UNIFORM APPROXIMATION OF CONTINUOUS MAPPINGS BY
SMOOTH MAPPINGS WITH NO CRITICAL POINTS ON
HILBERT MANIFOLDS

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Abstract. We prove that every continuous mapping from a separable infinite-
dimensional Hilbert space $X$ into $\mathbb{R}^m$ can be uniformly approximated by $C^\infty$ smooth mappings with no critical points. This kind of result can be regarded as a sort of strong approximate version of the Morse-Sard theorem. Some consequences of the main theorem are as follows. Every two disjoint closed subsets of $X$ can be separated by a one-codimensional smooth manifold which is a level set of a smooth function with no critical points. In particular, every closed set in $X$ can be uniformly approximated by open sets whose boundaries are $C^\infty$ smooth one-codimensional submanifolds of $X$. Finally, since every Hilbert manifold is diffeomorphic to an open subset of the Hilbert space, all of these results still hold if one replaces the Hilbert space $X$ with any smooth manifold $M$ modelled on $X$.

1. Introduction and main results

A fundamental result in differential topology and analysis is the Morse-Sard theorem [21, 22], which states that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is a $C^r$ smooth function, with $r > \max \{n - m, 0\}$, and $C_f$ stands for the set of critical points of $f$ (that is, the points $x$ at which the differential $df(x)$ is not surjective), then the set of critical values, $f(C_f)$, is of (Lebesgue) measure zero in $\mathbb{R}^m$. This result also holds true for smooth functions $f : X \to Y$ between two smooth manifolds of dimensions $n$ and $m$ respectively.

Several authors have dealt with the question as to what extent one can obtain a similar result for infinite-dimensional spaces or manifolds modelled on such spaces. Let us recall some of their results.

Smale [24] proved that if $X$ and $Y$ are separable connected smooth manifolds modelled on Banach spaces and $f : X \to Y$ is a $C^r$ Fredholm map (that is, every differential $df(x)$ is a Fredholm operator between the corresponding tangent spaces) then $f(C_f)$ is meager, and in particular $f(C_f)$ has no interior points, provided that $r > \max \{\text{index}(df(x)), 0\}$ for all $x \in X$; here $\text{index}(df(x))$ stands for the index of the Fredholm operator $df(x)$, that is, the difference between the dimension of the kernel of $df(x)$ and the codimension of the image of $df(x)$, which are both finite. However,
these assumptions are quite restrictive: for instance, if $X$ is infinite-dimensional then there is no Fredholm map $f : X \rightarrow \mathbb{R}$. In general, the existence of a Fredholm map $f$ from a manifold $X$ into another manifold $Y$ implies that $Y$ is infinite-dimensional whenever $X$ is.

On the other hand, one cannot dream of extending the Morse-Sard theorem to infinite dimensions without imposing strong restrictions. Indeed, as shown by Kupka’s counterexample [17], there are $C^\infty$ smooth functions $f : X \rightarrow \mathbb{R}$, where $X$ is a Hilbert space, so that their sets of critical values $f(C_f)$ contain intervals and in particular have non-empty interior.

More recently, S. M. Bates has carried out a deep study concerning the sharpness of the hypothesis of the Morse-Sard theorem and the geometry of the sets of critical values of smooth functions. In particular he has shown that the above $C^r$ smoothness hypothesis in the statement of the Morse-Sard theorem can be weakened to $C^{r-1,1}$; see [4, 5, 6, 7, 8]. C. G. Moreira and Bates have studied some generalizations of the Morse-Sard theorem related to Hausdorff measures and Hausdorff dimensions. They have also shown that the function $f$ as in Kupka’s counterexample can even be assumed to be a polynomial of degree three; see [9, 18].

Nevertheless, for many applications of the Morse-Sard theorem, it is often enough to know that any given continuous function can be uniformly approximated by a map whose set of critical values has empty interior. In this direction, Eells and McAlpin established the following theorem [15]: if $X$ is a separable Hilbert space, then every continuous function from $X$ into $\mathbb{R}$ can be uniformly approximated by a smooth function $f$ whose set of critical values $f(C_f)$ is of measure zero. This allowed them to deduce a version of this theorem for mappings between smooth manifolds $M$ and $N$ modelled on a separable infinite-dimensional Hilbert space $X$ and a Banach space $F$ respectively, which they called an approximate Morse-Sard theorem: every continuous mapping from $M$ into $N$ can be uniformly approximated by a smooth function $f : M \rightarrow N$ so that $f(C_f)$ has empty interior. However, this seemingly much more general version of the result is a bit tricky: as they already observed ([15], Remark 3A), when $F$ is infinite-dimensional, the function $f$ they obtain satisfies that $C_f = M$, although $f(M)$ has empty interior in $N$. Unfortunately, even though all the results of that paper seem to be true, some of the proofs are not correct.

In this paper we will prove a much stronger result: if $M$ is a $C^\infty$ smooth manifold modelled on a separable infinite-dimensional Hilbert space $X$ (in the sequel such a manifold will be called a Hilbert manifold), then every continuous mapping from $M$ into $\mathbb{R}^m$ can be uniformly approximated by $C^\infty$ smooth mappings with no critical points. This kind of result might be regarded as one of the strongest possible statements of approximate Morse-Sard theorems, when the target space is finite-dimensional.

As a by-product we also obtain the following: for every open set $U$ in a separable Hilbert manifold $M$ there is a $C^\infty$ smooth function $f$ whose support is the closure of $U$ and so that $df(x) \neq 0$ for every $x \in U$. This result could be summed up by saying that for every open subset $U$ of $M$ there is a function $f$ whose open support
is $U$ and which does not satisfy Rolle’s theorem; one should compare this result with
the main theorem from [2] (see also the references therein).

Either of these results has in turn interesting consequences related to smooth
approximation and separation of closed sets. For instance, every two disjoint closed
subsets in $M$ can be separated by a smooth one-codimensional submanifold of $M$
which is a level set of a smooth function with no critical points. In particular, every
closed subset of $M$ can be uniformly approximated by open sets whose boundaries
are smooth one-codimensional submanifolds of $M$.

So far these are some good consequences of our main result, all of them somehow
related to Morse-Sard type theorems. But there are some bad consequences as well,
perhaps the most noticeable one being that, since the set of smooth functions with
no critical points is dense in the set of continuous functions defined on a Hilbert
manifold, there are quite large sets of smooth functions for which no conceivable
Morse theory could be valid.

Let us now formally state our main results. For the sake of a convenient notation
in our proofs, when $\varphi$ takes real values we indistinctly use the symbols $d\varphi(x) = \varphi'(x)$
to denote the derivative of $\varphi$ at a point $x$, and we reserve $d\varphi(x)$ for the derivative
of a vector-valued function $\varphi : M \rightarrow \mathbb{R}^m$ at a point $x \in M$.

