OPERATOR INEQUALITIES. AGLER MODELS AND ERGODICITY

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Abstract. We discuss a kind of spectral theory for bounded linear operators T on a Hilbert space H satisfying

$$\alpha(T^*,T) := \sum_{n=0}^{\infty} \alpha_n T^{*n} T^n \geq 0 \quad \text{(convergence in SOT)}, \tag{1}$$

where $\alpha(t) = \sum \alpha_n t^n$ is an analytic function with $\alpha_n \in \mathbb{R}$ and $\alpha_0 = 1$. This type of conditions has been extensively analysed, starting from pioneering work by Agler in the 80's. There are also many papers on tuples of commuting operators. Put $k(t) = 1/\alpha(t) = \sum k_n t^n$. In 2018, Bickel, Clouâtre, Hartz and McCarthy proved (even for tuples of commuting operators) that if $\alpha_n < 0$ for every $n \geq 1$ and the quotients k_{n+1}/k_n tend to 1 then (1) holds if and only if T extends to $(B_k \otimes I_{\mathcal{R}}) \oplus S$, where B_k is a backward shift on the RKHS of analytic functions associated to k, $\mathcal{R}$ is an auxiliary Hilbert space and S is an isometry. As a complement of this result, we show that (V_D, W) provides a *minimal model* of T, where V_D is a contraction given by

$$V_D x(z) := D(I - zT)^{-1} x, \quad D := (\alpha(T^*,T))^{1/2}$$

and $W := (I_H - V_D^* V_D)^{1/2}$. So D plays the role of the defect operator of T. Let $\mathcal{A}_T$ be the set of analytic functions α with summable Taylor coefficients such that $\sum |\alpha_n| T^{*n} T^n$ converges in SOT and let $\mathcal{A}_T^0$ be the closure of the polynomials in $\mathcal{A}_T$. Define

$$\mathcal{C}_\alpha^w := \{T \in L(H) \,:\, \alpha \in \mathcal{A}_T, \alpha(T^*,T) \geq 0\}.$$

Using Gelfand Theory and a factorization lemma for polynomials, we prove the following result. Let $T \in L(H)$ with $\sigma(T) \subset \overline{\mathbb{D}}$. Suppose that $(1-t)^a \in \mathcal{A}_T$, where $a > 0$, and let $\beta \in \mathcal{A}_T^0$ satisfy that $\beta(t) > 0$ for every $t \in [0,1]$. If

$$\alpha(t) = (1-t)^a \beta(t)$$

and $T \in \mathcal{C}_\alpha^w$, then T is similar to an operator in $\mathcal{C}_{(1-t)^a}^w$.

This result applies to many functions α such that $k(t)$ has negative coefficients and Agler's techniques do not work. We will give some consequences for Ergodic Theory. In particular, we show that any operator in $\mathcal{C}_a^w$ is *quadratically* (C,b)-*bounded* for any $b > 1-a$.