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Extreme cases of weak type interpolation (with applications to Orlicz-Sobolev embeddings)

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ABSTRACT

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1. The L_p-spaces and the classical Sobolev spaces W_p^k turn out to be unsufficient for optimally sharp embedding theorems. Even in nonlimitting case $p < n/k$ the best embedding requires use of Lorentz spaces:

$$W_p^k \hookrightarrow L(q,p), \quad q = np/(n-kp) \qquad \text{(O'Neil [15], Dikarev [6]).}$$

The limitting case $p = n/k$ was studied in 60-th by Pohozhaev [17], Trudinger [20], Peetre [16] and some others, with the help of Orlicz spaces, but the best embedding $W_p^k \hookrightarrow X$ was proved only, when taking

$$X = \{f :\ \|f\|_X = \|t^{-1/p}\left(\ln\frac{e}{t}\right)^{-1} f^*(t)\|_{L_p} < \infty$$

(Maz'ya [12], Hansson [8], Brezis-Wainger [3]).

Involving various function spaces as a range for embedding, it is natural to use such spaces also in definition of Sobolev spaces in place of L_p. The corresponding embeddings can be studied then, using methods of weak interpolation. A detailed investigation of this problem can be found in a recent paper by Edmunds, Kerman, Pick [7]. Rather sharp results were obtained by Cianchi [4] for Orlicz-Sobolev spaces of the first order.

In this talk I want to present some new results on embeddings of generalized Sobolev spaces with applications to Orlicz spaces; a part of results was obtained jointly with M.Milman [13].

2. Let Ω be an open set in $I\!R^n$ with Lebesgue measure. A Banach space A of measurable functions $f:\ \Omega \mapsto I\!R$ is called *Banach function space* if

$$f \in A,\ |g(x)| \le |f(x)| \quad \Longrightarrow \quad g \in A,\ \|g\|_A \le \|f\|_A.$$

Let, as usual,

$$f^*(t) = \inf\{\lambda > 0:\ \text{mes}\,\{x \in \Omega:\ |f(x)| > \lambda\} \le t\}, \qquad t > 0.$$

If A is a rearrangement-invariant (r.i.) space on $(0,\infty)$, i.e., $\|f\|_A = \|f^*\|_A$, then $A(\Omega)$ consists of all measurable $f(x)$ such that $f^*(t) \in A$. Any r.i. space is characterized by

- fundamental function $\varphi_A(s) = \|\chi_{(0,s)}(t)\|_A$ and their extension indices

$$\pi_\varphi = \lim_{s\to 0} \frac{\ln m_\varphi(s)}{\ln s}, \qquad \rho_\varphi = \lim_{s\to\infty} \frac{\ln m_\varphi(s)}{\ln s},$$

where $m_\varphi(s) = \sup_t \varphi_A(ts)\Big/\varphi_A(t)$.

- dilation function $d_A(s) = \sup_{f\in A} \|f(t/s)\|_A\Big/\|f(t)\|_A$, defining the so-called *Boyd indices*

$$\pi_A = \lim_{s\to 0} \frac{\ln d_A(s)}{\ln s}, \qquad \rho_A = \lim_{s\to\infty} \frac{\ln d_A(s)}{\ln s}.$$

In general $0 \le \pi_\varphi \le \pi_A \le \rho_A \le \rho_\varphi \le 1$, but for many important spaces (L_p, Orlicz, Lorentz etc.) $\pi_A = \pi_\varphi$ and $\rho_A = \rho_\varphi$.

We will also need the following Hardy operators

$$Pf(t) = \frac{1}{t}\int_0^t f(s)\,ds, \qquad Qf(t) = \int_t^\infty f(s)\,\frac{ds}{s}$$

and the notation $f^{**}(t) = P(f^*)(t)$. We say the space A has (P) (or (Q)) property if the corresponding Hardy operator is bounded on A. The (P) property is equivalent to the inequality $\rho_A < 1$ and the (Q) property is equivalent to the inequality $\pi_A > 0$.

Let $k \in (0,n)$. We say the space A has $Q(k)$ property if

$$Q(k,A) = \int_1^\infty s^{k/n} d_A\left(\frac{1}{s}\right)\frac{ds}{s} < \infty.$$

This property is equivalent to the inequality $\pi_A > k/n$. If A has $Q(k)$ property then

$$\|t^{-k/n} Qf(t)\|_A \le Q(k,A)\|t^{-k/n} f(t)\|_A. \tag{1}$$

A similar inequality for operator P is valid without any requirements to A:

$$\|t^{-k/n} Pf(t)\|_A \le \frac{n}{k}\|t^{-k/n} f(t)\|_A. \tag{2}$$

Remark 1. *The dilation function and Hardy operators can be defined also for non-rearrangement invariant Banach function spaces as well as the* $(P), (Q)$ *and* $Q(k)$ *properties. The inequality* (1) *remains true; the inequality* (2) *is valid with another constant*

$$C = \int_0^1 s^{k/n} d_A\left(\frac{1}{s}\right) ds$$

in place of n/k.

