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THE MORSE-SARD THEOREM REVISITED

D. AZAGRA, J. FERRERA, AND J. GÓMEZ-GIL

Abstract. Let n, m, k be positive integers with $k = n - m + 1$. We establish an abstract Morse-Sard-type theorem which allows us to deduce, on the one hand, a previous result of De Pascale's for Sobolev $W^{k,p}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ functions with $p > n$ and, on the other hand, also the following new result: if $f \in C^{k-1}(\mathbb{R}^n, \mathbb{R}^m)$ has a Taylor expansion of order k at almost every point $x \in \mathbb{R}^n$ and satisfies $|f(x+h) - f(x) - Df(x)(h) - ... - D^{k-1}f(x)(h^{k-1})| = O(|h|^k)$ for every $x \in \mathbb{R}^n$, then the set of critical values of f is Lebesgue-null in $\mathbb{R}^m$. In particular this allows us to establish the Morse-Sard property for several classes of functions with badly behaved derivatives of order $k-1$ (for instance, functions whose derivatives of order $k-1$ are Stepanov functions) to which none of the previous generalizations of the Morse-Sard theorem can be applied. We also provide Dubovitskiĭ-type versions of the above results.

1. Introduction and main results

The Morse-Sard theorem [21, 27] states that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is of class C^k, where $k = n - m + 1$, then the set of critical values of f has measure zero in $\mathbb{R}^m$. A famous example of Whitney's [29] shows that this classical result is sharp within the classes of functions C^j. However, several generalizations of the Morse-Sard theorem for other classes of functions (notably Hölder and Sobolev spaces) have appeared in the literature; see [2, 3, 4, 6, 7, 8, 9, 12, 16, 14, 22, 17, 24, 25, 26, 30] and the references therein.

In this paper we will give a new, elementary proof of the Morse-Sard theorem which only requires f to be $k-1$ times continuously differentiable, to have a Taylor expansion of order k at almost every point $x \in \mathbb{R}^n$, and to satisfy $\limsup_{h\to 0} \frac{|f(x+h)-f(x)-Df(x)(h)-...-D^{k-1}f(x)(h^{k-1})|}{|h|^k} < \infty$ for every $x \in \mathbb{R}^n$. We will also see that this new version of the Morse-Sard theorem is neither weaker nor stronger than the recent results of Bourgain, Korobkov and Kristensen [7] for $W^{n,1}_{\mathrm{loc}}$ and $BV_{n,\mathrm{loc}}$ functions in the case $m = 1$, and it can be useful in the analysis of functions f of class $C^{k-1}(\mathbb{R}^n, \mathbb{R}^m)$ whose derivatives $D^{k-1}f$ behave somewhat badly near some points or regions, but nonetheless f has Taylor expansions almost everywhere. We will deduce our

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result of a new, abstract Morse-Sard-type theorem which will also allow us to easily recover De Pascale's theorem [8] for $W^{k,p}(\mathbb{R}^n,\mathbb{R}^m)$ Sobolev functions with $p>n$.

We will say that a function $f:\mathbb{R}^n\to\mathbb{R}^m$ has a Taylor expansion of order k at a point x provided there is a polynomial $P_x:\mathbb{R}^n\to\mathbb{R}^m$, of degree less than or equal to k, such that

$$\lim_{h\to 0}\frac{f(x+h)-P_x(h)}{|h|^k}=0. \tag{1.1}$$

Whenever such a P_x exists, it is unique, and we will denote $P_x=J^kf(x)$ (read the k-th order jet of f at x). Abusing notation, we will also write

$$J^kf(x)=(P_x^1,...,P_x^k),$$

where $P_x(h)=f(x)+P_x^1(h)+...+P_x^k(h)$, and P_x^j is the j-homogeneous polynomial component of P_x, for each $j=1,2,...,k$.

If f happens to be k times differentiable at x then it is known that f has a Taylor expansion P_x of order k at x, and in fact we have $P_x^j(h)=\frac{1}{j!}D^jf(x)(h^j)$ for every $j=1,...,k$. However, a function f may have a Taylor expansion of order k at x without being k times differentiable at x (or even two times differentiable at x, no matter how large k is).

The set of all functions $f:\mathbb{R}^n\to\mathbb{R}^m$ of class C^j such that f has a Taylor expansion of order k at each point $x\in\mathbb{R}^n$ is obviously a vector space, which we will denote by $\mathcal{C}^j\mathcal{P}^k(\mathbb{R}^n,\mathbb{R}^m)$ in this paper.

It should be noted that our definition of function admitting Taylor expansions on a set is much less demanding than other definitions appearing in the literature (compare e.g. with [31, Chapter 3.5]). Specifically, we do not require that the mappings $x\mapsto P_x$ be locally bounded, nor that the limits in (1.1) be locally uniform in x. Consequently, functions admitting Taylor expansions of order k at all points of a compact set are not, in general, restrictions of C^k functions.

The first of our main results is as follows.

Theorem 1.1. *Let $n\geq m$ be positive integers, $k:=n-m+1$ and let $f:\mathbb{R}^n\to\mathbb{R}^m$ be such that*

(1) *$f\in C^{k-1}(\mathbb{R}^n,\mathbb{R}^m)$;*
(2) *$\limsup_{h\to 0}\frac{|f(x+h)-f(x)-Df(x)(h)-...-D^{k-1}f(x)(h^{k-1})|}{|h|^k}<\infty$ for every $x\in\mathbb{R}^n$;*
(3) *f has a Taylor expansion of order k at almost every point $x\in\mathbb{R}^n$.*

Then $\mathcal{L}^m(f(C_f))=0$, where $C_f:=\{x\in\mathbb{R}^n:\mathrm{rank}\,(Df(x))<m\}$.
The same statement holds true if $\mathbb{R}^n$ is replaced with an open subset of $\mathbb{R}^n$.

Here, as in the rest of the paper, $\mathcal{L}^m$ denotes the Lebesgue outer measure in $\mathbb{R}^m$. Notice that for $n>m$ the critical set $C_f=\{x\in\mathbb{R}^n:\mathrm{rank}\,(Df(x))\leq m-1\}$ is defined as usual, as f must be at least of class C^1. In the case $n=m$, we note that condition (2) reads $\limsup_{h\to 0}\frac{|f(x+h)-f(x)|}{|h|}<\infty$ for

every $x \in \mathbb{R}^n$, which by itself already implies, by Stepanov's differentiability theorem, that f is differentiable almost everywhere (in particular f satisfies condition (3) of the above Theorem), and therefore we may define the critical set of f by $C_f = \{x \in \mathbb{R}^n : Df(x) \text{ exists and } \operatorname{rank}(Df(x)) \leq n-1\}$. In this case the above Theorem tells us that $\mathcal{L}^n(f(C_f)) = 0$. See also Proposition 4.1 below. For $n > m$, however, condition (2) gives us some control of $|D^{k-1}f(x+h)(h^{k-1}) - D^{k-1}f(x)(h^{k-1})|$, but none of $\|D^{k-1}f(x+h) - D^{k-1}f(x)\|$, and for this reason we cannot obtain that $\limsup_{h\to 0} \frac{\|D^{k-1}f(x+h)-D^{k-1}f(x)\|}{|h|} < \infty$ for every $x \in \mathbb{R}^n$ and then apply Stepanov's theorem to deduce that $D^{k-1}f$ is almost everywhere differentiable and in particular (3) holds. That is why we must demand that (3) holds in addition to (2) whenever $n > m$.

Nevertheless, if we require that $\limsup_{h\to 0} \frac{\|D^{k-1}f(x+h)-D^{k-1}f(x)\|}{|h|} < \infty$ for every x (that is, the derivative $D^{k-1}f$ satisfies the hypothesis of Stepanov's differentiability theorem everywhere), then f will satisfy (2) and (3). Therefore we have the following.

Theorem 1.2. *Assume that $f \in C^{k-1}(\mathbb{R}^n, \mathbb{R}^m)$, with $k = n - m + 1 \geq 1$, and that $D^{k-1}f$ satisfies*

$$\limsup_{h\to 0} \frac{\|D^{k-1}f(x+h) - D^{k-1}f(x)\|}{|h|} < \infty$$

for every $x \in \mathbb{R}^n$. Then $\mathcal{L}^m(f(C_f)) = 0$.
The same statement holds true if $\mathbb{R}^n$ is replaced with an open subset of $\mathbb{R}^n$.

It is clear that the above Theorem generalizes the main result of [3]. On the other hand, observe that if $J^k f(x) = (P_x^1, ..., P_x^k)$ and $f \in C^{k-1}(\mathbb{R}^n, \mathbb{R}^m)$ then

$$\limsup_{h\to 0} \frac{|f(x+h) - f(x) - Df(x)(h) - ... - D^{k-1}f(x)(h^{k-1})|}{|h|^k} = \|P_x^k\| < \infty.$$

Thus, another consequence of Theorem 1.1 is the following.

Corollary 1.3. *Let n, m be positive integers with $n \geq m$, and set $k = n - m + 1$. If $f \in \mathcal{C}^{k-1}\mathcal{P}^k(\mathbb{R}^n, \mathbb{R}^m)$ then the set of critical values of f is of Lebesgue measure zero in $\mathbb{R}^m$.*
The same statement holds true if $\mathbb{R}^n$ is replaced with an open subset of $\mathbb{R}^n$.

Obviously, this Corollary also implies the classical version of the Morse-Sard theorem for functions $f : \mathbb{R}^n \to \mathbb{R}^m$ of class C^k. It is natural to ask whether condition (2) of Theorem 1.1 could be replaced with the same limit condition holding for a.e. x instead of all x. The answer is negative, as can be ascertained by examining Whitney's classical example [29]. Nonetheless, it is perhaps worth noticing that the proof of Theorem 1.1 shows that it is sufficient that (2) be satisfied on C_f. See Section 5 below for more details, and for some variants of Theorem 1.1 in the case $m = 1$ in connection with this question and the results from [22].

We will deduce Theorem 1.1 from the following abstract, more powerful Morse-Sard-type result.