**Theorem 1.1.** Let $U$ be an open subset of a separable infinite-dimensional Hilbert
space $X$. Then, for every continuous mapping $f : U \rightarrow \mathbb{R}^m$ and for every con-
tinuous positive function $\varepsilon : U \rightarrow (0, +\infty)$, there exists a $C^\infty$ smooth mapping
$\psi : U \rightarrow \mathbb{R}^m$ such that $\|f(x) - \psi(x)\| \leq \varepsilon(x)$ and $d\psi(x)$ is surjective for all $x \in X$
(that is, $\psi$ has no critical points).

We will prove this result in the following section. Let us now establish the
announced consequences of Theorem 1.1.

**Theorem 1.2.** Let $M$ be a separable Hilbert manifold. Then, for every continuous
mapping $f : M \rightarrow \mathbb{R}^m$ and every continuous positive function $\varepsilon : M \rightarrow (0, +\infty)$,
there exists a $C^\infty$ smooth mapping $\psi : M \rightarrow \mathbb{R}^m$ such that $\|f(x) - \psi(x)\| \leq \varepsilon(x)$ and $d\psi(x)$ is surjective for all $x \in X$
(that is, $\psi$ has no critical points).

Proof. One could adapt the ideas in the proof of Theorem 1.1 to extend it to the
setting of Hilbert manifolds but, for simplicity, we will instead use another approach.
Indeed, bearing in mind a fundamental result on Hilbert manifolds due to Eells and
Elworthy [14] that every separable Hilbert manifold can be $C^\infty$ embedded as an
open subset of the Hilbert space, it is a triviality to observe that Theorem 1.1 still
holds if we replace $U$ with a separable Hilbert manifold. \(\square\)

We will say that an open subset $U$ of a Hilbert manifold $M$ is smooth
provided that its boundary $\partial U$ is a smooth one-codimensional submanifold of $M$.

**Corollary 1.3.** Let $M$ be a separable Hilbert manifold. Then, for every two disjoint
closed subsets $C_1$, $C_2$ of $M$, there exists a $C^\infty$ smooth function $\varphi : M \rightarrow \mathbb{R}$ with
no critical points, such that the level set $N = \varphi^{-1}(0)$ is a $1$-codimensional $C^\infty$
smooth submanifold of $M$ that separates $C_1$ and $C_2$, in the following sense. Define
$U_1 = \{x \in M : \varphi(x) < 0\}$ and $U_2 = \{x \in M : \varphi(x) > 0\}$; then $U_1$ and $U_2$ are disjoint $C^\infty$ smooth open sets of $M$ with common boundary $\partial U_1 = \partial U_2 = N$, so that $C_i \subset U_i$ for $i = 1, 2$.

**Proof.** By Urysohn’s lemma there exists a continuous function $f : M \rightarrow [-1, 1]$ so that $C_1 \subset f^{-1}(-1)$ and $C_2 \subset f^{-1}(1)$. Taking $\varepsilon = 1/3$ and applying Theorem 1.2 we get a $C^\infty$ smooth function $\varphi : M \rightarrow \mathbb{R}$ which has no critical points and is so that $|f(x) - \varphi(x)| \leq 1/3$ for all $x \in M$; in particular

$$C_1 \subseteq f^{-1}(-1) \subseteq \varphi^{-1}(-\infty, 0) := U_1,$$

and

$$C_2 \subseteq f^{-1}(1) \subseteq \varphi^{-1}(0, +\infty) := U_2.$$ The open sets $U_1$ and $U_2$ are smooth because their common boundary $N = \varphi^{-1}(0)$ is a smooth one-codimensional submanifold of $M$ (thanks to the implicit function theorem and the fact that $d\varphi(x) \neq 0$ for all $x$).

A trivial consequence of this result is that every closed subset of $M$ can be uniformly approximated by smooth open subsets of $M$. In fact,

**Corollary 1.4.** Every closed subset of a separable Hilbert manifold $M$ can be approximated by smooth open subsets of $M$ in the following sense: for every closed set $C \subset M$ and every open set $W$ containing $C$ there is a $C^\infty$ smooth open set $U$ so that $C \subset U \subseteq W$.

Finally, the following result, which also implies the above corollary, tells us that for every open set $U$ in $M$ there always exists a function whose open support is $U$ and which does not satisfy Rolle’s theorem.

**Theorem 1.5.** For every open subset $U$ of a Hilbert manifold $M$ there is a continuous function $f$ on $M$ whose support is the closure of $U$, so that $f$ is $C^\infty$ smooth on $U$ and yet $f$ has no critical point in $U$.

**Proof.** For the same reasons as in the proof of Theorem 1.2 we may assume that $U$ is an open subset of the Hilbert space $X = \ell_2$. Let $\varepsilon : X \rightarrow [0, +\infty)$ be the distance function to $X \setminus U$, that is,

$$\varepsilon(x) = \operatorname{dist}(x, X \setminus U) = \inf\{\|x - y\| : y \in X \setminus U\}.$$ The function $\varepsilon$ is continuous on $X$ and satisfies that $\varepsilon(x) > 0$ if and only if $x \in U$. According to Theorem 1.1, and setting $f(x) = 2\varepsilon(x)$, there exists a $C^\infty$ smooth function $\psi : U \rightarrow \mathbb{R}$ which has no critical points on $U$ and which $\varepsilon$-approximates $f$ on $U$, that is, $|2\varepsilon(x) - \psi(x)| \leq \varepsilon(x)$ for all $x \in U$. This inequality implies that $\lim_{x \to z} \psi(x) = 0$ for every $z \in \partial U$. Therefore, if we set $\psi = 0$ on $X \setminus U$, the extended function $\psi : X \rightarrow [0, +\infty)$ is continuous on the whole of $X$, is $C^\infty$ smooth on $U$ and has no critical points on $U$. On the other hand, $\psi(x) \geq \varepsilon(x) > 0$ for all $x \in U$, hence the support of $\psi$ is $\overline{U}$. \qed
2. Proof of the main result

The main idea behind the proof of Theorem 1.1 is as follows. First we use a perturbed smooth partition of unity to approximate the given continuous mapping $f$. The summands of this perturbed partition of unity are functions supported on scalloped balls and carefully constructed in such a way that the critical set $C_\varphi$ of the approximating sum $\varphi$ is locally compact.

Then we have to eliminate all the critical points without losing much of the approximation. To this end we compose the approximating mapping $\varphi$ with a deleting diffeomorphism $h : U \rightarrow U \setminus C_\varphi$ which extracts the critical points $C_\varphi$ and is as close to the identity as we want. The existence of such a diffeomorphism is guaranteed by the following quite elaborated result of West's [25]. We say that a mapping $g$ from a subset $A$ of $M$ is limited by an open cover $G$ of $M$ if the collection $\{\{x, g(x)\} : x \in A\}$ refines $G$.