3. In the last years it was revealed that various first order differential properties of functions $f(x)$ can be characterized, using some special differences $f^*(\varepsilon t) - f^*(t)$, $0 < \varepsilon < 1$, and $f^{**}(t) - f^*(t)$. They can be connected by the relations

$$f^*(t/2) - f^*(t) \le 2(f^{**}(t) - f^*(t)),$$

$$f^{**}(t) - f^*(t) \le P\Big(f^*(\cdot/2) - f^*(\cdot)\Big)(t) + (f^*(t/2) - f^*(t)).$$

Thus both differences have equivalent norms in any space A with (P) property. The first of differences was studied and used much more than the second — by V.Kolyada [9], Sagher-Shvartsman [19], Malý-Pick [11] and many others. The second was used by Ulyanov [21], then by Bennett-DeVore-Sharpley [2] in connection with BMO-spaces and recently by Bastero-Milman-Ruiz [1]. However it is more perspective in the embedding theory due to relation $(f^{**})'(t) = \big(f^{**}(t) - f^*(t)\big)/t$, whence

$$f^{**}(t) = Q(f^{**} - f^*)(t) \quad \text{for } f^{**}(\infty) = 0. \tag{3}$$

Our starting point will be the inequality

$$t^{-1/n}\big(f^{**}(t) - f^*(t)\big) \le c\,(\nabla f)^{**}(t) \tag{4}$$

which is valid for any smooth function with $f^{**}(\infty) = 0$. Here, as usual, ∇ stands for gradient and $(\nabla f)^{**}$ means $|\nabla f|^{**}$. It can be derived from some general results by Kolyada for the first kind of differences. Another way is to prove (4) first for spherically symmetric (radial) functions and then to use the Polya-Szegö symmetrization principle. For any Banach function space A, we obtain immediately

$$\|t^{-1/n}\big(f^{**}(t) - f^*(t)\big)\|_A \le c\|(\nabla f)^{**}(t)\|_A \tag{5}$$

If the space A has $Q(1)$ property than by (1) and (3) we obtain that

$$\|t^{-1/n} f^{**}(t)\|_A \le c\|(\nabla f)^{**}(t)\|_A$$

and, taking $|\nabla f|$ in place of f, we come to the inequality

$$\|t^{-2/n}\big(f^{**}(t) - f^*(t)\big)\|_A \le c_1\|t^{-1/n}(\nabla f)^{**}(t)\|_A \le c_2\|(D^2 f)^{**}(t)\|_A$$

for any $f \in C^2(\Omega)$ with $f^{**}(\infty) = (\nabla f)^{**}(\infty) = 0$. Here D^2 stands for vector composed by all second derivatives.

Proceeding inductively, we come to the following main assertion.

Theorem 2. *Let A be a Banach function spaces and Ω be an open set in $I\!R^n$. If A has $Q(k-1)$ property and a function $f \in C^k(\Omega)$ is such that $f^{**}(\infty) = (\nabla f)^{**}(\infty) = \cdots = (D^{k-1}f)^{**}(\infty) = 0$, then*

$$\|t^{-k/n}\big(f^{**}(t) - f^{*}(t)\big)\|_A \leq c\,\|(D^k f)^{**}(t)\|_A. \tag{6}$$

If A is a r.i. space with (P) property then we may omit two stars in the right-hand term. And if the boundary $\partial\Omega$ satisfies some minimal smooth conditions sufficient for the extension theorems, then we may replace $D^k f$ by various reduced expression, for example, by k-th order gradient

$$\nabla^k f = \begin{cases} \Delta^{k/2} f & \text{for even } k \\ \nabla(\Delta^{(k-1)/2}) f & \text{for odd } k, \end{cases}$$

where Δ means the standard Laplace operator. We arrive at the inequality

$$\|t^{-k/n}\big(f^{**}(t) - f^{*}(t)\big)\|_A \leq c\,\|(\nabla^k f)\|_A, \tag{7}$$

which in such a form was obtained together with M.Milman in [13].

The result of Theorem 2 is optimal in the following sense.