Theorem 1.4. *For all positive integers $n \geq m \geq 1$, $2 \leq j \leq n$, and open sets U and V in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, let $\mathcal{C}^j_{n,m}(U,V)$ be classes of mappings $f : U \to V$ such that:*

(MS1) $C^j(U,V) \subset \mathcal{C}^j_{n,m}(U,V) \subset C^{j-1}(U,V)$;

(MS2) *if $n \geq m+1$, $f \in \mathcal{C}^{n-m+1}_{n,m}(U,V)$, A is a subset of $\{x : D^i f(x) = 0$ for $i = 1, ..., n-m\}$ and $\mathcal{L}^n(A) = 0$, then $\mathcal{L}^m(f(A)) = 0$;*

(MS3) *if $n \geq m+1$ then every $f \in \mathcal{C}^{n-m+1}_{n,m}(U,V)$ has a Taylor expansion of order $k = n-m+1$ at almost every $x \in \mathbb{R}^n$;*

(MS4) *if $f \in \mathcal{C}^j_{n,n}(U,V)$ and f maps U diffeomorphically onto V then $f^{-1} \in \mathcal{C}^j_{n,n}(V,U)$;*

(MS5) *if $f \in \mathcal{C}^j_{n,m}(U,V)$, $g \in \mathcal{C}^j_{n,n}(W,U)$, and either f is of class C^j or g is a diffeomorphism, then $f \circ g \in \mathcal{C}^j_{n,m}(W,V)$;*

(MS6) *if $g \in \mathcal{C}^j_{n,m}(U,V)$, $1 \leq i \leq m-1$, then $g_y \in \mathcal{C}^j_{n-i,m}(U_y, \mathbb{R}^m)$ for $\mathcal{L}^i$-a.e. $y \in \mathbb{R}^i$ with $U_y := \{z \in \mathbb{R}^{n-i} : (y,z) \in U\} \neq \emptyset$, where $g_y : U_y \to \mathbb{R}^m$ is defined by $g_y(z) = g(y,z)$;*

(MS7) *if $f \in \mathcal{C}^j_{n,i}(U,V)$ and $g \in \mathcal{C}^j_{n,s}(U,V')$ with $i+s = m$, then $\varphi \in \mathcal{C}^j_{n,m}(U, V \times V')$, where φ is defined by $\varphi(x) = (f(x), g(x))$.*

Assume that $n \geq m+1$. Then $\mathcal{L}^m(f(C_f)) = 0$ for every $f \in \mathcal{C}^{n-m+1}_{n,m}(U,V)$.

Moreover, in the special case $m = 1$, the classes $C^j_{n,1}(U,V)$ may be defined only for $j = n$, and the conditions (MS1) with $j = n$, (MS2) and (MS3) alone are sufficient to ensure that $\mathcal{L}^1(f(C_f)) = 0$ for every $f \in \mathcal{C}^n_{n,1}(U,V)$.

Another consequence of the above result is the following theorem of De Pascale [8] for Sobolev spaces (see also [12] for a simpler proof).

Theorem 1.5 (De Pascale, 2001)**.** *Let n, m, k be positive integers with $n \geq m$, $k = n-m+1$, and let p be a real number with $p > n$. Then $\mathcal{L}^m(f(C_f)) = 0$ for every f in the Sobolev space $W^{k,p}_{loc}(\mathbb{R}^n, \mathbb{R}^m)$.*

We will also establish the following Dubovitskiĭ-Sard-type versions of Theorems 1.4 and 1.1.

Theorem 1.6. *For all positive integers $n \geq m \geq 1$, $2 \leq j \leq n$, and open sets U and V in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, let $\mathcal{C}^j_{n,m}(U,V)$ be classes of mappings $f : U \to V$ such that:*

(DS1) $C^j(U,V) \subset \mathcal{C}^j_{n,m}(U,V) \subset C^{j-1}(U,V)$;

(DS2) *if $n \geq m+1$, $2 \leq k \leq n-m+1$, $\ell = n-m-k+1$, $f \in \mathcal{C}^k_{n,m}(U,V)$, A is a subset of $\{x : D^j f(x) = 0$ for $j = 1, ..., k-1\}$ and $\mathcal{L}^n(A) = 0$, then $\mathcal{H}^\ell(A \cap f^{-1}(y)) = 0$ for a.e. $y \in \mathbb{R}^m$;*

(DS3) *if $n \geq m+1$ and $2 \leq k \leq n-m+1$ then every $f \in \mathcal{C}^k_{n,m}(U,V)$ has a Taylor expansion of order k at almost every $x \in \mathbb{R}^n$;*

(DS4) *if* $f \in \mathcal{C}^j_{n,n}(U,V)$ *and* f *maps* U *diffeomorphically onto* V *then* $f^{-1} \in \mathcal{C}^j_{n,n}(V,U)$*;*

(DS5) *if* $f \in \mathcal{C}^j_{n,m}(U,V)$*,* $g \in \mathcal{C}^j_{n,n}(W,U)$*, and either* f *is of class* C^j *or* g *is a diffeomorphism, then* $f \circ g \in \mathcal{C}^j_{n,m}(W,V)$*;*

(DS6) *if* $g \in \mathcal{C}^j_{n,m}(U,V)$*,* $1 \le i \le m-1$*, then* $g_y \in \mathcal{C}^j_{n-i,m}(U_y,\mathbb{R}^m)$ *for* $\mathcal{L}^i$*-a.e.* $y \in \mathbb{R}^i$ *with* $U_y := \{z \in \mathbb{R}^{n-i} : (y,z) \in U\} \neq \emptyset$*, where* $g_y : U_y \to \mathbb{R}^m$ *is defined by* $g_y(z) = g(y,z)$*;*

(DS7) *if* $f \in \mathcal{C}^j_{n,i}(U,V)$ *and* $g \in \mathcal{C}^j_{n,s}(U,V')$ *with* $i+s=m$*, then* $\varphi \in \mathcal{C}^j_{n,m}(U,V\times V')$*, where* φ *is defined by* $\varphi(x) = (f(x),g(x))$*.*

Assume that $n \ge m+1$*,* $2 \le k \le n-m+1$*, and* $\ell = n-m-k+1$*. Then, for every* $f \in \mathcal{C}^k_{n,m}(U,V)$ *we have*

$$\mathcal{H}^\ell(C_f \cap f^{-1}(y)) = 0 \text{ for a.e. } y \in \mathbb{R}^m.$$

Moreover, in the special case $m=1$*, the conditions (DS1)-(DS3) alone are sufficient to ensure that, if* $2 \le k \le n-m+1$ *and* $\ell = n-m-k+1$*, then for every* $f \in \mathcal{C}^n_{n,1}(U,V)$ *one has* $\mathcal{H}^\ell(C_f \cap f^{-1}(y)) = 0$ *for a.e.* $y \in \mathbb{R}$*.*

Here, as in the rest of the paper, $\mathcal{H}^\ell$ denotes the ℓ-dimensional outer Hausdorff measure in $\mathbb{R}^n$.

Theorem 1.7. *Let* $n \ge m$ *be positive integers,* $1 \le k \le n-m+1$*, and let* $f : \mathbb{R}^n \to \mathbb{R}^m$ *be such that*

(1) $f \in C^{k-1}(\mathbb{R}^n,\mathbb{R}^m)$*;*

(2) $\limsup_{h\to 0} \frac{|f(x+h)-f(x)-Df(x)(h)-\ldots-D^{k-1}f(x)(h^{k-1})|}{|h|^k} < \infty$ *for every* $x \in \mathbb{R}^n$*;*

(3) f *has a Taylor expansion of order* k *at almost every* $x \in \mathbb{R}^n$*.*

Then, for $\ell = n-m-k+1$*, we have that* $\mathcal{H}^\ell(C_f \cap f^{-1}(y)) = 0$ *for a.e.* $y \in \mathbb{R}^m$*.*

It is clear that Theorems 1.6 and 1.7 include Theorems 1.4 and 1.1 as particular cases for $k = n-m+1$, or equivalently $\ell = 0$. We chose to state them separately for expository reasons.

Another consequence of Theorem 1.6 is the following Theorem of P. Hajłasz and S. Zimmerman [14]

Theorem 1.8 (Hajłasz-Zimmerman)**.** *Fix* $n,m,k \in \mathbb{N}$*. Suppose* $\Omega \subseteq \mathbb{R}^n$ *is open and* $f \in W^{k,p}_{loc}(\Omega,\mathbb{R}^m)$ *for some* $n < p < \infty$*. If* $\ell = \max\{n-m-k+1, 0\}$ *then* $\mathcal{H}^\ell(C_f \cap f^{-1}(y)) = 0$ *for a.e.* $y \in \mathbb{R}^m$*.*

Both Theorems 1.7 and 1.8 are generalizations of the so-called Dubovitskiĭ-Sard theorem [9], which guarantees the same conclusion under the more stringent assumption that $f \in C^k(\mathbb{R}^n,\mathbb{R}^m)$.

The rest of this paper is organized as follows. In Section 2 we will collect several known or easy results that will facilitate the proof of Theorem 1.4, which we will provide in Section 3. In section 4 we will explain how Theorem

1.1 is deduced from Theorem 1.4. In Section 5 we will explain how Theorem 1.5 can be deduced from Theorem 1.4 with little effort, we will study some examples that will clarify the relations of the above results to other authors' work, and we will state and prove some variants of Theorem 1.1 for $m = 1$. Finally, in Section 6 we will explain what ingredients have to be added to the proofs of Theorems 1.4 and 1.1 in order to obtain Theorems 1.6 and 1.7 (again, we choose to separate proofs for expository reasons and because the Dubovitskiĭ-Sard-type results require using more advanced tools, such as the upper integral and Hausdorff measures).

2. Some tools

In the proof of our main results it will be very convenient to use the Kneser-Glaeser Rough Composition Theorem, which we next restate (see [1, Theorem 14.1] for its proof, based on an application of Whitney's Extension Theorem [28]). Let us recall that, given a positive integer s, a map f is said to be s-flat on a set A if $D^j f(x) = 0$ for every $x \in A$ and $j = 1, ..., s$.

Theorem 2.1 (Kneser-Glaeser)**.** *Let $W \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open sets; $A^* \subset W$ and $A \subset V$, with A closed relative to V, $f : V \to \mathbb{R}^p$ of class C^r on V and s-flat on A, $g : W \to V$ of class C^{r-s} with $g(A^*) \subset A$. Then there is a map $H : W \to \mathbb{R}^p$ of class C^r satisfying:*

(1) *$H(x) = f(g(x))$ for $x \in A^*$;*
(2) *H is s-flat on A^*.*

We next state and prove seven lemmas that we will also need in the next two sections.