**Theorem 2.1** (West). Let $C$ be a closed, locally compact subset of a Hilbert manifold $M$, $U$ an open subset of $M$ with $C \subset U$, and $G$ an open cover of $M$. Then there is a $C^\infty$ diffeomorphism $h$ of $M$ onto $M \setminus C$ which is the identity outside $U$ and is limited by $G$.

In this way we will obtain a smooth mapping $\psi$ which has no critical points, and which happens to approximate the function $\varphi$ (which in turn approximates the original $f$) because the perturbation brought on $\varphi$ by the composition with $h$ is not very important (recall that $h$ is arbitrarily close to the identity).

The following proposition shows the existence of a function $\varphi$ with the above properties. Recall that $C_\varphi$ stands for the set of critical points of $\varphi$.

**Proposition 2.2.** Let $U$ be an open subset of the separable Hilbert space $X$. Let $f : U \rightarrow \mathbb{R}^m$ be a continuous mapping, and $\varepsilon : U \rightarrow (0, \infty)$ a continuous positive function. Then there exist a $C^\infty$ smooth mapping $\varphi : U \rightarrow \mathbb{R}^m$ so that

(a) $C_\varphi$ is locally compact and closed (relatively to $U$);
(b) $\|\varphi(x) - f(x)\| \leq \varepsilon(x)/2$ for all $x \in U$.

Assume for a while that Proposition 2.2 is already established, and let us see how we can deduce Theorem 1.1.

**Proof of Theorem 1.1**

For the given continuous mappings $f$ and $\varepsilon$, take a mapping $\varphi$ with the properties of Proposition 2.2. Since $\varphi$ and $\varepsilon$ are continuous, for every $z \in U$ there exists $\delta_z > 0$ so that if $x, y \in B(z, \delta_z)$ then $\|\varphi(y) - \varphi(x)\| \leq \varepsilon(z)/4 \leq \varepsilon(x)/2$.

Let $G = \{B(x, \delta_x) : x \in U\}$, $M = U$, and for the critical set $C = C_\varphi$, use Theorem 2.1 to find a $C^\infty$ diffeomorphism $h : U \rightarrow U \setminus C$ so that $h$ is limited by $G$. Define $\psi = \varphi \circ h$.

Since $h$ is limited by $G$ we have that, for any given $x \in U$, there exists $z \in U$ such that $x, h(x) \in B(z, \delta_z)$, and therefore $\|\varphi(h(x)) - \varphi(x)\| \leq \varepsilon(z)/4$, that is, we have that

$\|\psi(x) - \varphi(x)\| \leq \varepsilon(z)/4 \leq \varepsilon(x)/2$. 

Hence, by combining this inequality with (b) of Proposition 2.2, we obtain that
\[ \|\psi(x) - f(x)\| \leq \varepsilon(x) \] (1)
for all \( x \in U \).

Finally, it is clear that \( \psi \) does not have any critical point. Indeed, since \( h(x) \notin C = C_\varphi \), we have that the linear map \( d\varphi(h(x)) \) is surjective, and \( dh(x) \) is a linear isomorphism, so \( d\psi(x) = d\varphi(h(x)) \circ dh(x) \) is a linear surjection from \( X \) onto \( \mathbb{R}^m \), for every \( x \in U \).

\[ \Box \]

**Remark 2.3.** In the case when \( f : U \rightarrow \mathbb{R} \) we do not need to use the full power of West’s result (whose proof is quite involved and extremely technical) and we can obtain a more self-contained proof which we briefly outline next.

When the function \( f \) takes values in the real line, one can show that \( C_\varphi \) is contained in a countable union of small compact sets \( K_n \) which are separated from one another by pairwise disjoint small open sets \( U_n \), that is, \( C_\varphi \subseteq \bigcup_{n=1}^\infty K_n \subseteq \bigcup_{n=1}^\infty U_n \), with \( U_n \cap U_m = \emptyset \) whenever \( n \neq m \), and the oscillation of \( \varphi \) on each \( U_n \) is very small. The proof of this stronger statement of Proposition 2.2 is similar to the one we give below (Case I), but requires taking two extra precautions to define the \( K_n \)’s and \( U_n \)’s. First, one has to slightly perturb the radii \( r_n \) in such a way that, for any finite selection of centers \( y_n \), the spheres that are the boundaries of the balls \( B(y_n, r_n) \) have empty intersection with the affine subspace spanned by those centers, and no center belongs to any other sphere (of course, these are generic properties of radii and centers). And second, one has to use Remark 2.7 below in order to inductively select the \( \lambda_n \)’s in the proof of Proposition 2.2 in a way that allows to define the \( K_n \)’s and \( U_n \)’s (making a picture with only three balls will help to guess how these sets can be defined by induction together with the \( \lambda_n \)’s).

Thanks to this more accurate alternative statement of 2.2 we can replace West’s theorem with a much more elementary result which tells us that for every compact subset \( K \) and every open subset \( U \) of \( X \) with \( K \subseteq U \), there exists a \( C^\infty \) diffeomorphism \( h : X \rightarrow X \setminus K \) such that \( h \) restricts to the identity outside \( U \) (this fact is well known and was probably first proved by Moulis [20]. We gave another proof in a former version of the present paper which has recently been extended to the class of Banach spaces having Schauder basis and \( C^p \) smooth bump functions, see [3]).

In this case, to eliminate the critical points of the approximating function \( \varphi \), we may compose \( \varphi \) with a sequence of deleting diffeomorphisms \( h_n : X \rightarrow X \setminus K_n \) which extract each of the compact sets of critical points \( K_n \) and restrict to the identity outside each of the open sets \( U_n \). The infinite composition of deleting diffeomorphisms with our function, \( \psi = \varphi \circ \bigcirc_{n=1}^\infty h_n \), is locally finite, in the sense that only a finite number (in fact at most one) of the diffeomorphisms are acting on some neighborhood of each point, while all the rest restrict to the identity on that neighborhood. As in the proof above, it follows that \( \psi \) has no critical points (the same argument works locally), and still approximates \( f \) (recall that each \( h_n \) restricts to the identity outside the set \( U_n \), on which \( \varphi \) has a very small oscillation, and the \( U_n \) are pairwise disjoint).
Proof of Proposition 2.2

We will assume that \( U = X \), since the proof is completely analogous in the case of a general open set. One only has to take some (easy but rather rambling) technical precautions in order to make sure that the different balls considered in the argument are in \( U \).

In order to avoid bearing an unnecessary burden of notation, we will make the proof of this proposition for the case of a constant \( \varepsilon > 0 \). Later on we will briefly explain what additional technical precautions must be taken in order to deduce the general form of this result (see Remark 2.9 below).

Let \( B(x, r) \) and \( \overline{B}(x, r) \) stand for the open ball and closed ball, respectively, of center \( x \) and radius \( r \), with respect to the usual hilbertian norm \( \| \cdot \| \) of \( X \).

