DIFFERENTIABILITY VERSUS CONTINUITY: RESTRICTION AND EXTENSION THEOREMS AND MONSTROUS EXAMPLES

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Abstract. This talk is based on the first part of a 2019 BAMS survey, with the same title, written with Juan B. Seoane–Sepúlveda. Its aim is to revisit the centuries old discussion on the interrelations between continuous and differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. The new angle of this presentation is influenced by a series of very recent results in this research area.

This is presented in an narrative that answers two classical questions: (1) *To what extend a continuous function must be differentiable?* and (2) *How strong is the assumption of differentiability of a function?*

Question (2) will be interpreted as: *To what extent the derivative F' of an $F\colon \mathbb{R} \to \mathbb{R}$ must be continuous?* Here we recall some well known properties of the derivatives (large set of points of continuity, Darboux property) as well as newer (e.g., a finite composition of derivatives from $I = [0,1]$ to I has fixed point property). We will also provide a very easy new construction of *everywhere differentiable nowhere monotone map.*

Concerning question (1): we indicate a simple new proof that *for every continuous $f\colon \mathbb{R} \to \mathbb{R}$ there is a perfect set $Q \subset \mathbb{R}$ such that $f \restriction Q$ is differentiable*; discuss Jarník and Whitney differentiable extension theorems; deduce that *for every continuous $f\colon \mathbb{R} \to \mathbb{R}$ there is a C^1 map $g\colon \mathbb{R} \to \mathbb{R}$ such that $f \cap g$ is uncountable.* We will also present a new seemingly paradoxical example a differentiable function $F\colon \mathbb{R} \to \mathbb{R}$ (which can be nowhere monotone) and of compact perfect $\mathfrak{X} \subset \mathbb{R}$ such that $F'(x) = 0$ for all $x \in \mathfrak{X}$ while $F[\mathfrak{X}] = \mathfrak{X}$; thus, the map $\mathfrak{f} = F \restriction \mathfrak{X}$ is shrinking at every point while, paradoxically, not globally.