Lemma 2.2. *Let $n \geq m$, $k = n - m + 1$, and $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function. Assume that N is a subset of $\{x : \limsup_{h\to 0} \frac{|f(x+h)-f(x)|}{|h|^k} < \infty\}$, and that $\mathcal{L}^n(N) = 0$. Then $\mathcal{L}^m(f(N)) = 0$ as well.*

Proof. For each $j \in \mathbb{N}$ we define

$$A_j = \{x \in \mathbb{R}^n : \limsup_{h\to 0} \frac{|f(x+h) - f(x)|}{|h|^k} \leq j\}.$$

It is clear that $N \subset \bigcup_{j=1}^{\infty} A_j$, hence it is enough to see that $\mathcal{L}^m(f(N \cap A_j)) = 0$ for each j.

Let us fix $j \in \mathbb{N}$, $\varepsilon > 0$. Since $\mathcal{L}^n(N) = 0$, for each $i \in \mathbb{N}$ there exists a sequence $\{C_{i\alpha}\}_\alpha$ of cubes with $\operatorname{diam}(C_{i\alpha}) \leq 1/i$, $N \subset \bigcup_\alpha C_{i\alpha}$, and

$$\sum_\alpha \operatorname{vol}(C_{i\alpha}) < \frac{\varepsilon}{n^{n/2}\alpha(m)(j+1)^m},$$

where $\alpha(m)$ denotes the volume of the unit ball of $\mathbb{R}^m$. Define now

$$D_i = \{x \in A_j \cap N : f(x+h) \geq f(x) - (j+1)|h|^k \text{ whenever } |h| < \frac{1}{i}\}.$$

It is easily checked that $D_i \subseteq D_{i+1}$ for every i, and $A_j \cap N = \bigcup_{i=1}^{\infty} D_i$.

If $x, y \in D_i \cap C_{i\alpha}$ we have $|f(x) - f(y)| \leq (j+1)|x-y|^k$ and therefore, using that $km = (k-1)(m-1) + n \geq n$, we get

$$|f(x) - f(y)|^m \leq (j+1)^m \mathrm{diam}(C_{i\alpha})^{km} \leq (j+1)^m \mathrm{diam}(C_{i\alpha})^n,$$

which implies that

$$\mathrm{diam}(f(D_i \cap C_{i\alpha})) \leq (j+1)^m \mathrm{diam}(C_{i\alpha})^n,$$

and consequently

$$\mathcal{L}^m(f(D_i \cap C_{i\alpha})) \leq (j+1)^m n^{n/2} \alpha(m) \mathrm{vol}(C_{i\alpha})$$

as well. By [10, Theorem 1.1.2, p.5] we obtain

$$\mathcal{L}^m(f(A_j \cap N)) = \lim_{i\to\infty} \mathcal{L}^m(f(D_i)) \leq \limsup_{i\to\infty} \sum_\alpha \mathcal{L}^m(f(D_i \cap C_{i\alpha})) \leq$$
$$\limsup_{i\to\infty} \sum_\alpha (j+1)^m n^{n/2} \alpha(m) \mathrm{vol}(C_{i\alpha}) \leq \varepsilon,$$

and by letting ε go to 0 we conclude that $\mathcal{L}^m(f(N \cap A_j)) = 0$. □

Lemma 2.3. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function, $n \geq m$, $k = n - m + 1$. Then*

$$\mathcal{L}^m\left(f\left(\{x \in \mathbb{R}^n : (0, ..., 0) \in J^k f(x)\}\right)\right) = 0.$$

Proof. Let us denote $A = \{x \in \mathbb{R}^n : (0, ..., 0) \in J^k f(x)\}$. We want to see that $\mathcal{L}^m(f(A)) = 0$. By using obvious truncation arguments we may assume, without loss of generality, that $A \subset B := [-R, R]^n$ for some fixed $R \in \mathbb{N}$. Let $\varepsilon > 0$. For every $i \in \mathbb{N}$ we define

$$D_i = \{x \in B : f(x+h) \geq f(x) - \varepsilon |h|^k \text{ whenever } |h| \leq \frac{\sqrt{n}}{i}\}.$$

Note that $\{D_i\}$ is an increasing sequence of sets such that $A \subset \bigcup_j D_j$ (indeed, for every $x \in A$ we have

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{|h|^k} = 0,$$

and consequently $f(x+h) - f(x) \geq -\varepsilon |h|^k$ if $|h| \leq \frac{\sqrt{n}}{i}$ for i big enough). For each i, we may decompose B as the union of a family of cubes $\{C_{i\alpha}\}_{\alpha=1}^{N(i)}$ of diameter $\sqrt{n}/i$ (hence of volume $1/i^n$) with pairwise disjoint interiors. In particular $\sum_\alpha \mathrm{vol}(C_{i\alpha}) = \sum_\alpha \frac{1}{i^n} = \mathrm{vol}(B)$.

If $x, y \in D_i \cap C_{i\alpha}$ then $|x - y| \leq \frac{\sqrt{n}}{i}$, hence

$$|f(y) - f(x)| \leq \varepsilon |x-y|^k \leq \frac{\varepsilon n^{k/2}}{i^k}, \tag{2.1}$$

and (recalling that $km \geq n$)

$$|f(y) - f(x)|^m \leq \frac{\varepsilon^m n^{km/2}}{i^{km}} \leq \frac{\varepsilon^m n^{km/2}}{i^n} = \varepsilon^m \, n^{km/2} \mathrm{vol}(C_{i\alpha}). \tag{2.2}$$

It follows that

$$\mathcal{L}^m\left(f(D_i \cap C_{i\alpha})\right) \leq \varepsilon^m \alpha(m) n^{km/2} \mathrm{vol}(C_{i\alpha}),$$

and therefore

$$\begin{aligned}\mathcal{L}^m(f(D_i)) &\leq \sum_\alpha \mathcal{L}^m(f(D_i \cap C_{i\alpha})) \\ &\leq \varepsilon^m \alpha(m) n^{km/2} \sum_\alpha \mathrm{vol}(C_{i\alpha}) = \varepsilon^m \alpha(m) n^{km/2} \mathrm{vol}(B).\end{aligned}$$

Using [10, Theorem 1.1.2, p.5] we obtain

$$\mathcal{L}^m(f(A)) \leq \lim_{i\to\infty} \mathcal{L}^m(f(D_i)) \leq \varepsilon^m \alpha(m) n^{km/2} \mathrm{vol}(B),$$

and by letting ε go to 0 we conclude that $\mathcal{L}^m(f(A)) = 0$. □

Lemma 2.4. *Let $k \geq 2$ be a positive integer and $\varphi : U \to V$ be a C^{k-1} diffeomorphism between two open subsets of $\mathbb{R}^n$. Assume that φ has a Taylor expansion of order k at x. Then the inverse diffeomorphism φ^{-1} has a Taylor expansion of order k at $y = \varphi(x)$.*

In particular, if φ has a Taylor expansion of order k almost everywhere then so does φ^{-1}.

Proof. Without loss of generality we may assume that $x = 0$ and $y = \varphi(0) = 0$. Write $\varphi(h) = P(h) + R(h)$, where $R(h) = o(|h|^k)$ and P is a polynomial of degree less than or equal to k. We have

$$h = \varphi(\varphi^{-1}(h)) = P(\varphi^{-1}(h)) + R(\varphi^{-1}(h)).$$

Since $DP(0) = D\varphi(0)$ is invertible, there is an open neighborhood of 0, which we may assume to be U, on which P is a C^{k-1} diffeomorphism, with an inverse denoted by P^{-1}. It follows that

$$\varphi^{-1}(h) = P^{-1}(P(\varphi^{-1}(h))) = P^{-1}(h - R(\varphi^{-1}(h))),$$

hence

$$\varphi^{-1}(h) - P^{-1}(h) = P^{-1}(h - R(\varphi^{-1}(h))) - P^{-1}(h) = o(|h|^k),$$

because φ^{-1} and P^{-1} are locally bi-Lipschitz, and $R(v) = o(|v|^k)$. Now, P^{-1} need not be a polynomial, but if we define Q as the Taylor polynomial of order k of P^{-1} we then have

$$P^{-1}(h) - Q(h) = o(|h|^k),$$

and by summing the last two equations we obtain that

$$\varphi^{-1}(h) - Q(h) = o(|h|^k),$$

thus proving the first assertion of the Lemma. The second assertion is a consequence of the first one and of the fact that a C^1 diffeomorphism between open subsets of $\mathbb{R}^n$ maps $\mathcal{L}^n$-null sets onto $\mathcal{L}^n$-null sets. □

Lemma 2.5. *Let $k \geq 2$ be a positive integer and $\varphi : U \to V$ be a C^1 diffeomorphism between two open subsets of $\mathbb{R}^n$. Assume that φ satisfies*

$$\limsup_{h\to 0} \frac{|\varphi(x+h) - P(h)|}{|h|^k} < \infty$$

for a point $x \in U$ and a polynomial P of degree up to $k-1$. Then the inverse φ^{-1} satisfies

$$\limsup_{h\to 0} \frac{|\varphi^{-1}(y+h) - Q(h)|}{|h|^k} < \infty$$

for $y = \varphi(x)$ and some polynomial Q of order up to $k-1$.

Proof. Replace $o(|h|^k)$ with $O(|h|^k)$, and P with the Taylor polynomial of order $k-1$ of φ at x, in the proof of Lemma 2.4. □

Lemma 2.6. *Assume that $f : \mathbb{R}^d \to \mathbb{R}^n$ has a Taylor expansion of order k at x, and that $g : \mathbb{R}^n \to \mathbb{R}^m$ has a Taylor expansion of order k at $y = f(x)$. Then $g \circ f$ has a Taylor expansion of order k at x.*

Proof. We may assume that $x = y = 0$, and write $f = Q + S$, $g = P + R$, where P, Q are polynomials of degree less than or equal to k, $S(h) = o(|h|^k)$, and $R(h) = o(|h|^k)$. Then, using that P and Q are locally Lipschitz, it is easy to check that

$$\begin{aligned} g(f(h)) &= P(f(h)) + R(f(h)) \\ &= P(Q(h) + S(h)) + R(Q(h) + S(h)) \\ &= P(Q(h)) + (P(Q(h) + S(h)) - P(Q(h))) + R(Q(h) + S(h)) \\ &= P(Q(h)) + o(|h|^k) + o(|h|^k), \end{aligned}$$

that is

$$g(f(h)) - P(Q(h)) = o(|h|^k).$$

Although the polynomial $P \circ Q$ is, in general, of order greater than k, we may define T as the Taylor polynomial of order k of $P \circ Q$ at 0, so that

$$P(Q(h)) - T(h) = o(|h|^k),$$

and by summing the last two equalities we get $g(f(h)) - T(h) = o(|h|^k)$. □

Lemma 2.7. *Assume that $f : \mathbb{R}^d \to \mathbb{R}^n$ satisfies*

$$\limsup_{h\to 0} \frac{|f(x+h) - Q(h)|}{|h|^k} < \infty$$

for some $x \in \mathbb{R}^d$ and a polynomial Q of degree up to $k-1$, and that $g : \mathbb{R}^n \to \mathbb{R}^m$ satisfies

$$\limsup_{h\to 0} \frac{|g(y+h) - P(h)|}{|h|^k} < \infty$$

for $y = f(x)$ and a polynomial P of degree up to $k-1$. Then

$$\limsup_{h\to 0} \frac{|(g \circ f)(x+h) - T(h)|}{|h|^k} < \infty,$$

where T is the Taylor polynomial of order $k-1$ of $P \circ Q$.

