THE MACKEY-ARENS THEOREM IN THE CONTEXT OF TOPOLOGICAL ABELIAN GROUPS

ELENA MARTÍN PEINADOR
COMPLUTENSE UNIVERSITY OF MADRID

Abstract. The locally quasi-convex groups were defined by Vilenkin in the 50's of the last century. They are an important class of topological abelian groups which encloses as a subclass the locally convex topological vector spaces. Thus, central results of Functional Analysis might have extended versions for locally quasi-convex groups. This is not a straightforward process, and usually the obstructions which appear make the theory reacher. In order to deal with local quasi-convexity several authors have developed techniques based on Numerical Analysis, and there has been a great activity in this field in the last 25 years.

The Mackey-Arens Theorem -a relevant result of linear Functional Analysis- asserts that for a real topological vector space (X, τ), the set $\mathrm{LCT}(X, \tau)$ of all compatible locally convex topologies on X has a maximum, subsequently called the Mackey topology. In [1] the Mackey-Arens Theorem was studied within the category of abelian topological groups. The main problem left open in the mentioned paper was the existence of the analogue to the Mackey topology for abelian groups. Explicitly, if (G, τ) is an abelian topological group, is there a maximum in the family $\mathrm{LQC}(G, \tau)$ of all the locally quasi-convex topologies compatible with τ?. The existence of the Mackey topology for a broad class of topological groups, including the locally compact and the complete metrizable abelian groups, was already established in [1].

Finally, in 2018 the question has been solved in the negative. Außenhofer and Gabriyelyan (simultaneously) provided an example of a topological group which does not admit a Mackey topology: namely the free abelian topological group on a convergent sequence. Some other examples have just appeared.

In this lecture the Mackey theory for groups will be presented, stressing the similarities and differences with the classical Mackey theory for locally convex spaces.

References

[1] M.J. Chasco, E. Martín-Peinador, V. Tarieladze. *On Mackey topology for groups*, Stud. Math. 132, **3**, (1999) 257–284.

[2] T. Borsich, M. J. Chasco, X. Domínguez, E. Martín-Peinador. *On g-barrelled groups and their permanence properties* . J. Math. Anal. Appl. 473 (2019), no. 2, 1203?1214

[3] J.M. Díaz Nieto, E. Martín-Peinador. *Characteristics of the Mackey topology for abelian topological groups*, Descriptive Topology and Functional Analysis (Chapter 7), Springer Proc. Math. Stat., 80 pp. 119–144 (2014).

[4] L. Außenhofer. *On the non-existence of the Mackey topology for locally quasi-convex groups*, Forum Math. **30 (5)** (2018), 1119–1128.

[5] S. Gabriyelyan. *A locally quasi-convex abelian group without a Mackey group topology*, Proc. Amer. Math. Soc. **146** (2018), no. 8, 3627–3632.

[6] X. Domínguez, E. Martín-Peinador, V. Tarieladze. *On ultrabarrelled spaces, their group analogs and Baire spaces* . Descriptive Topology and Functional Analysis. II, 77-87, Springer Proc. Math. Stat., 286, 2019