**Case I.** We will first consider the case of a real valued function \( f : U \rightarrow \mathbb{R} \). Fix \( \varepsilon > 0 \). By continuity, for every \( x \in X \) there exists \( \delta_x > 0 \) so that \( |f(y) - f(x)| \leq \varepsilon/200 \) whenever \( y \in B(x, 2\delta_x) \). Since \( X = \bigcup_{x \in X} B(x, \delta_x/2) \) is separable, there exists a countable subcovering, \( X = \bigcup_{n=1}^{\infty} B(x_n, r_n/2) \), where \( r_n = \delta_{x_n} \), for some sequence of centers \( \{x_n\} \). By induction (and using the fact that every finite-dimensional subspace of \( X \) has empty interior in \( X \)), we can choose a sequence of *linearly independent* vectors \( \{y_n\} \), with \( y_n \in B(x_n, r_n/2) \), so that

\[
X = \bigcup_{n=1}^{\infty} B(y_n, r_n). \tag{3}
\]

Moreover, we have that

\[
|f(y) - f(y_n)| \leq \varepsilon/100 \quad \text{whenever} \quad \|y - y_n\| \leq r_n. \tag{4}
\]

Now we define the scalloped balls \( B_n \) that are the basis for our perturbed partition of unity: set \( B_1 = B(y_1, r_1) \), and for \( n \geq 2 \) define

\[
B_n = B(y_n, r_n) \setminus \left( \bigcup_{j=1}^{n-1} \overline{B}(y_j, \lambda_n r_j) \right);
\]

where \( 1/2 < \lambda_2 < \lambda_3 < ... < \lambda_n < \lambda_{n+1} < ... < 1 \), with \( \lim_{n \to \infty} \lambda_n = 1 \).

Taking into account that \( \lim_{n \to \infty} \lambda_n = 1 \), it is easily checked that the \( B_n \) form a locally finite open covering of \( X \), with the nice property that \( |f(y) - f(y_n)| \leq \varepsilon/100 \) whenever \( y \in B_n \).

Next, pick a \( C^\infty \) smooth function \( g_1 : \mathbb{R} \rightarrow [0, 1] \) so that:

(i) \( g_1(t) = 1 \) for \( t \leq 0 \),
(ii) \( g_1(t) = 0 \) for \( t \geq \frac{r_1}{2} \),
(iii) \( g_1'(t) < 0 \) if \( 0 < t < \frac{r_1}{2} \);

and define then \( \varphi_1 : X \rightarrow \mathbb{R} \) by \( \varphi_1(x) = g_1(\|x - y_1\|^2) \) for all \( x \in X \). Note that \( \varphi_1 \) is a \( C^\infty \) smooth function whose open support is \( B_1 \), and \( B_1 \cap C_{\varphi_1} = \{y_1\} \), that is, \( y_1 \) is the only critical point of \( \varphi_1 \) that lies inside \( B_1 \).

Now, for \( n \geq 2 \), pick \( C^\infty \) smooth functions \( \theta_{(n,j)} : \mathbb{R} \rightarrow [0, 1] \), \( j = 1, ..., n \), with the following properties. For \( j = 1, ..., n - 1 \), \( \theta_{(n,j)} \) satisfies that
Moreover, under the same conditions as in (iii) just above we have that

(i) $\theta_{(n,j)}(t) = 0$ for $t \leq (\lambda_n r_j)^2$,
(ii) $\theta_{(n,j)}(t) = 1$ for $t \geq r_j^2$,
(iii) $\theta'_{(n,j)}(t) > 0$ if $(\lambda_n r_j)^2 < t < r_j^2$;

while for $j = n$ the function $\theta_{(n,n)}$ is such that

(i) $\theta_{(n,n)}(t) = 1$ for $t \leq 0$,
(ii) $\theta_{(n,n)}(t) = 0$ for $t \geq r_n^2$,
(iii) $\theta'_{(n,n)}(t) < 0$ if $0 < t < r_n^2$.

Then define the function $g_n : \mathbb{R}^n \rightarrow [0,1]$ as

$$g_n(t_1, \ldots, t_n) = \prod_{i=1}^{n} \theta_{(n,i)}(t_i)$$

for all $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$. This function is clearly $C^\infty$ smooth on $\mathbb{R}^n$ and satisfies the following properties:

(i) $g_n(t_1, \ldots, t_n) > 0$ if and only if $t_j > (\lambda_n r_j)^2$ for all $j = 1, \ldots, n - 1$, and $t_n < r_n^2$; and $g_n$ vanishes elsewhere;
(ii) $g_n(t_1, \ldots, t_n) = \theta_{(n,n)}(t_n)$ whenever $t_j \geq r_j^2$ for all $j = 1, \ldots, n - 1$;
(iii) $\nabla g_n(t_1, \ldots, t_n) \neq 0$ provided $(\lambda_n r_j)^2 < t_j$ for all $j = 1, \ldots, n - 1$, and $0 < t_n < r_n^2$.

Moreover, under the same conditions as in (iii) just above we have that

$$\frac{\partial g_n}{\partial t_n}(t_1, \ldots, t_n) = \frac{\partial \theta_{(n,n)}}{\partial t_n}(t_n) \prod_{i=1}^{n-1} \theta_{(n,i)}(t_i) < 0,$$  

(5)

since no function in this product vanishes on the specified set, while for $j < n$, according to the corresponding properties of the functions $\theta_{(n,j)}$ we have that

$$\frac{\partial g_n}{\partial t_j}(t_1, \ldots, t_n) = \frac{\partial \theta_{(n,j)}}{\partial t_j}(t_j) \prod_{i=1, i \neq j}^{n} \theta_{(n,i)}(t_i) > 0.$$  

(6)

If we are not in the conditions of (iii) then the corresponding inequalities do still hold but are not strict.

Let us now define $\varphi_n : X \rightarrow [0,1]$ by

$$\varphi_n(x) = g_n(\|x - y_1\|^2, \ldots, \|x - y_n\|^2).$$

It is clear that $\varphi_n$ is a $C^\infty$ smooth function whose open support is precisely the scalloped ball $B_n$.

As above, let us denote by $C_{\varphi_n}$ the critical set of $\varphi_n$, that is, $C_{\varphi_n} = \{x \in X : \varphi'_n(x) = 0\}$. Since our norm $\| \cdot \|$ is hilbertian we have that, if $x \in C_{\varphi_n} \cap B_n$, then $x$ belongs to the affine span of $y_1, \ldots, y_n$. Indeed, if $x \in B_n$,

$$\varphi'_n(x) = \sum_{j=1}^{n} \frac{\partial g_n}{\partial t_j}(\|x - y_1\|^2, \ldots, \|x - y_n\|^2) 2(x - y_j) = 0,$$  

(7)
then, since the $y_j$'s are independent and the coefficients of the $x - y_j$'s are not all zero by (5), it follows that $x$ is in the affine span of $y_1, ..., y_n$. Here, as is usual, we identify the Hilbert space $X$ with its dual $X^*$, and we make use of the fact that the derivative of the function $x \mapsto \|x\|^2$ is the mapping $x \mapsto 2x$.