Proof. Replace $o(|h|^k)$ with $O(|h|^k)$, and P, Q, T with polinomials of order $k-1$ in the proof of the preceding Lemma. □

Lemma 2.8. *Let m, n, k, i be positive integers, with $i \leq n-1$. Assume that $f : \mathbb{R}^n \to \mathbb{R}^m$ has a Taylor expansion of order k at $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$. Then, for $\mathcal{L}^i$-a.e. $y \in \mathbb{R}^i$, the function $g_y : \mathbb{R}^{n-i} \to \mathbb{R}^m$ has a Taylor expansion of order k at $\mathcal{L}^{n-i}$-a.e. $z \in \mathbb{R}^{n-i}$, where g_y is defined by $g_y(z) = g(y, z)$.*

Proof. Using the hypothesis and Fubini's theorem, for $\mathcal{L}^i$-a.e. $y \in \mathbb{R}^i$ we have that, for $\mathcal{L}^{n-i}$-a.e. $z \in \mathbb{R}^{n-i}$, the function g has a Taylor expansion of order k at $x = (y, z)$. This obviously implies that the function g_y has a Taylor expansion of order k at z, for such $y \in \mathbb{R}^i$, $z \in \mathbb{R}^{n-i}$. □

3. Proof of Theorem 1.4

Throughout this section we will assume that $f \in \mathcal{C}^k_{n,m}(U, \mathbb{R}^m)$, where $U \subseteq \mathbb{R}^n$ is open and $n > m \geq 1$ are positive integers, and k will be defined by $k := n - m + 1$; note in particular that $k \geq 2$. We will also use $\mathcal{C}^j_{n,m}$ as an abbreviation for $\mathcal{C}^j_{n,m}(W, V)$ when the sets W, V are understood.

Let $C = \{x \in U \,:\, \operatorname{rank}(Df(x)) < m\}$ be the set of critical points of f, and set

$$A_j = \{x \in C \,:\, D^i f(x) = 0 \text{ for } 1 \leq i \leq j\}, \;\; 1 \leq j \leq n - m,$$

and

$$K = \{x \in C : 1 \leq \operatorname{rank}(Df(x)) \leq m - 1\}.$$

Then we have

$$C = K \cup (A_1 \setminus A_2) \cup (A_2 \setminus A_3) \cup \ldots \cup (A_{n-m-1} \setminus A_{n-m}) \cup A_{n-m}. \tag{3.1}$$

The proof of Theorem 1.4 can be carried out in three steps following the general plan of [1, Lemma 15.2] and [12, Theorem 5]. In the first step, by using conditions (MS4)-(MS7) and a standard argument going back to [27] we show that it suffices to prove the theorem in the case $K = \emptyset$, that is, $C = \{x : Df(x) = 0\}$. In the second step we prove that $\mathcal{L}^m(f(A_{n-m})) = 0$. This is where our proof really differs from others; the key point is noticing that for a function $\varphi : \mathbb{R} \to \mathbb{R}$ the set $\{t \in \mathbb{R} : (0, a_t) \in J^2\varphi(t) \text{ for some } a_t > 0\}$ is always (at most) countable. Finally, in the third step we see how, thanks to the Implicit Function Theorem and the Kneser-Glaeser Theorem, one can reduce the dimension from n to $n-1$, which by an induction argument finishes the proof. This third step is almost identical to Step 3 in the proof of [12, Theorem 5] or [1, Lemma 15.2], but since the argument is short we will include its proof here for completeness (and for the benefit of those readers who might not yet be familiar with this scheme).

Claim 3.1. First Step. *We may assume $K = \emptyset$.*

Proof. Observe that f is at least of class C^1, because $n > m$ and $\mathcal{C}^k_{n,m} = \mathcal{C}^{n-m+1}_{n,m} \subset C^{n-m}$, by (MS1). Define

$$K_i = \{x \in \mathbb{R}^n : \text{ rank}(D(f(x)) = i\}, \ \ 1 \leq i \leq m-1.$$

Take $\bar{x} \in K_i$. We may assume that

$$\det\left(\frac{\partial(f_1, ..., f_i)}{\partial(x_1, ..., x_i)}\right)(\bar{x}) \neq 0.$$

Then, in some relatively compact neighborhood V of $\bar{x}$, we can take as coordinates $(y_1, ..., y_n) = Y(x) = (f_1(x), ..., f_i(x), x_{i+1}, ..., x_n)$. Therefore, defining $X = Y^{-1}$, f is of the form

$$f(X(y)) = (y_1, ..., y_i, g(y_1, ..., y_n)).$$

By using properties (MS1), (MS4), (MS6) and (MS7) of the classes $\mathcal{C}^i_{j,\ell}$, it is easy to see that $X \in \mathcal{C}^k_{n,n}$. Then, by using property (MS5) we get $f \circ X \in \mathcal{C}^k_{n,m}$, and by using again (MS5) and (MS1) it is immediately clear that $g \in \mathcal{C}^k_{n,m-i}(\widetilde{V}, \mathbb{R}^{m-i})$, where $\widetilde{V} = Y(V)$. Moreover, in these new coordinates we have

$$D(f \circ X)(y) = \begin{pmatrix} I_i & 0 \\ * & D(g_{|_{(y_1,...,y_i)}}) \end{pmatrix},$$

where $g_{|_{(y_1,...,y_i)}}(y_{i+1}, ..., y_n) := g(y_1, ..., y_n)$, and $\widetilde{V}_{(y_1,...,y_i)} = \{(z_1, ..., z_{n-i}) \in \mathbb{R}^{n-i} : (y_1, ..., y_i, z_1, ..., z_{n-i}) \in \widetilde{V}\}$.

Now, by using property (MS6) and observing that $\text{rank}(D(f \circ X)) = \text{rank}(Df) = i$ on K_i, we have that, for $\mathcal{L}^i$-a.e. $(y_1, ..., y_i)$,

$$g_{|_{(y_1,...,y_i)}} \in \mathcal{C}^k_{n-i,m-i},$$

and

$$D(g_{|_{(y_1,...,y_i)}}) = 0 \text{ on } Y(K_i) \cap \widetilde{V}_{(y_1,...,y_i)}.$$

Therefore, if we prove that $f(A_1 \setminus A_2)$, ..., $f(A_{n-m-1} \setminus A_{n-m})$, $f(A_{n-m})$ are of measure zero (which we will indeed do in the second and third steps), by applying that part of the proof to the function $g_{|_{(y_1,...,y_i)}}$, we will get

$$\mathcal{L}^{m-i}(g_{|_{(y_1,...,y_i)}}(Y(K_i) \cap \widetilde{V}_{(y_1,...,y_i)})) = 0 \text{ for } \mathcal{L}^i\text{-a.e. } (y_1, ..., y_i),$$

and by Fubini's theorem we will conclude that $0 = \mathcal{L}^m(f \circ X(\widetilde{V} \cap Y(K_i)))$ $= \mathcal{L}^m(f(V \cap K_i))$, for every $i \in \{1, ..., m-1\}$, hence that $\mathcal{L}^m(f(K)) = 0$ as well. □

Let us now proceed with the **Second Step** of the proof of Theorem 1.4 and show that $\mathcal{L}^m(f(A_{n-m})) = 0$. We can distinguish two subsets of A_{n-m}, namely,

$$C_{n-m} = \{x \in A_{n-m} \, : \, (0, ..., 0) \in J^k f(x)\},$$

and

$$B_{n-m} = A_{n-m} \setminus C_{n-m}.$$

By Lemma 2.3 we already know that $\mathcal{L}^m(f(C_{n-m})) = 0$. Therefore it is enough to see that $\mathcal{L}^m(f(B_{n-m})) = 0$. In turn, because $B_{n-m} \subset A_{n-m}$, and thanks to property (MS2) of the class $\mathcal{C}^k_{n,m}$, this will be established once we prove the following.

Claim 3.2. $\mathcal{L}^n(B_{n-m}) = 0$.

Proof. Property (MS3) tells us that

$$\mathcal{L}^n\left(\{x \in \mathbb{R}^n : J^k f(x) = \emptyset\}\right) = 0.$$

Therefore, thanks again to property (MS2), we may assume that

$$B_{n-m} \subset \{x \in A_{n-m} : J^k f(x) \neq \emptyset\}.$$

Since C_{n-m} is disjoint with B_{n-m} we then have that, for every $x \in B_{n-m}$,

$$J^k f(x) = (0, ..., 0, P^k_x) \text{ for some } P^k_x \neq 0.$$

Let $\{v_j\}_{j=1}^\infty$ and $\{w_j\}_{j=1}^\infty$ be dense sequences in the unit spheres $\mathbb{S}^{n-1}$ and $\mathbb{S}^{m-1}$ of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Define, for every $(\alpha, \beta) \in \mathbb{N} \times \mathbb{N}$,

$$E^+_{\alpha\beta} = \{x \in B_{n-m} : \langle P^k_x(v_\alpha), w_\beta\rangle > 0\},$$

and

$$E^-_{\alpha\beta} = \{x \in B_{n-m} : \langle P^k_x(v_\alpha), w_\beta\rangle < 0\}.$$

If $x \in B_{n-m}$ then $P^k_x(v) \neq 0$ for some $v \in \mathbb{S}^{n-1}$, and by continuity of P^k_x and density of $\{v_j\}_{j=1}^\infty$, $\{w_j\}_{j=1}^\infty$ there exist $\alpha, \beta \in \mathbb{N}$ such that $\langle P^n_x(v_\alpha), v_\beta\rangle \neq 0$, and therefore $x \in E^+_{\alpha\beta} \cup E^-_{\alpha\beta}$. This shows that

$$B_{n-m} = \bigcup_{\alpha,\beta\in\mathbb{N}} \left(E^+_{\alpha\beta} \cup E^-_{\alpha\beta}\right).$$

Thus it is sufficient to see that $\mathcal{L}^n(E^+_{\alpha\beta}) = 0$ and $\mathcal{L}^n(E^-_{\alpha\beta}) = 0$ for each $(\alpha, \beta) \in \mathbb{N} \times \mathbb{N}$. Let us fix $\alpha, \beta \in \mathbb{N}$ and see that $\mathcal{L}^n(E^+_{\alpha\beta}) = 0$ (the proof that $\mathcal{L}^n(E^-_{\alpha\beta}) = 0$ is completely analogous).