Theorem 3. *Let B be a Banach function space such that $d_B(s) < \infty$ for any $s > 0$ (for example, B is a r.i. space). If under conditions of Theorem 1*

$$\|f\|_B \leq c\,\|(D^k f)^{**}\|_A$$

for all admissible f, then

$$\|f\|_B \leq c\,\|t^{-k/n}\big(f^{**}(t) - f^{*}(t)\big)\|_A.$$

An analogous assertion holds for the case of inequality (7) with $\nabla^k f$ in place of $D^k f$.

4. Let now A be an Orlicz space generated by some Young function $\Phi(t)$. Then $\varphi_A(t) \approx 1/\Phi^{-1}(1/t)$, i.e., up to equivalence of norms, any Orlicz space is uniquely defined by his fundamental function $\varphi(t) = \varphi_A(t)$ which may be taken as a parameter defining A. Moreover, we can give an explicit formula for the norm in A expressed via φ [18]:

$$\|f\|_A = \sup_{\|g\|_{L_1} \leq 1} \int_\Omega |f(x)||g(x)|\varphi\left(\frac{1}{|g(x)|}\right) dx.$$

The Orlicz-Sobolev space $W^k_A(\Omega)$ is defined as set of all functions $f \in A$ such that all their weak derivatives up to order k exist and also belong to A. This space is usually normed by

$$\|f\|_{W^k_A(\Omega)} = \sum_{|\alpha|=0}^{k} \left\| \frac{\partial^{|\alpha|} f}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n} \right\|_A, \ |\alpha| = \alpha_1 + \cdots + \alpha_n.$$

There are numerous results about embeddings of Sobolev-Orlicz spaces; as usual, one tries to embed them into other Orlicz spaces. As we will show, if even such a result is extremely sharp, it almost never is optimal among all possible embeddings into r.i.

spaces. Generally speaking, the optimal embedding $W_A^k(\Omega) \hookrightarrow X$ with r.i. space X requires involving non-Orlicz spaces.

Recall that the Boyd indices of an Orlicz space A coincide with the extension indices π_φ and ρ_φ.

Theorem 4. *Let $\pi_\varphi > k/n$ where φ is the fundamental function of the Orlicz space A. Then $W_A^k(\Omega) \hookrightarrow X$ with*

$$\|f(x)\|_X = \|t^{-k/n} f^{**}(t)\|_A$$

and the space X is the best possible among all r.i. spaces.

Proof. The conditions of the theorem imply that A has $Q(k)$ property and this is enough not only for Theorems 1 and 2 but also for the inequality (1), whence

$$\begin{aligned}\|t^{-k/n} f^{**}(t)\|_A &= \|t^{-k/n} Q(f^{**} - f^*)(t)\|_A \\ &\le c\,\|t^{-k/n}(f^{**}(t) - f^*(t))\|_A \le c\,\|(D^k f)^{**}(t)\|_A.\end{aligned}$$

Optimality of this embedding follows from Theorem 2. □

Investigations of Lorentz [10], Montgomery-Smith [14] and some others show that the space X is not an Orlicz space.

The case of $\pi_\varphi = k/n$ may be called "limitting case of embedding", it is equivalent to the case $p = n/k$ for L_p spaces. Consider an operator

$$Q_k f(t) = \int_t^\infty s^{k/n} f(s)\,\frac{ds}{s}.$$

We will need the following auxiliary assertion.

Lemma 5. *Let a Banach function space A have $Q(k-1)$ property and let another space X be such that $Q_k:\ A \to X$. Then $\|f^{**}\|_X \le c\,\|(D^k f)^{**}\|_A$.*

Proof. Taking $h(t) = t^{-k/n}\big(f^{**}(t) - f^*(t)\big)$, we obtain immediately that

$$Q_k h(t) = \int_t^\infty \big(f^{**}(s) - f^*(s)\big)\,\frac{ds}{s} = f^{**}(t),$$

whence

$$\|f^{**}(t)\|_X = \|Q_k h(t)\|_X \le c\,\|t^{-k/n}\big(f^{**}(t) - f^*(t)\big)\|_A \le c\,\|(D^k f)^{**}\|_A.$$

□

Now all things are reduced to the study of operator Q_k on a given Orlicz space A. This operator is *of joint weak type*

$$\left(1,\ \frac{n}{n-k}\ ;\ \frac{n}{k}\ ,\ \infty\right)$$

and can be studied by the methods of weak type interpolation.