Similarly, it can be shown that $x \in C_{f_1 + \cdots + f_m} \cap (B_1 \cup \cdots \cup B_m)$ implies that $x$ belongs to the affine span of $y_1, ..., y_m$. Indeed, write $(\varphi_1 + \cdots + \varphi_m)'(x)$ as in (7) and choose $j$ so that $x \in B_j$ and $x \notin B_k$ for $j < k \leq m$. Then the coefficient of $x - y_j$ in the sum is nonzero by (5).

In order that our approximating function has a small critical set we cannot use the standard approximation provided by the partition of unity associated with the functions $(\varphi_j)_{j \in \mathbb{N}}$, namely

$$x \mapsto \sum_{n=1}^{\infty} \alpha_n \varphi_n(x),$$

where $\alpha_n = f(y_n)$. Indeed, such a function would have a huge set of critical points since it would be constant (equal to $\alpha_n$) on a lot of large places (at least on each $B_n$ minus the union of the rest of the $B_j$). Instead, we will modify this standard approximation by letting the $\alpha_n$ be functions (and not mere numbers) of very small oscillation and with only one critical point (namely $y_n$). So, for every $n \in \mathbb{N}$ let us pick a $C^\infty$ smooth real function $\alpha_n : [0, +\infty) \longrightarrow \mathbb{R}$ with the following properties:

(i) $\alpha_n(0) = f(y_n)$;
(ii) $\alpha_n'(t) < 0$ whenever $t > 0$;
(iii) $|\alpha_n(t) - \alpha_n(0)| \leq \varepsilon/100$ for all $t \geq 0$;

and define $\alpha_n : X \longrightarrow \mathbb{R}$ by $\alpha_n(x) = \alpha_n(\|x - y_n\|^2)$ for every $x \in X$. It is clear that $\alpha_n$ is a $C^\infty$ smooth function on $X$ whose only critical point is $y_n$, and

$$|\alpha_n(x) - f(y_n)| \leq \varepsilon/100$$

for all $x \in X$. Since the sums are locally finite, it is clear that $\varphi$ is a well-defined $C^\infty$ smooth function.

**Fact 2.4.** The function $\varphi$ approximates $f$ nicely. Namely, we have that

(i) $|\varphi(x) - f(x)| \leq \varepsilon/50$ for all $x \in X$, and
(ii) $|\varphi(y) - f(x)| \leq \varepsilon/25$ for all $x, y \in B(y_n, r_n)$ and each $n \in \mathbb{N}$.

**Proof.** For every $n$ we have that $|\alpha_n(x) - f(y_n)| \leq \varepsilon/100$ for all $x \in X$. On the other hand, by (4) above we know that $|f(x) - f(y_n)| \leq \varepsilon/100$ whenever $x \in B(y_n, r_n)$. Then, by the triangle inequality, it follows that

$$|\alpha_n(x) - f(x)| \leq \varepsilon/50$$

(8)
whenever \( x \in B(y_n, r_n) \). Since \( \varphi_m(y) = 0 \) when \( y \notin B(y_m, r_m) \), from (8) we get that
\[
|\varphi(x) - f(x)| = \left| \sum_{m=1}^{\infty} (\alpha_m(x) - f(x))\varphi_m(x) \right| \leq \sum_{m=1}^{\infty} \frac{\varepsilon/50}{\varphi_m(x)} = \varepsilon/50
\]
for all \( x \in X \), which shows (i).

The proof of (ii) is trivial: \( |\varphi(y) - f(x)| \leq |\varphi(y) - f(y)| + |f(y) - f(y_n)| + |f(y_n) - f(x)| \). The first term is smaller than \( \varepsilon/50 \) by (i) and each of the other terms is smaller than \( \varepsilon/100 \) by (4).

Now let us have a look at the derivative of \( \varphi \). To this end let us introduce the auxiliary functions \( f_n \) defined by
\[
f_n(x) = \frac{\sum_{k=1}^{n} \alpha_k(x)\varphi_k(x)}{\sum_{k=1}^{n} \varphi_k(x)}, \quad \text{for all } x \in \bigcup_{i=1}^{n} B_i.
\]
Since the \( B_j \)'s form a locally finite cover, it follows that for each \( x \) there are a neighborhood \( V_x \) and \( n = n_x \) so that \( V_x \subseteq B_n \) and \( V_x \cap B_j = \emptyset \) for \( j > n \). In particular \( \varphi = f_n = f_m \) for \( m \geq n \) on \( V_x \).

Then the expression for the derivative of \( \varphi \) is given by
\[
\varphi'(x) = \frac{\sum_{j=1}^{n} [\alpha_j'(x)\varphi_j(x) + \alpha_j(x)\varphi_j'(x)] \sum_{i=1}^{n} \varphi_i(x) - \sum_{j=1}^{n} \varphi_j'(x) \sum_{i=1}^{n} \alpha_i(x)\varphi_i(x)}{(\sum_{j=1}^{n} \varphi_j(x))^2}.
\]
Therefore, for \( x \) and \( n = n_x \) as above, we have that \( \varphi'(x) = 0 \) if and only if
\[
\sum_{j=1}^{n} \sum_{i=1}^{n} \varphi_i(x) \left[ \alpha_j'(x)\varphi_j(x) + (\alpha_j(x) - \alpha_i(x))\varphi_j'(x) \right] = 0. \tag{9}
\]

**Notation 2.5.** In the sequel \( A[z_1, \ldots, z_k] \) stands for the affine subspace spanned by a finite sequence of points \( z_1, \ldots, z_k \in X \).

**Fact 2.6.** Let \( x \) and \( n = n_x \) be as above. Then there are numbers \( \beta_j(x) \), not all of them zero, such that
\[
\varphi'(x) = \sum_{k=1}^{n} \beta_k(x)(x - y_k).
\]
In particular, if \( x \in C_\varphi \), then \( x \in A_n := A[y_1, \ldots, y_n] \).

**Proof.** As above, in all the subsequent calculations, we will identify the Hilbert space \( X \) with its dual \( X^* \), and the derivative of \( \| \cdot \|^2 \) with the mapping \( x \mapsto 2x \). To save notation, let us simply write
\[
\frac{\partial g_n}{\partial H_j}(\|x - y_1\|^2, \ldots, \|x - y_n\|^2) = \mu_{(n, j)};
\]
and
\[
a_j'(\|x - y_j\|^2) = \eta_j.
\]
Notice that, according to (5) and (6) above, \( \mu_{(n,j)} \geq 0 \) for \( j = 1, ..., n - 1 \), while \( \mu_{(n,n)} \leq 0 \); and \( \mu_{(n,n)} \neq 0 \) provided \( x \in B_n \) and \( x \neq y_n \); on the other hand it is clear that \( \eta_j < 0 \) for all \( j \) unless \( x = y_j \) (in which case \( \eta_j = 0 \)).