Let $[v_\alpha]^\perp$ stand for the orthogonal complement of the line spanned by the direction v_α. For each $y \in [v_\alpha]^\perp$ let us define

$$R_y = \{y + tv_\alpha : t \in \mathbb{R}\}, \text{ and } G_y = R_y \cap E^+_{\alpha\beta}.$$

We will show that G_y is (at most) countable for every $y \in [v_\alpha]^\perp$. Once we have checked this, by applying Fubini's theorem we will immediately deduce that $\mathcal{L}^n(E^+_{\alpha\beta}) = 0$ (note that we may indeed apply Fubini's theorem because the sets $E^\pm_{\alpha\beta}$ are measurable; indeed, A_{n-m} is closed, and we can write

$$C_{n-m} = \bigcap_{i=1}^\infty \bigcup_{j=1}^\infty \bigcap_{|h|\leq 1/j} \{x \in A_{n-m} : |f(x+h) - f(x)| \leq \frac{1}{i}|h|^k\},$$

so that C_{n-m} is a countable intersection of F_σ sets, hence B_{n-m} is measurable, and

$$E^+_{\alpha\beta} = B_{n-m} \cap \left(\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{0\le t\le 1/j} \{x : \langle f(x+tv_\alpha), w_\beta\rangle - \langle f(x), w_\beta\rangle \ge \frac{1}{i} t^k\} \right),$$

so that $E^+_{\alpha\beta}$, being the intersection of B_{n-m} with an F_σ set, is measurable as well. Similarly, $E^-_{\alpha\beta}$ is measurable).

Fix $y \in [v_\alpha]^\perp$, set $\widetilde{G_y} = \{t \in \mathbb{R} : y + tv_\alpha \in G_y\}$, and consider the auxiliary function $\varphi : \widetilde{G_y} \to \mathbb{R}$ defined by

$$\varphi(t) = \langle D^{k-2} f(y+tv_\alpha)(v_\alpha^{k-2}), w_\beta\rangle = D^{k-2}\langle f, w_\beta\rangle(y+tv_\alpha)(v_\alpha^{k-2})$$

(in the case $k=2$ we understand that $D^0 f = f$). For each $t \in \widetilde{G_y}$ we have

$$J^2\varphi(t) = \Big(\langle D^{k-1} f(y+tv_\alpha)(v_\alpha^{k-1}), w_\beta\rangle,\ \langle P^k_{y+tv_\alpha}(v_\alpha), w_\beta\rangle\Big) = (0, \langle P^k_{y+tv_\alpha}(v_\alpha), w_\beta\rangle).$$

In particular, for every $t \in \widetilde{G_y}$ we have

$$J^2\varphi(t) = (0, a_t) \text{ for some } a_t > 0.$$

We have to see that $\widetilde{G_y}$ is countable. In order to do so, let us define, for every $i, j, \ell \in \mathbb{N}$, the sets

$$D_\ell = \{t \in \widetilde{G_y} : J^2\varphi(t) = (0, a_t),\ a_t \ge \frac{1}{\ell}\},$$

and

$$G_{ij\ell} = \{t \in \widetilde{G_y} \,:\, \varphi(t+h) - \varphi(t) - \frac{1}{\ell}t^2 \ge -\frac{1}{j}t^2 \text{ whenever } |h| \le \frac{1}{i}\}.$$

Obviously we have $\widetilde{G_y} = \bigcup_{\ell=1}^{\infty} D_\ell$, hence we only have to check that D_ℓ is countable for each $\ell \in \mathbb{N}$. Fix ℓ and notice that

$$D_\ell \subseteq \bigcap_{j=1}^{\infty} \left(\bigcup_{i=1}^{\infty} G_{ij\ell} \right)$$

(indeed, for every $t \in D_\ell$ and for every $j \in \mathbb{N}$, since $\lim_{h\to 0} \frac{\varphi(t+h)-\varphi(t)-a_t h^2}{h^2} = 0$ there exists $i \in \mathbb{N}$ so that $|\varphi(t+h) - \varphi(t) - a_t h^2| \le h^2/j$ if $|h| \le 1/i$. Hence, using that $a_t \ge 1/\ell$ because $t \in D_\ell$, we have

$$\frac{1}{\ell}h^2 \le a_t h^2 \le \varphi(t+h) - \varphi(t) + \frac{1}{j}h^2 \text{ for } |h| \le 1/i,$$

and in particular $t \in G_{ij\ell}$).

Now, if $t, s \in G_{ij\ell}$ and $|s-t| \le 1/i$ then we have

$$\varphi(t+s-t) - \varphi(t) - \frac{1}{\ell}|t-s|^2 \ge -\frac{1}{j}|t-s|^2$$

and

$$\varphi(s+t-s) - \varphi(s) - \frac{1}{\ell}|t-s|^2 \ge -\frac{1}{j}|t-s|^2.$$

By summing the last two inequalities we get

$$-\frac{2}{\ell}|t-s|^2 \geq -\frac{2}{j}|t-s|^2,$$

and for $j > \ell$ this is possible only if $t = s$. Therefore, if $j > \ell$, the set $G_{ij\ell}$ contains only isolated points (as the distance between two of its points must be greater than $1/i$), and in particular is countable. Then, if we set $j = 2\ell$ for instance, it follows that

$$D_\ell \subset \bigcup_{i=1}^{\infty} G_{i(2\ell)\ell}$$

is countable. □

Let us now finally make the **Third Step** of the proof of Theorem 1.4. Note that the above argument already proves the Theorem for $n = m+1$ (or equivalently $k = 2$). This allows us to start an induction argument on the dimension n. Assuming that the result is true for functions f in the class $\mathcal{C}^{n-m}_{n-1,m}$ with $K = \emptyset$, we have to check that it is also true for functions f in the class $\mathcal{C}^{n-m+1}_{n,m}$ with $K = \emptyset$. By the second step we know that $\mathcal{L}^m(f(A_{n-m})) = 0$. Therefore, bearing in mind equation (3.1), we only have to show the following.

Claim 3.3. *$\mathcal{L}^m(f(A_{s-1} \setminus A_s)) = 0$ for $2 \leq s \leq k-1$.*

Proof. Fix a point $x \in A_{s-1} \setminus A_s$. It is sufficient to see that there exists an open neighborhood V of x such that $\mathcal{L}^m(f((A_{s-1} \setminus A_s) \cap V)) = 0$. By the standing hypotheses, f is of class C^{k-1}, and f is $(s-1)$-flat at x, but some partial derivative of order s of f is not zero. We may assume for instance that

$$\frac{\partial w}{\partial x_n}(x) \neq 0,$$

where

$$w(x) = \partial^\alpha f(x) = 0$$

and α is a multi-index with $|\alpha| = s-1$. Since w is of class C^{k-s}, by the Implicit Function Theorem there exists an open neighborhood V of x such that $V \cap \{w = 0\}$ is an $(n-1)$-dimensional graph of class C^{k-s}. In particular there exist an open subset W in $\mathbb{R}^{n-1}$ and a function $g : W \to \mathbb{R}^n$ of class C^{n-s} such that $V \cap A_{s-1} \subset g(W)$.

Now let us consider $A^* := \{z \in W : g(z) \in A_{s-1}\}$. By Theorem 2.1, there exists $F : W \to \mathbb{R}$ of class C^{k-1} such that $F = f \circ g$ on A^*, and $DF(x) = 0$ on A^*. In particular we have $f(A_{s-1} \cap V) \subset F(C_F \cap W)$, where C_F stands for the critical set of F. But, since F is defined on an open subset of $\mathbb{R}^{n-1}$ and is of class C^{k-1}, and by property (MS1) we have $C^{k-1} \subset \mathcal{C}^{k-1}_{n-1,m} = \mathcal{C}^{n-m}_{n-1,m}$, we may use the induction hypothesis to conclude that $\mathcal{L}^m(F(C_F \cap W)) = 0$, and therefore $\mathcal{L}^m(f(A_{s-1} \cap V)) = 0$ as well. The proof is complete.

Finally, notice that conditions (MS4)-(MS7) are used only in the First Step of the above proof. Since in the special case $m = 1$ we always have

$K = \emptyset$, the First Step of the proof may be omitted. Therefore it is not necessary to define the classes $C^j_{n,1}(U,V)$ for $j \neq n$, and the conditions (MS1) for $j = n$, (MS2) and (MS3) alone are sufficient to ensure that $\mathcal{L}^1(f(C_f)) = 0$ for every $f \in \mathcal{C}^n_{n,1}(U,V)$. □

4. Proof of Theorem 1.1

The case $n = m$ of Theorem 1.1 is a consequence of the following.

Proposition 4.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ a function, define*

$$W = \{x \in \mathbb{R}^n : \limsup_{h\to 0} \frac{|f(x+h) - f(x)|}{|h|} < \infty\},$$

and assume that $W \neq \emptyset$. Then $\mathcal{L}^n(f(C_f)) = 0$, where

$$C_f := \{x \in \mathbb{R}^n : Df(x) \text{ exists and rank}(Df(x)) < n\}.$$

Note that, by Stepanov's theorem (see [18, 15, 20] for instance), $Df(x)$ exists for almost every $x \in W$. On the other hand, if $Df(x)$ exists then $x \in W$. Therefore C_f is always a subset of W.

Proof. **Case 1.** Assume first that f is bounded on $\mathbb{R}^n$. Define $A_j = \{x \in \mathbb{R}^n : |f(x+h) - f(x)| \leq j|h| \text{ for all } h \in \mathbb{R}^n\}$, for each $j \in \mathbb{N}$. We have that

$$W = \bigcup_{j=1}^{\infty} A_j.$$

Indeed, if $x \in W$, there exist $M > 0$ and $\delta > 0$ so that $|f(x+h) - f(x)| \leq M|h|$ whenever $|h| \leq \delta$. On the other hand, if $|h| \geq \delta$ then $|f(x+h) - f(x)| \leq 2\|f\|_\infty \leq \frac{2\|f\|_\infty}{\delta}|h|$. In any case we get $x \in A_j$ for all $j > \max\{M, 2\|f\|_\infty/\delta\}$.