5. We will investigate the extreme case $\pi_A = k/n$, involving Lorentz representation of Orlicz spaces. For the simplicity, we consider only the case of a bounded set Ω, assuming that $|\Omega| = 1$. Let $\omega(t) = \phi_A(t)t^{-1/p}$, then $\pi_\omega = 0$, i.e., $\omega(t)$ is a slowly varying function. We consider only the case when $\omega(t)$ satisfies the Lorentz condition:

$$\exists\, \delta > 0 : \qquad \int_0^1 \frac{dt}{\omega^{-1}(\delta\omega(t))} < \infty. \tag{8}$$

For instance, we obtain all Orlicz spaces generated by the Young functions

$$\Phi(u) \approx u^p (\ln u)^{\gamma_1} (\ln\ln u)^{\gamma_2} \cdots (\ln\ln\ldots\ln u)^{\gamma_n},\ u \geq u_0, \tag{9}$$

where $\gamma_1, \gamma_2, \ldots \gamma_n$ may be arbitrary real numbers. We obtain also some more quickly increasing functions such as

$$\Phi(u) \approx u^p\, e^{\ln^\varepsilon u} \qquad \text{for}\ \ 0 < \varepsilon < 1, \quad u \geq u_0.$$

As was proved by Lorentz [10], *the condition* (8) *is necessary and sufficient for an Orlicz space A with the fundamental function $\varphi_A(t) = \omega(t)t^{1/p}$, $p > 1$, to have an equivalent (quasi)norm*

$$\|f\|_A \approx \|\omega(t)f^*(t)\|_{L_p}. \tag{10}$$

Weak type interpolation for Orlicz spaces, generated by the functions (9) with one or two logarithmic factors, were studied by Bennett-Rudnick and by a group of Czech mathematicians; optimal spaces in this interpolation were described by Cwikel-Pustylnik [5]. The results can be plainly extended to any number of logarithmic factors and even to arbitrary fundamental functions $\varphi_A(t) = \omega(t)t^{1/p}$ with the property $\omega(t^2) \approx \omega(t)$. This leads to the following embedding theorem.

Theorem 6. *Let an Orlicz space A be such that $\varphi_A(t) = \omega(t)t^{1/p}$, $\omega(t^2) \approx \omega(t)$, $p = n/k$. Then $W^k_A(\Omega) \hookrightarrow X$, where X is a r.i. space with the (quasi)norm*

$$\|f\|_X = \|\omega(t)(1 - \ln t)^{-1} t^{-1/p} f^*(t)\|_{L_p}. \tag{11}$$

Moreover, this space X is best possible among all r.i. spaces, giving such an embedding for the space $W^k_A(\Omega)$.

Remark 7. *The space X with the norm* (11) *does not satisfy the Lorentz condition* (8) *and thus is not an Orlicz space. This once again justifies our assertion that the optimal target spaces for embeddings of Orlicz-Sobolev spaces are not Orlicz spaces.*

Note also that if $\omega(t) \equiv 1$, the norm (11) *coincides with the above mentioned result of Maz'ya, Hansson and Brezis-Wainger for a sharp embedding of Sobolev space W^k_p in the limitting case $p = n/k$. By the way, Theorem 4 gives optimality of this embedding.*

6. The target spaces of weak type interpolation for Lorentz spaces with the function $\omega(t)$, satisfying the condition (8) but increasing faster then in Theorem 4, never were studied before. We will use the following new property of the operator Q_k in extreme cases of weak type interpolation.

Theorem 8. *Let the norm in a space A satisfy inequality* (10) *with $p = n/k$ and an increasing function $\omega(t)$ such that*

$$\int_t^1 \frac{ds}{s\omega(s)} \le \frac{c}{t\omega'(t)}.$$

Then $Q_k : A \to Y$, where the (quasi)norm in Y is defined by

$$\|f\|_Y = \|t\omega'(t)t^{-1/p}f^*(t)\|_{L_p}$$

and the space Y is best possible among all Lorentz spaces giving the same result.

Corollary 9. *If A is an Orlicz space satisfying the conditions of Theorem 5 then $W_A^k(\Omega) \hookrightarrow Y$.*

Let me give an example of application of the last result. We consider $\omega(t) = \exp\{-(1 - \ln t)^\varepsilon\}$ with arbitrary $0 < \varepsilon < 1$. It is easy to see that $\omega(t^2)/\omega(t) \to 0$ as $t \to 0$, thus we cannot apply Theorem 6. But the conditions of Theorem 8 are fulfilled, which gives that $W_A^k(\Omega) \hookrightarrow Y$, where

$$\|f\|_Y = \|(1 - \ln t)^{\varepsilon-1} \exp\{-(1 - \ln t)^\varepsilon\}t^{-1/p}f^*(t)\|_{L_p}.$$

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