Assuming \( x \in C_f \cap B_n \), and taking into account the expression for \( \varphi_j(x) \) from (7) and the fact that \( \alpha_j'(x) = 2\eta_j(x - y_j) \), we can write condition (9) above in the form

\[
2 \sum_{j=1}^{n} \sum_{i=1}^{n} \varphi_i(x) \left[ \eta_j \varphi_j(x) (x - y_j) + (\alpha_j(x) - \alpha_i(x)) \sum_{t=1}^{j} \mu_{(j,t)} (x - y_t) \right] = 0,
\]

which in turn is equivalent (taking the common factors of each \( x - y_j \)) to the following one

\[
\sum_{j=1}^{n} \beta_j (x - y_j) = 0, \quad \text{where}
\]

\[
\beta_j := \left[ \eta_j \varphi_j(x) \sum_{i=1}^{n} \varphi_i(x) + \sum_{k=j}^{n} \left( \sum_{i=1}^{n} (\alpha_k(x) - \alpha_i(x)) \varphi_i(x) \right) \mu_{(k,j)} \right].
\]

We next show that at least one of the \( \beta_j \)'s is strictly negative, hence nonzero. We can obviously assume that \( x \) is not any of the points \( y_1, ..., y_n \) (which are already in \( A_n \)). In this case we have that \( \mu_{(n,n)} < 0 \) and \( \eta_j < 0 \) for all \( j = 1, ..., n \). For simplicity, we will only make the argument in the case \( n = 3 \); giving a proof in a more general case would be as little instructive as tedious to read.

Let us first assume that \( \varphi_j(x) \neq 0 \) for \( j = 1, 2, 3 \). We begin by looking at

\[
\beta_3 := \eta_3 \varphi_3(x) \sum_{i=1}^{3} \varphi_i(x) + \sum_{i=1}^{3} (\alpha_3(x) - \alpha_i(x)) \varphi_i(x) \mu_{(3,3)}.
\]

If \( \sum_{i=1}^{3} (\alpha_3(x) - \alpha_i(x)) \varphi_i(x) \geq 0 \) we are done, since in this case we easily see that \( \beta_3 < 0 \) (remember that \( \mu_{(3,3)} \leq 0 \), \( \eta_3 < 0 \), and \( \varphi_3(x) > 0 \)). Otherwise we have that

\[
\sum_{i=1}^{3} (\alpha_3(x) - \alpha_i(x)) \varphi_i(x) < 0,
\]

and then we look at the term

\[
\beta_2 := \eta_2 \varphi_2(x) \sum_{i=1}^{3} \varphi_i(x) + \sum_{k=2}^{3} \left( \sum_{i=1}^{3} (\alpha_k(x) - \alpha_i(x)) \varphi_i(x) \right) \mu_{(k,2)}.
\]

Now, since \( \mu_{(3,3)} \geq 0 \), we have \( \sum_{i=1}^{3} (\alpha_3(x) - \alpha_i(x)) \varphi_i(x) \mu_{(3,2)} \leq 0 \), and on the other hand \( \eta_2 \varphi_2(x) \sum_{i=1}^{3} \varphi_i(x) < 0 \) so that, if \( \sum_{i=1}^{3} (\alpha_2(x) - \alpha_i(x)) \varphi_i(x) \) happens to be nonnegative, then we also have \( \sum_{i=1}^{3} (\alpha_2(x) - \alpha_i(x)) \varphi_i(x) \mu_{(2,2)} \leq 0 \), and then we are done since \( \beta_2 \), being a sum of negative terms (one of them strictly negative) must
be negative as well. Otherwise,
\[ \sum_{i=1}^{3} \left( \alpha_2(x) - \alpha_i(x) \right) \varphi_i(x) \]
is negative, and then we finally pass to the term
\[ \beta_1 := \eta_1 \varphi_1(x) \sum_{i=1}^{3} \varphi_i(x) + \sum_{k=1}^{3} \left( \sum_{i=1}^{3} \left( \alpha_k(x) - \alpha_i(x) \right) \varphi_i(x) \right) \mu_{(k,1)}. \]

Here, by the assumptions we have made so far and taking into account the signs of \( \mu_{(k,j)} \) and \( \eta_j \), we see that \( \sum_{i=1}^{3} \left( \alpha_k(x) - \alpha_i(x) \right) \varphi_i(x) \mu_{(k,1)} \leq 0 \) for \( k = 2, 3 \). Having arrived at this point, it is sure that \( \sum_{i=1}^{3} \left( \alpha_k(x) - \alpha_i(x) \right) \varphi_i(x) \mu_{(k,1)} \) must be nonnegative (otherwise the numbers \( \sum_{i=1}^{3} \left( \alpha_k(x) - \alpha_i(x) \right) \varphi_i(x) \) should be strictly negative for all \( k = 1, 2, 3 \), which is impossible if one takes \( \alpha_k(x) \) to be the maximum of the \( \alpha_i(x) \)), and now we can deduce as before that \( \beta_1 < 0 \).

Finally let us consider the case when some of the \( \varphi_i(x) \) vanish, for \( i = 1, 2 \) (remember that \( \varphi_3(x) \neq 0 \) since \( x \in B_3 \), the open support of \( \varphi_3 \)). From the definitions of \( \mu_{(k,j)} \), \( g_n \) and \( \varphi_n \), it is clear that \( \mu_{(k,j)} = 0 \) for \( j > k \), and bearing this fact in mind we can simplify equality (10) to a great extent by dropping all the terms that now vanish.

If \( \varphi_2(x) = \varphi_3(x) = 0 \) then (10) reads
\[ \varphi_3(x)^2 \eta_3 \left( x - y_3 \right) = 0, \]
which cannot happen since we assumed \( x \neq y_j \) (this means that the only critical point that \( f_n \) can have in \( B_3 \setminus (B_1 \cup B_2) \) is \( y_3 \)).