Let us denote by $f_{|A_j}$ the restriction of f to A_j. It is clear that $f_{|A_j}$ is j-Lipschitz on A_j. By Kirszbraun's theorem (see [11, section 2.10.43] for instance) there exists a j-Lipschitz extension of $f_{|A_j}$ to all of $\mathbb{R}^n$; let us call such an extension f_j. Next, by Rademacher's theorem, f_j is differentiable on $\mathbb{R}^n \setminus N_j$, where N_j is an $\mathcal{L}^n$-null set. Similarly, there exists Z_j with $\mathcal{L}^n(Z_j) = 0$ such that the distance function $d(\cdot, A_j)$ is differentiable on $\mathbb{R}^n \setminus Z_j$. Then the set

$$N := \bigcup_{j=1}^{\infty} N_j \cup Z_j$$

is $\mathcal{L}^n$-null, and we have that, for every $x \in \mathbb{R}^n \setminus N$, the functions f_j and $d(\cdot, A_j)$ are differentiable at x, for all $j \in \mathbb{N}$.

Now, it is an easy and well known consequence of the area (or the coarea) formula (see [10] for instance) that if $g : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz then $\mathcal{L}^n(g(\{x : Dg(x) \text{ exists and } \operatorname{rank}(Dg(x)) < n\})) = 0$. If we define

$$C_j = \{x \in \mathbb{R}^n : Df_j(x) \text{ exists and } \operatorname{rank}(Df_j(x)) < n\},$$

we thus have that $\mathcal{L}^n(f_j(C_j)) = 0$ for every j.

Claim 4.2. $f(C_f) \subseteq f(N) \cup \bigcup_{j=1}^{\infty} f(C_j)$.

Assuming the Claim, we then have, by Lemma 2.2, that $\mathcal{L}^n(f(N)) = 0$, and we deduce that

$$\mathcal{L}^n(f(C_f)) \leq \mathcal{L}^n(f(N)) + \sum_{j=1}^{\infty} \mathcal{L}^n(f(C_j)) = 0,$$

concluding the proof.

In order to prove the Claim, we use an idea of [5, pp. 518-519], see also [20, Lemma 2.3]. Let $x \in C_f \setminus N$. Then f_i and $d(\cdot, A_i)$ are differentiable at x for every $i \in \mathbb{N}$. On the other hand, since $\mathbb{R}^n = \bigcup_{i=1}^{\infty} A_i$ there exists $j \in \mathbb{N}$ such that $x \in A_j$. For every $h \in \mathbb{R}^n$ we can find $y_h \in A_j$ such that $|x+h-y_h| \leq 2d(x+h, A_j)$. Then we have (using that $x \in A_j$, $f_j(x) = f(x)$, and f_j is j-Lipschitz on $\mathbb{R}^n$) that

$$\begin{aligned}&|f(x+h) - f_j(x+h)| \leq |f(x+h) - f(y_h)| + |f_j(y_h) - f_j(x+h)| \leq \\ &j|x+h-y_h| + j|y_h - (x+h)| = 2j|x+h-y_h| \leq 4jd(x+h, A_j),\end{aligned}$$

and since the function $\psi_j(z) := d(z, A_j)$ is differentiable at x and $D\psi_j(x) = 0$ (because $\psi_j \geq 0$ and $\psi_j(x) = 0$), we deduce that

$$\begin{aligned}&|f(x+h) - f(x) - Df_j(x)(h)| \\ &\leq |Df_j(x)(h) + f_j(x) - f_j(x+h)| + 4j|\psi_j(x+h) - \psi_j(x)| = o(h),\end{aligned}$$

which proves that $Df(x) = Df_j(x)$. Since, by assumption, $x \in C_f$, we have $\operatorname{rank}(Df(x)) < n$, hence $\operatorname{rank}(Df_j(x)) < n$ as well, that is, $x \in C_j$, and in particular $f(x) \in f(C_j)$.

Case 2. Assume now that f is unbounded, and for each $j \in \mathbb{N}$ define

$$g_j(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq j, \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $\limsup_{h\to 0} \frac{|f(x+h)-f(x)|}{|h|} < \infty$ (and in particular, if f is differentiable at x) then f is bounded on an open neighborhood V_x of x, hence there exists $j = j_x$ such that $f = g_j$ on V_x, and it follows that

$$f(C_f) \subseteq \bigcup_{j=1}^{\infty} g_j(C_{g_j}).$$

For each $j \in \mathbb{N}$, the function g_j is bounded, so we have $\mathcal{L}^n(g_j(C_{g_j})) = 0$ by Case 1. We may then conclude that $\mathcal{L}^n(f(C_f)) = 0$. □

Now, in order to prove Theorem 1.1 in the case $n > m$, we define, for each U open in $\mathbb{R}^i$ and V open in $\mathbb{R}^j$ with $i \geq j$, the class $\mathcal{C}^s_{i,j}(U, V)$ as the set of all functions $f : U \to V$ such that:

(1) $f \in C^{s-1}(U, V)$;
(2) $\limsup_{h\to 0} \frac{|f(x+h)-f(x)-Df(x)(h)-\ldots-D^{s-1}f(x)(h^{s-1})|}{|h|^s} < \infty$ for every $x \in \mathbb{R}^n$;

(3) f has a Taylor expansion of order s at almost every $x \in U$.

By using Lemmas 2.2, 2.4, 2.5, 2.6, 2.7, and 2.8, and bearing in mind that diffeomorphisms map (Lebesgue) null sets onto null sets, it is easy to verify that the classes $\mathcal{C}^s_{i,j}$ satisfy the properties (MS1)–(MS7) of Theorem 1.4. Then Theorem 1.1 for $n > m$ follows at once.

5. Examples and relations to other work

Let us begin by briefly explaining how De Pascale's Theorem 1.5 (which, to the best of our knowledge, remains the finest known Morse-Sard-type theorem for Sobolev vector-valued functions) also follows, with little effort, from Theorem 1.4. In the case $n > m$, we may define, for each U open in $\mathbb{R}^i$ and V open in $\mathbb{R}^j$, with $i \geq j$, $s \geq 1$, the classes $\mathcal{C}^s_{i,j}(U,V) := W^{s,p}_{\mathrm{loc}}(U,V)$. By using standard results and techniques of Sobolev space theory (Change of Variables, Slicing Theorem, etc) it is not difficult to check that these classes satisfy properties (MS4)-(MS7). Property (MS1) is part of Morrey's inequality, and property (MS2) is the easy part of Step 2 in the proof of [12, Theorem 5] (to check this, one may combine Taylor's theorem, Morrey's inequality, and Young's inequality to see that

$$|f(y) - f(x)|^m \leq C \int_{B(x,r)} (1 + |D^k f(z)|^p) dz,$$

for $x \in A$ and $|y-x| \leq r \leq 1$, and then conclude by a covering argument, see [12] for details). Finally, property (MS3) can be easily checked as follows: consider the map $\mathbb{R}^n \ni x \mapsto g(x) = D^{k-1}f(x) \in \mathbb{R}^M$, where M is the dimension of the space $\mathcal{L}^{k-1}_s(\mathbb{R}^n, \mathbb{R}^m)$ of $(k-1)$-linear symmetric maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ and we identify $\mathcal{L}^{k-1}_s(\mathbb{R}^n, \mathbb{R}^m)$ with $\mathbb{R}^M$. Using that $f \in W^{k,p}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^m)$, an easy calculation shows that the coordinate functions g_j of $g = (g_1, ..., g_M)$ have first order weak derivatives which are in L^p. That is, $g_j \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ for every $j = 1, ..., M$. By [10, Theorem 6.2.1], g_j is then differentiable almost everywhere in $\mathbb{R}^n$, for each $j = 1, ..., M$. It follows that g is differentiable almost everywhere in $\mathbb{R}^m$, which means that f is k times differentiable at a.e. $x \in \mathbb{R}^n$, and in particular f has a Taylor expansion at a.e. $x \in \mathbb{R}^n$. Thus we may apply Theorem 1.4 to deduce that every $f \in W^{k,p}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ has the Morse-Sard property if $k = n - m + 1 \geq 2$ and $p > n$.

As for the case $n = m$, this is an easy consequence of the coarea formula for Sobolev mappings [19] (or an immediate consequence of Proposition 4.1 above and the mentioned fact that functions of $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ are differentiable almost everywhere when $p > n$).

Let us now see why Theorems 1.1 and 1.2, and Corollary 1.3 are not weaker than the recent Bourgain-Korobkov-Kristensen generalizations [7] of the Morse-Sard theorem in the case of real-valued functions for the spaces $W^{n,1}(\mathbb{R}^n, \mathbb{R})$ and $BV_{n,\mathrm{loc}}(\mathbb{R}^n)$. Since $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n) \subset BV_{\mathrm{loc}}(\mathbb{R}^n)$, the results of [7] are stronger than all the previous generalizations of the Morse-Sard

theorem for Sobolev spaces in the case $m = 1$, and are also stronger than [24] (even in the case $n = 2$, because the results of [7] do not require that the function be Lipschitz). Thus, in order to make our point, it will be enough to exhibit an example of a function $f \in \mathcal{C}^1\mathcal{P}^2(\mathbb{R}^2, \mathbb{R})$ such that $f \notin BV_{2,\mathrm{loc}}(\mathbb{R}^2)$.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} x^4 \sin\left(\frac{1}{x^2}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We have $\frac{\partial f}{\partial y} = 0$ everywhere, and

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} 4x^3 \sin\left(\frac{1}{x^2}\right) - 2x\cos\left(\frac{1}{x^2}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

so that $f \in C^1(\mathbb{R}^2)$. On the other hand, it is clear that f is C^∞ on $\{(x,y) : x \neq 0\}$, and for every (x_0, y_0) with $x_0 = 0$ we have that $(0,0) \in J^2 f(0, y_0)$, because

$$\lim_{(x,y)\to(0,0)} \frac{x^4 \sin\left(\frac{1}{x^2}\right)}{x^2 + y^2} = 0.$$

Hence $f \in \mathcal{C}^1\mathcal{P}^2(\mathbb{R}^2, \mathbb{R})$. However, $g := \partial f/\partial x \notin BV(\mathbb{R}^2)$. Indeed, defining $g_y(x) = \frac{\partial f}{\partial x}(x,y)$, it is easy to see that

$$V_0^1 g_y = \infty$$

for each $y \in \mathbb{R}$ (here we use the notation from [10, 5.11]), and because g_y is continuous this implies

$$\mathrm{ess}V_0^1 g_y = \infty$$

for every y, hence

$$\int_{-1}^{1} \mathrm{ess}V_0^1 g_y = \infty,$$

and by [10, Theorem 5.11.2] this implies that $g \notin BV_{\mathrm{loc}}(\mathbb{R}^2)$, hence $f \notin BV_{2,\mathrm{loc}}(\mathbb{R}^2)$ either.

