APPROXIMATING CONTINUOUS FUNCTIONS WITH SMOOTH ONES WITHOUT CRITICAL POINTS

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Abstract. The Morse-Sard theorem states that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is of class C^k, $k = n-m+1$ then the set of critical values of f has Lebesgue measure zero in $\mathbb{R}^m$. This theorem fails in infinite dimension, although S. Smale proved a version of the theorem for differentibale functions whose derivatives were Fredholm operators. This is a very strong hypothesis that restrics the range of applications drastically. However in many cases it is not important if a function has the Morse-Sard property or not but if it can be approximated by smooth functions with the Morse-Sard property.

In 2004 Azagra and Cepedllo proved that every continous function from l_2 to $\mathbb{R}^m$ can be approximated by a C^∞-function without any critical point. In 2007 Azagra and Jimńez-Sevilla proved the same for continuous $f : E \to \mathbb{R}$ where E is a Banach space with separable dual. In this talk we will extend these results to show that the same is true for continuous $f : E \to F$ where E is a Banach space from a large class (including all classcial Banach spaces) and the target space F is a quotient of E.

Theorem 1. *Let E be one of the classical Banach spaces c_0, ℓ_p or L^p, $1 < p < \infty$. Let F be a Banach space, and assume that there exists a bounded linear operator from E onto F. Then, for every continuous mapping $f : E \to F$ and every continuous function $\varepsilon : E \to (0, \infty)$ there exists a C^k mapping $g : E \to F$ such that $\|f(x) - g(x)\| \leq \varepsilon(x)$ and $Dg(x) : E \to F$ is a surjective linear operator for every $x \in E$. (Here k denotes the order of smoothness of the space E).*

The proof will rely in part on a new result about extractibility of subsets from the space E. Namely that given a closed subset X in E that is locally contained in graphs of continuous functions defined in subspaces of infinite codimension in E (and taking values in their orthonal complements), given an open set U with $X \subseteq U$ and given an open cover $\mathcal{G}$, then there exists a C^∞-diffeomorphism from $E \setminus X$ onto E which is the identity outside U and refines $\mathcal{G}$.