If \( \varphi_1(x) = 0 \) and \( \varphi_2(x) \neq 0 \) then the term \( \beta_1 \) accompanying \( (x - y_1) \) in (10) vanishes, and hence (10) is reduced to
\[ \sum_{j=2}^{3} \left[ \eta_j \varphi_j(x) \sum_{i=2}^{3} \varphi_i(x) + \sum_{k=j}^{3} \left( \sum_{i=2}^{3} \left( \alpha_k(x) - \alpha_i(x) \right) \varphi_i(x) \right) \mu_{(k,j)} \right] (x - y_j) = 0. \]

Since at least one of the numbers \( \sum_{i=2}^{3} \left( \alpha_k(x) - \alpha_i(x) \right) \varphi_i(x) \), \( k = 2, 3 \), is nonnegative, the same reasoning as in the first case allows us to conclude that either \( \beta_3 \) or \( \beta_2 \) is strictly negative. Finally, in the case \( \varphi_1(x) \neq 0 \) and \( \varphi_2(x) = 0 \), it is \( \beta_2 \) that vanishes, and (10) reads \( \beta_1 \left( x - y_1 \right) + \beta_3 \left( x - y_3 \right) = 0 \), where
\[ \beta_3 = \eta_3 \varphi_3(x) \sum_{i=1, i \neq 2}^{3} \varphi_i(x) + \sum_{i=1, i \neq 2}^{3} \left( \alpha_3(x) - \alpha_i(x) \right) \varphi_i(x) \mu_{(3,3)}, \]

and
\[ \beta_1 = \eta_1 \varphi_1(x) \sum_{i=1, i \neq 2}^{3} \varphi_i(x) + \sum_{k=1, i \neq 2}^{3} \sum_{i=1, i \neq 2}^{3} \left( \alpha_k(x) - \alpha_i(x) \right) \varphi_i(x) \mu_{(k,1)}. \]
Again, at least one of the numbers $\sum_{i=1,i\neq 2}^{3}(a_k(x) - a_i(x))\varphi_i(x)$, $k = 1, 3$, is non-negative, and the same argument as above applies. □

Remark 2.7. The above discussion actually shows the following inclusions:

- $C_{f_3} \cap B_3 \subseteq \mathcal{A}[y_1, y_2, y_3]$;
- $C_{f_3} \cap (B_3 \setminus B_1) \subseteq \mathcal{A}[y_2, y_3]$; and
- $C_{f_3} \cap (B_3 \setminus (B_1 \cup B_2)) \subseteq \mathcal{A}[y_3]$.

More generally, for each $n \in \mathbb{N}$ and for every finite sequence of positive integers $k_1 < k_2 < \ldots < k_m < n$, we have $C_{f_n} \cap (B_n \setminus \bigcup_{j=1}^{m} B_{k_j}) \subseteq \mathcal{A}[\{y_1, \ldots, y_n\} \setminus \{y_{k_1}, \ldots, y_{k_m}\}]$. This is a crucial fact when one wants to give a self-contained proof of Theorem 1.1 (in the case $m = 1$); see Remark 2.3 above.

Since $\varphi$ has a continuous derivative, it is obvious that its critical set $C_\varphi$ is closed in $U$. From Fact 2.6 it follows that $C_\varphi$ is locally contained in finite dimensional subspaces, that is, for each $x \in C_\varphi$ there is an open bounded neighborhood $V_x$ of $x$ so that $C_\varphi \cap V_x$ is contained in a finite dimensional subspace $F_x$ of $X$ and hence is compact (as is closed and bounded as well). This means that $C_\varphi$ is locally compact, and concludes the proof of Proposition 2.2 in the case $m = 1$.

Case II. Let us now deal with the case when $f : X \to \mathbb{R}^m$ with $m \geq 2$. We denote $f = (f^1, \ldots, f^m)$, where $f^1, \ldots, f^m$ are the coordinate functions of $f$. In this case we have to construct $C^\infty$ smooth functions $\varphi^1, \ldots, \varphi^m$ so that each $\varphi^j$ uniformly approximates $f^j$ and the set of points $x \in X$ at which the derivatives $df^j(x)$, ... $d\varphi^m(x)$ are linearly dependent is locally compact. If we succeed in doing so then it is clear that the function $\varphi = (\varphi^1, \ldots, \varphi^m) : X \to \mathbb{R}^m$ will approximate $f$ and its set $C_\varphi$ of critical points will be closed and locally compact.

Let us define $\varepsilon_j = \varepsilon/\sqrt{4m}$, $j = 1, \ldots, m$. As each of the functions $f^j$, with $j = 1, \ldots, m$, is continuous, for every $x \in X$ there exists $\delta^j_x > 0$ so that

$$|f^j(y) - f^j(x)| \leq \varepsilon_j/200 \quad \text{for all} \quad y \in B(x, 2\delta_x).$$

Since $X = \bigcup_{x \in X} B(x, \delta^j_x/2)$ is separable, we may take a countable subcovering,

$$X = \bigcup_{n=1}^{\infty} B(x^n_j, r^n_j/2),$$

where $r^n_j = \delta^j_n$, for each $j = 1, \ldots, m$.

Now, we can slightly perturb the centers $x^n_j$ of the balls so that the union of all the $m$ sequences of centers forms a set of linearly independent vectors. Indeed, bearing in mind that the complement of every finite dimensional subspace of $X$ is dense in the infinite dimensional space $X$, we may inductively choose (taking $m$ points $y^n_1, \ldots, y^n_m$ at each $k$-th step of the induction process) sequences of points $(y^n_j)_{n=1}^{\infty}$, $j = 1, \ldots, m$, with $y^n_j \in B(x^n_j, r^n_j/2)$, so that:

(i) $\{y^n_j : n \in \mathbb{N}, j = 1, \ldots, m\}$ is a set of linearly independent vectors;

(ii) $X = \bigcup_{n=1}^{\infty} B(y^n_j, r^n_j)$ for every $j = 1, \ldots, m$; and

(iii) $|f^j(y) - f^j(y^n_j)| \leq \varepsilon_j/100$ whenever $\|y - y^n_j\| \leq r^n_j$. 

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Next, for each collection of balls \( \{ B(y^i_n, r^i_n) \}_{n \in \mathbb{N}} \) and each function \( f^j \), define scalloped balls \( B^j_n \) and construct a function \( \varphi^j \) exactly as in Case I above, so that

\[
| \varphi^j(y) - f^j(y) | \leq \varepsilon_j
\]

for all \( y \in B(y^i_n, r^i_n) \).

This function \( \varphi^j \) is of the form

\[
\varphi^j(x) = \sum_{n=1}^{\infty} \frac{\alpha^j_n(x) \varphi^j_n(x)}{\sum_{n=1}^{\infty} \varphi^j_n(x)} = \lim_{n \to \infty} f^j_n(x),
\]

where

\[
f^j_n(x) = \sum_{k=1}^{n} \frac{\alpha^j_k(x) \varphi^j_k(x)}{\sum_{k=1}^{n} \varphi^j_k(x)}, \text{ for all } x \in \bigcup_{i=1}^{m} B^j_i,
\]

the domains of the \( f^j_n \) form increasing towers of open sets whose union is \( X \), and, for each \( x \in X \) there is some open neighborhood \( V^j_n \) of \( x \) and some \( n^j \in \mathbb{N} \) so that \( \varphi^j(y) = f^j_n(y) \) for all \( y \in V^j_n \) and all \( n \geq n^j \).