It is also easy to see that f is not semiconcave on any neighborhood of $(0,0)$, which also shows that, even in the case $n = 2$, Theorem 1.1 is not weaker than [26, Theorem 8].

On the other hand, it is clear that the results of [7, 24] and, in the case $n = 2$, [26] are not weaker than Theorem 1.1 either. For instance, it is easy to produce examples of delta-convex, and of locally semiconcave, functions $f : \mathbb{R}^2 \to \mathbb{R}$ which are not of class C^1.

Let us finish this section with some remarks about Theorem 1.1. A natural question is whether condition (2) of Theorem 1.1 could be replaced with a weaker condition in which the limit would hold for a.e. x instead of all x. The answer is negative. Indeed, consider Whitney's example from [29], where an arc C in $\mathbb{R}^2$ and a function $f : \mathbb{R}^2 \to \mathbb{R}$ are constructed in such a way that f is critical on C (meaning that $C \subseteq C_f$) and f is not constant on C; in particular $f(C_f)$ contains an open interval, and f does not have

the Morse-Sard property. See also [23, 13] and the references therein for more information about Whitney-type examples. Although not explicitly stated in Whitney's paper, these f and C satisfy two important additional properties:

(a) f is of class C^∞ on $\mathbb{R}^2 \setminus C$, and
(b) $\mathcal{L}^2(C) = 0$.

Property (b) follows easily from the definition of C, while (a) is a consequence of the facts that f is constructed by applying the Whitney Extension Theorem (WET) to a function defined on C, and that the WET provides us with extensions which are always of class C^∞ outside the closed set on which the functions to be extended are initially defined. Then, since f has derivatives of all orders which are locally Lipschitz on the open set $\mathbb{R}^2 \setminus C$, it is clear that $\limsup_{h\to 0} \frac{|f(x+h)-f(x)-Df(x)(h)|}{|h|^2} < \infty$ for all $x \in \mathbb{R}^2 \setminus C$, and in particular for a.e. $x \in \mathbb{R}^2$. On the other hand f clearly satisfies conditions (1) and (3) of Theorem 1.1. However, $\mathcal{L}^1(f(C_f)) > 0$.

Thus, refining the question about condition (2) of Theorem 1.1, one could ask: how small must a set $N \subset \mathbb{R}^n$ be in order that Theorem 1.1 still holds true if we replace condition (2) with a new condition in which the limit holds for every $x \in \mathbb{R}^n \setminus N$? This question is of course much more difficult to answer. By combining Theorem 1.4 for $m = 1$ with [22, Theorem 2], we can nevertheless obtain a partial answer as follows.

Theorem 5.1. *Let $n \geq 2$, $0 < \alpha < 1$, and let $f : \mathbb{R}^n \to \mathbb{R}$ be such that*

(1) $f \in C^{n-1,\alpha}(\mathbb{R}^n, \mathbb{R}^m)$;
(2) $\limsup_{h\to 0} \frac{|f(x+h)-f(x)-Df(x)(h)-\ldots-D^{n-1}f(x)(h^{n-1})|}{|h|^n} < \infty$ *for every* $x \in \mathbb{R}^n \setminus N$, *for some* N *with* $\mathcal{H}^{n-1+\alpha}(N) = 0$;
(3) f *has a Taylor expansion of order* n *at almost every point* $x \in \mathbb{R}^n$.

Then $\mathcal{L}^1(f(C_f)) = 0$.

Here $C^{n-1,\alpha}$ denotes the subset of C^{n-1} defined by all functions whose derivatives of order $n-1$ satisfy Hölder-continuity conditions of order α on compact sets.

Proof. Let us define, for each U open in $\mathbb{R}^n$ the class $\mathcal{C}^n_{n,1}(U, \mathbb{R})$ as the set of all functions $f : U \to \mathbb{R}$ such that:

(1) $f \in C^{n-1,\alpha}(U, \mathbb{R})$;
(2) $\limsup_{h\to 0} \frac{|f(x+h)-f(x)-Df(x)(h)-\ldots-D^{n-1}f(x)(h^{n-1})|}{|h|^n} < \infty$ for every $x \in U \setminus N$, for some $N \subset \mathbb{R}^n$ with $\mathcal{H}^{n-1+\alpha}(N) = 0$;
(3) f has a Taylor expansion of order n at almost every $x \in U$.

It is easy clear that the classes $\mathcal{C}^n_{n,1}$ satisfy properties (MS1) and (MS3) of Theorem 1.4 for $m = 1, j = n$. As for property (MS2), we know by [22, Theorem 2(ii)] that every $f \in C^{n-1+\alpha}(\mathbb{R}^n, \mathbb{R})$ maps $\mathcal{H}^{n-1+\alpha}$-null sets onto $\mathcal{L}^1$-null sets. Therefore, by Lemma 2.2, (MS2) will be satisfied as long as we

ask that $|f(x+h)-f(x)-Df(x)(h)-...-D^{n-1}f(x)(h^{n-1})| = O(|h|^n)$ for every x outside an $\mathcal{H}^{n-1+\alpha}$-null set. □

Remark 5.2. The reason why we cannot similarly apply Theorem 1.4 in the case $m \geq 2$ is that we cannot check condition (MS6) due to the lack of a Fubini theorem for Hausdorff measures

By using the same argument, and taking into account that $C^{n-1} \subset C^{n-2,\alpha}$, one may also prove the following.

Theorem 5.3. *Let $n \geq 2$, $0 < \alpha < 1$, and let $f : \mathbb{R}^n \to \mathbb{R}$ be such that*

(1) *$f \in C^{n-1}(\mathbb{R}^n, \mathbb{R}^m)$;*
(2) *$\limsup_{h\to 0} \frac{|f(x+h)-f(x)-Df(x)(h)-...-D^{n-1}f(x)(h^{n-1})|}{|h|^n} < \infty$ for every $x \in \mathbb{R}^n \setminus N$, for some N with $\mathcal{H}^{n-2+\alpha}(N) = 0$;*
(3) *f has a Taylor expansion of order k at almost every point $x \in \mathbb{R}^n$.*

Then $\mathcal{L}^1(f(C_f)) = 0$.

The reader is invited to consider other classes of functions (e.g. $C^{k+\beta+}$ in [22]) and use Theorem 1.4 to formulate other variants of Theorem 1.1.

6. Proofs of Theorems 1.6, 1.7, and 1.8

The proof of Theorem 1.6 follows the same plan as that of Theorem 1.4. Given a function $f \in \mathcal{C}^k_{n,m}$, we define $C = \{x \in U \,:\, \operatorname{rank}(Df(x)) < m\}$, and set

$$A_j = \{x \in C \,:\, D^i f(x) = 0 \text{ for } 1 \leq i \leq j\}, \;\; 1 \leq j \leq k-1,$$

and

$$K = \{x \in C : 1 \leq \operatorname{rank}(Df(x)) \leq m-1\},$$

so that

$$C = K \cup (A_1 \setminus A_2) \cup (A_2 \setminus A_3) \cup ... \cup (A_{k-2} \setminus A_{k-1}) \cup A_{k-1}. \tag{6.1}$$

Steps 1 and 3 of the proof of Theorem 1.4 can be rewritten, with obvious changes and no difficulty, to see that one can always assume $K = \emptyset$ and that, once one has checked that $\mathcal{H}^\ell(A_{k-1} \cap f^{-1}(y)) = 0$ for a.e. y, one can also use the Implicit Function Theorem and the Kneser-Glaeser Theorem in order to reduce the dimension from n to $n-1$ and apply the induction argument.

So, the only really different point of the proof is Step 2. Let us see that $\mathcal{H}^\ell(A_{k-1} \cap f^{-1}(y)) = 0$ for a.e. $y \in \mathbb{R}^m$. We can distinguish two subsets of A_{k-1}, namely,

$$C_{k-1} = \{x \in A_{k-1} \,:\, (0,...,0) \in J^k f(x)\},$$

and

$$B_{k-1} = A_{k-1} \setminus C_{k-1}.$$

Claim 6.1. *$\mathcal{H}^\ell(C_{k-1} \cap f^{-1}(y)) = 0$ for almost every $y \in \mathbb{R}^m$.*

Proof. It suffices to prove the Claim in the case $\ell \geq 1$, as the case $\ell = 0$ is just Theorem 1.4. We may assume that $C_{k-1} \subset B := [-R, R]^n$ for some fixed $R \in \mathbb{N}$. Let $\varepsilon > 0$. For every $i \in \mathbb{N}$ we define D_i as in the proof of Lemma 2.3, so we have $C_{k-1} \subset \bigcup_j D_j$ and $D_i \subset D_{i+1}$. For each i, we decompose B as the union of a family of cubes $\{Q_{ij}\}_{j=1}^{N(i)}$ of diameter $\sqrt{n}/i$ with pairwise disjoint interiors. If $x, y \in D_i \cap Q_{ij}$ equation (2.2) in the proof of Lemma 2.3 shows that

