{U})$ the set of all mappings $\varphi \in \mathcal{H}(V, E)$ such that for each $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $\varphi(V_k) \subseteq (\widehat{U_{n_k}})_{\mathcal{H}^\infty(\mathcal{U})}$.

(ii) Let $\varphi \in \mathcal{H}^\infty(\mathcal{V},\mathcal{U})$. Then the operator $C_\varphi : \mathcal{H}^\infty(\mathcal{U}) \longrightarrow \mathcal{H}^\infty(\mathcal{V})$ defined by $C_\varphi(f) = f \circ \varphi$, for all $f \in \mathcal{H}^\infty(\mathcal{U})$, is called composition operator.

We say that a Banach space E is a *Tsirelson-like space* if E is reflexive and $\mathcal{P}_f(^mE)$ is dense in $\mathcal{P}(^mE)$, for all $m \in \mathbb{N}$.

Example 5.3. In [19], B. Tsirelson constructed a reflexive Banach space E, with an unconditional Schauder basis, that does not contain any subspace which is isomorphic to c_0 or to any l_p. R. Alencar, R. Aron and S. Dineen proved in [1] that $\mathcal{P}_f(^mE)$ is dense in $\mathcal{P}(^mE)$, for all $m \in \mathbb{N}$.

J. Mujica shows in [15], Lemma 2.1, that if E is a Tsirelson-like space and U is a convex and balanced open subset of E, then the spectra of $\mathcal{H}^\infty(\mathcal{U})$ is identified with U. Using this result, Theorem 5.1 and the same arguments of Theorem 2.2, we can prove the following Banach-Stone theorem.

Theorem 5.4. *Let E and F be Tsirelson-like spaces. Let $U \subseteq E$ and $V \subseteq F$ be convex and balanced open subsets, and let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ and $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ be regular covers for U and V respectively, such that each U_n and V_n are bounded and circular. Then the following conditions are equivalent*

(i) *There exists a bijective mapping $\varphi : V \longrightarrow U$ such that $\varphi \in \mathcal{H}^\infty(\mathcal{V},\mathcal{U})$ and $\varphi^{-1} \in \mathcal{H}^\infty(\mathcal{U},\mathcal{V})$.*

(ii) *The algebras $\mathcal{H}^\infty(\mathcal{U})$ and $\mathcal{H}^\infty(\mathcal{V})$ are topologically isomorphic.*

By combining Theorems 5.4 and 2.2, we have the following extension result.

Corollary 5.5. *Let E and F be Tsirelson-like spaces. Let $U \subseteq E$ and $V \subseteq F$ be convex and balanced open subsets, and let $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ and $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ be regular covers for U and V respectively, such that each U_n and V_n are bounded and circular. If the algebras $\mathcal{H}^\infty(\mathcal{U})$ and $\mathcal{H}^\infty(\mathcal{V})$ are topologically isomorphic then the algebras $(\mathcal{H}(U), \tau)$ and $(\mathcal{H}(V), \tau)$ are topologically isomorphic, for $\tau = \tau_0$, τ_ω and τ_δ.*

To end this section, let us define the Fréchet space $\mathcal{H}_b(U, F)$. If $d_U(x)$ denotes the distance from x to the boundary of U, then the following open sets

$$U_n = \{x \in U : \|x\| < n \text{ and } d_U(x) > 2^{-n}\}$$

form a sequence $(U_n)_{n\in\mathbb{N}}$ of bounded open sets which cover U and $d_{U_{n+1}}(U_n) > 0$, for each $n \in \mathbb{N}$. In the particular case when $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$, U_n defined above, the space $\mathcal{H}^\infty(\mathcal{U}, F)$ is denoted by $\mathcal{H}_b(U, F)$ and its elements are called *holomorphic mappings of bounded type*. If $F = \mathbb{C}$ we write $\mathcal{H}_b(U)$ instead of $\mathcal{H}_b(U, F)$ and in this case we have that $\mathcal{H}_b(U)$ is a Fréchet algebra. Let $V \subseteq F$ be an open subset of a Banach space F, and $\mathcal{V} = (V_n)$ defined in the similar way as above. Then for $\mathcal{U}$ and $\mathcal{V}$ the set $\mathcal{H}^\infty(\mathcal{V}, \mathcal{U})$ will be denoted by $\mathcal{H}_b(V, U)$.

We say that a subset $A \subseteq U$ is *U-bounded* if A is bounded and there exists $\epsilon > 0$ such that $A + B(0, \epsilon) \subseteq U$. It is easy to see that $\mathcal{H}_b(U)$ is the set of all holomorphic functions $f : U \longrightarrow \mathbb{C}$ which are bounded in each U-bounded subset; and that $\mathcal{H}_b(V, U)$ is the set of all mappings $\varphi \in \mathcal{H}(V, E)$ such that $\varphi(V) \subseteq U$ and $\varphi(B)$ is U-bounded, for each B a V-bounded subset, provided that U is convex.

The following corollary is a Banach-Stone theorems for algebras of holomorphic functions of bounded type.

Corollary 5.6. *Let E and F be Tsirelson-like spaces. Let $U \subseteq E$ and $V \subseteq F$ be convex and balanced open subsets. Then the following conditions are equivalent.*

(i) *There exists a bijective mapping $\varphi : V \longrightarrow U$ such that $\varphi \in \mathcal{H}_b(V, U)$ and $\varphi^{-1} \in \mathcal{H}_b(U, V)$.*

(ii) *The algebras $\mathcal{H}_b(U)$ e $\mathcal{H}_b(V)$ are topologically isomorphic.*

And we also have an extension result for algebras of holomorphic functions of bounded type.

Corollary 5.7. *Let E and F be Tsirelson-like spaces. Let $U \subseteq E$ and $V \subseteq F$ be convex and balanced open subsets. If the algebras $\mathcal{H}_b(U)$ and $\mathcal{H}_b(V)$ are topologically isomorphic then the algebras $(\mathcal{H}(U), \tau)$ and $(\mathcal{H}(V), \tau)$ are topologically isomorphic, for $\tau = \tau_0$, τ_ω and τ_δ.*

In section we have established our main results pseudoconvex domains. In Sections 4, however, the results are valid for absolutely convex domains, and we do not know if the results are true for more general domains, for instance for polynomially convex domains. It is clear that a positive answer to this question implies a positive answer to our question.

In [23] we present Banach-Stone theorems for algebras of germs of holomorphic functions.

Preliminary versions of some of the results in Sections 1, 2, 3 and 4 were announced in [20], [21] and [22].

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