Now define the mappings \( \varphi : X \to \mathbb{R}^m \) and \( f_n : \bigcap_{j=1}^{m} \bigcup_{i=1}^{n} B^j_i \to \mathbb{R}^m \) by

\[
\varphi(x) = (\varphi^1(x), ..., \varphi^m(x)), \quad \text{and} \quad f_n(x) = (f^1_n(x), ..., f^m_n(x)).
\]

By the choice of the \( \varepsilon_j \) and the construction of the functions \( \varphi^j \), it is clear that

\[
||\varphi(x) - f(x)|| \leq \varepsilon/2, \text{ for all } x \in X,
\]

that is, \( \varphi \) approximates \( f \) as is required.

**Fact 2.8.** If \( x \in C_{f_n} \cap \bigcup_{j=1}^{m} \bigcup_{i=1}^{n} B^j_i \) then \( x \in A[y^i_j : 1 \leq i \leq n, 1 \leq j \leq m] \).

**Proof.** According to Fact 2.6, for each \( j \) and each \( x \in \bigcup_{i=1}^{n} B^j_i \) we can assign numbers \( \beta^j_1(x), ..., \beta^j_n(x) \) such that at least one of them does not vanish, and

\[
df^j_n(x) = \sum_{k=1}^{n} \beta^j_k(x)(x - y^j_k).
\]

Suppose now that \( x \in \bigcap_{j=1}^{m} \bigcup_{i=1}^{n} B^j_i \) and that the linear map \( df_n(x) : X \to \mathbb{R}^m \) is not surjective (that is, \( x \) is a critical point of \( f_n \)); this means that there are numbers \( \gamma_1(x), ..., \gamma_m(x) \), not all of them zero, such that

\[
\sum_{j=1}^{m} \gamma_j(x) df^j_n(x) = 0.
\]

Then, by combining (11) and (12) we get that

\[
\sum_{j=1}^{m} \sum_{k=1}^{n} \gamma_j(x) \beta^j_k(x)(x - y^j_k) = 0,
\]

where not all of the numbers \( \gamma_j(x) \beta^j_k(x) \) vanish. Since the vectors \( y^j_k \) are all linearly independent, it follows from (13) that \( x \) is in the affine span of the vectors \( y^j_k \) with \( j = 1, ..., m; k = 1, ..., n \).
As \( d\phi \) is continuous, the set of critical points \( C_\varphi \) is closed in \( U \). Now we can easily show that \( C_\varphi \) is locally compact as well. Indeed, take \( x \in X \). For every \( j = 1, ..., m \) we know that there exists a neighborhood \( V^j_x \) of \( x \) and some \( n^j_x \in \mathbb{N} \) so that \( \varphi^j(y) = f^j_n(y) \) for all \( y \in V^j_x \) and every \( n \geq n^j_x \). Fix \( n = n_x := \max\{n^1_x, ..., n^m_x\} \), and take \( W_x \) an open bounded neighborhood of \( x \) so that \( W_x \subset V_x := \bigcap_{j=1}^m V^j_x \).

Then we have that
\[
\varphi(y) = (\varphi^1(y), ..., \varphi^m(y)) = (f^1_n(y), ..., f^m_n(y)) = f_n(y)
\]
for all \( y \in V_x \), and in particular \( V_x \subset \bigcap_{j=1}^m \bigcup_{i=1}^n B^j_i \). Now, according to Fact 2.8, it follows that \( C_\varphi \cap V_x = C_{f_n} \cap V_x \) is contained in an affine subspace of dimension \( mn \). In particular \( C_\varphi \cap \overline{W_x} \) is compact, because it is closed, bounded, and is contained in a finite-dimensional subspace. \( \square \)

**Remark 2.9.** Let us say a few words as to the way one has to modify the above proof in order to establish Proposition 2.2 when \( \varepsilon \) is a continuous positive function. At the beginning of the proof of Case I of Proposition 2.2, before choosing the \( \delta_x \)'s, we have to take some number \( \alpha_x > 0 \) so that \( |\varepsilon(y) - \varepsilon(x)| \leq \varepsilon(x)/4 \) whenever \( \|y - x\| \leq 2\alpha_x \) and then we can find some \( \delta_x \leq \alpha_x \) so that \( |f(y) - f(x)| \leq \varepsilon(x)/200 \) whenever \( y \in B(x, 2\delta_x) \). In particular, after choosing the \( r_n = \delta_{x_n} \) as in the proof of Case I above, we have that
\[
|f(y) - f(y_n)| \leq \varepsilon(y_n)/200, \quad \varepsilon(y) \leq \frac{4}{3} \varepsilon(y)
\]
for all \( y \in B(y_n, r_n) \). Then we can go on with the proof, with appropriate modifications, to construct the functions \( \varphi \) and \( f_n \). Some obvious changes must be made in the definition of the functions \( a_n \) and \( \alpha_n \). Fact 2.4 now tells us that
\[
|\varphi(y) - f(y_n)| \leq \varepsilon(y_n)/4
\]
for all \( y \in B(y_n, r_n) \). Then, by combining (14) and (15) we get that
\[
|\varphi(y) - f(y)| \leq |\varphi(y) - f(y_n)| + |f(y_n) - f(y)| \leq \frac{\varepsilon(y_n)}{4} + \frac{\varepsilon(y_n)}{8} = \frac{3}{8} \varepsilon(y_n) \leq \frac{\varepsilon(y)}{2}
\]
for all \( y \in B(y_n, r_n) \) and, since these balls cover \( X \), this proves that \( |\varphi(y) - f(y)| \leq \varepsilon(y)/2 \) for all \( y \in X \).

In Case II it is enough to define the functions \( \varepsilon_j(x) = \varepsilon(x)/\sqrt{4m} \), for \( j = 1, ..., m \). The rest of the proof applies just replacing \( \varepsilon_j \) and \( \varepsilon \) with \( \varepsilon_j(x) \) and \( \varepsilon(x) \), and making some obvious minor modifications as in Case I. \( \square \)

**Open Problem 2.10.** The above proof works only in Hilbert space (because Fact 2.6 fails whenever the norm is not Hilbertian). We do not know if Theorem 1.1 is true for more general Banach spaces \( X \).

**Acknowledgements**

We are indebted to Tadeusz Dobrowolski, who read the first version of this paper and suggested a simplification of the original proof (hinted at in Remark 2.3) by pointing out to us West’s paper [25]. We also wish to thank Pilar Cembranos, José
Mendoza, Tijani Pakhrrou and Raúl Romero, who helped us to realize that Fact 2.6 fails whenever the norm is not hilbertian. This research was in part carried out during a stay of the first-named author in the Mathematics Department of University College London; this author wishes to thank the members of that Department and very especially David Preiss for their kind hospitality and advice. Finally we thank the referee for a thorough report which made us improve the exposition.

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