(6.2)
$$\mathcal{H}^m(f(D_i \cap Q_{ij})) \leq C(m,n)\mathrm{diam}(f(D_i \cap Q_{ij}))^m \leq \varepsilon^m C(m,n)\mathrm{diam}(Q_{ij})^{km},$$

where $C(m,n)$ denotes a constant only depending on m, n. For all $s \geq i$, the sets $\{D_s \cap Q_{sj} \cap f^{-1}(y)\}_j$ form a covering of $D_i \cap f^{-1}(y)$ by sets of diameter less than or equal to $\sqrt{n}/s$, and $\mathrm{diam}(D_s \cap Q_{sj} \cap f^{-1}(y)) \leq \mathrm{diam}(Q_{sj})\chi_{f(D_s \cap Q_{sj})}(y)$. So we have, for all $\alpha \geq s \geq i$,

$$\begin{aligned}\mathcal{H}^\ell_{\sqrt{n}/s}(D_i \cap f^{-1}(y)) &\leq C(\ell)\sum_j \mathrm{diam}(D_\alpha \cap Q_{\alpha j} \cap f^{-1}(y))^\ell \\ &\leq C(\ell)\sum_j \mathrm{diam}(Q_{\alpha j})^\ell \chi_{f(D_\alpha \cap Q_{\alpha j})}(y),\end{aligned}$$

hence

$$\mathcal{H}^\ell_{\sqrt{n}/s}(D_i \cap f^{-1}(y)) \leq \liminf_{\alpha \to \infty} C(\ell)\sum_j \mathrm{diam}(Q_{\alpha j})^\ell \chi_{f(D_\alpha \cap Q_{\alpha j})}(y),$$

and by letting $s \to \infty$ we get

$$\mathcal{H}^\ell(D_i \cap f^{-1}(y)) \leq \liminf_{\alpha \to \infty} C(\ell)\sum_j \mathrm{diam}(Q_{\alpha j})^\ell \chi_{f(D_\alpha \cap Q_{\alpha j})}(y),$$

and consequently

$$\begin{aligned}\mathcal{H}^\ell(C_{k-1} \cap f^{-1}(y)) &\leq \sup_i \mathcal{H}^\ell(D_i \cap f^{-1}(y)) \\ &\leq \liminf_{\alpha \to \infty} C(\ell)\sum_j \mathrm{diam}(Q_{\alpha j})^\ell \chi_{f(D_\alpha \cap Q_{\alpha j})}(y).\end{aligned}$$

We now use a variation of an idea in [14, Claim 3.1]: by taking the upper integral with respect to $d\mathcal{H}^m(y)$ on both sides of the above inequality (see [10] for the definition of the upper integral), using Fatou's Lemma, plugging

(6.2), and observing that $\ell + km = n + (m-1)(k-1) \geq n$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^m}^{*} \mathcal{H}^\ell(C_{k-1} \cap f^{-1}(y)) d\mathcal{H}^m(y) \\
&\leq C(m,n) \liminf_{i\to\infty} \sum_j \operatorname{diam}(Q_{ij})^\ell \mathcal{H}^m(f(D_i \cap Q_{ij})) \\
&\leq C(\ell,m,n) \liminf_{i\to\infty} \sum_j \operatorname{diam}(Q_{ij})^\ell \varepsilon^m \operatorname{diam}(Q_{i,j})^{km} \\
&\leq C(\ell,m,n) \liminf_{i\to\infty} \sum_j \varepsilon^m \operatorname{diam}(Q_{i,j})^n \leq C(\ell,m)(2R)^n \varepsilon^m.
\end{aligned}$$

By sending ε to 0 we thus have

$$\int_{\mathbb{R}^m}^{*} \mathcal{H}^\ell(C_{k-1} \cap f^{-1}(y)) d\mathcal{H}^m(y) = 0.$$

This means that there exists a sequence $\{\varphi_j\}$ of simple $\mathcal{H}^m$-measurable functions such that $\varphi_j(y) \geq \mathcal{H}^\ell(C_{k-1} \cap f^{-1}(y))$ and $\int_{\mathbb{R}^m} \varphi_j(y) d\mathcal{H}^m(y) \to 0$. So $\varphi_j \to 0$ in $L^1(\mathbb{R}^m, \mathcal{H}^m) = L^1(\mathbb{R}^m)$, hence there is a subsequence φ_{j_i} such that $\lim_{i\to\infty} \varphi_{j_i}(y) = 0$ for $\mathcal{L}^m$-a.e. y, which implies $\mathcal{H}^\ell(C_{k-1} \cap f^{-1}(y)) = 0$ for $\mathcal{L}^m$-a.e. $y \in \mathbb{R}^m$. □

Now, notice that the proof of Claim 3.2 also shows (just by replacing $n-m$ with $k-1$) that $\mathcal{L}^n(B_{k-1}) = 0$ in the current setting. Therefore, by using condition (DS2) we also get $\mathcal{H}^\ell(B_{k-1} \cap f^{-1}(y)) = 0$ for a.e. $y \in \mathbb{R}^m$, which, together with Claim 6.1 yields $\mathcal{H}^\ell(A_{k-1} \cap f^{-1}(y)) = 0$ for a.e. $y \in \mathbb{R}^m$. The proof of Theorem 1.6 is complete.

Next, in order to deduce Theorem 1.7 from Theorem 1.6, we define, for each U open in $\mathbb{R}^i$ and V open in $\mathbb{R}^j$ with $i \geq j$, and $1 \leq s \leq n$, the class $\mathcal{C}^s_{i,j}(U,V)$ as the set of all functions $f : U \to V$ such that:

(1) $f \in C^{s-1}(\mathbb{R}^i, \mathbb{R}^j)$;
(2) $\limsup_{h\to 0} \frac{|f(x+h) - f(x) - Df(x)(h) - \ldots - D^{s-1}f(x)(h^{s-1})|}{|h|^s} < \infty$ for every $x \in \mathbb{R}^n$;
(3) f has a Taylor expansion of order s at almost every $x \in \mathbb{R}^n$.

By adding the following Lemma to Lemmas 2.2, 2.4, 2.5, 2.6, 2.7, 2.8, and using the fact that diffeomorphisms map null sets onto null sets it is easy to check that the classes $\mathcal{C}^s_{i,j}$ satisfy properties (DS1)–(DS7) of Theorem 1.6; thus Theorem 1.7 follows from Theorem 1.6 (note that the case $n = m$ corresponds to $k = 1$ and $\ell = 0$, a situation which is already covered by Theorem 1.1).

Lemma 6.2. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function, $1 \leq k \leq n-m+1$, and define $\ell = n-m-k+1$. Then, for every subset N of $\{x : \limsup_{h\to 0} \frac{|f(x+h)-f(x)|}{|h|^k} < \infty\}$ such that $\mathcal{L}^n(N) = 0$, we have that $\mathcal{H}^\ell(N \cap f^{-1}(y)) = 0$ for almost every $y \in \mathbb{R}^m$.*

Proof. In the case $\ell = 0$, which corresponds to $k = n - m + 1$, this Lemma tells us the same thing as Lemma 2.2. Therefore we may assume $\ell \geq 1$. For each $j \in \mathbb{N}$ let us define A_j as in the proof of Lemma 2.2. It is enough to see that $\mathcal{H}^\ell(A_j \cap N \cap f^{-1}(y)) = 0$ for every $j \in \mathbb{N}$. So fix $j \in \mathbb{N}$ and $\varepsilon > 0$, and for each $\alpha \in \mathbb{N}$ choose a sequence of cubes $\{Q_{\alpha\beta}\}_\beta$ such that $\mathrm{diam}(Q_{\alpha\beta}) \leq 1/\alpha$, $N \subset \bigcup_{\beta=1}^\infty Q_{\alpha\beta}$, and $\sum_{\beta=1}^\infty \mathrm{diam}(Q_{\alpha\beta})^n \leq \varepsilon$. Define now D_i as in the proof of Lemma 2.2, so that $D_i \subset D_{i+1}$ and $A_j \cap N = \bigcup_{i=1}^\infty D_i$. If $x, y \in D_i \cap Q_{i\beta}$ we have, as in the proof of Lemma 2.2, that

$$|f(x) - f(y)| \leq (j+1)|x-y|^k,$$

which implies that

$$\mathcal{H}^m(f(D_i \cap Q_{i\beta})) \leq C(m)(j+1)^m \mathrm{diam}(Q_{i\beta})^{km}.$$

Now, as in the proof of Lemma 6.1, for every $\alpha \geq i$ the sets $\{D_\alpha \cap Q_{\alpha\beta} \cap f^{-1}(y)\}_\beta$ form a covering of $A_j \cap N \cap D_i \cap f^{-1}(y)$ by sets of diameter less than or equal to $\sqrt{n}/\alpha$, and $\mathrm{diam}(D_\alpha \cap Q_{\alpha\beta} \cap f^{-1}(y)) \leq \mathrm{diam}(Q_{\alpha\beta})\chi_{f(D_\alpha \cap Q_{\alpha\beta})}(y)$ so, by the same argument as in the proof of Lemma 6.1, we have

$$\mathcal{H}^\ell(A_j \cap N \cap f^{-1}(y)) \leq \liminf_{\alpha\to\infty} C(\ell) \sum_\beta \mathrm{diam}(Q_{\alpha\beta})^\ell \chi_{f(D_\alpha \cap Q_{\alpha\beta})}(y).$$

Then, similarly to the proof of Lemma 6.1, we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^m}^* \mathcal{H}^\ell(A_j \cap N \cap f^{-1}(y)) d\mathcal{H}^m(y) \\
&\leq C(m,n) \liminf_{\alpha\to\infty} \sum_\beta \mathrm{diam}(Q_{\alpha\beta})^\ell \mathcal{H}^m(f(D_\alpha \cap Q_{\alpha\beta})) \\
&\leq C(\ell,m,n) \liminf_{\alpha\to\infty} \sum_\beta \mathrm{diam}(Q_{\alpha\beta})^\ell (j+1)^m \mathrm{diam}(Q_{\alpha\beta})^{km} \\
&\leq C(\ell,m,n) \liminf_{\alpha\to\infty} \sum_j (j+1)^m \mathrm{diam}(Q_{\alpha\beta})^n \leq C(\ell,m,n)(j+1)^m \varepsilon^m.
\end{aligned}$$

By letting ε go to 0 we thus have $\int_{\mathbb{R}^m}^* \mathcal{H}^\ell(A_j \cap N \cap f^{-1}(y)) d\mathcal{H}^m(y) = 0$, and consequently that $\mathcal{H}^\ell(A_j \cap N \cap f^{-1}(y)) = 0$ for a.e. $y \in \mathbb{R}^m$. □

Finally, let us say a few words about how one can deduce Theorem 1.8 from Theorem 1.6. One can define, for each U open in $\mathbb{R}^i$ and V open in $\mathbb{R}^j$, with $i \geq j$, $s \geq 1$, the classes $\mathcal{C}^s_{i,j}(U,V) := W^{s,p}_{\mathrm{loc}}(U,V)$. By using standard results and techniques of Sobolev space theory it is not difficult to check that these classes satisfy properties (DS4)-(DS7). Property (DS1) follows from Morrey's inequality, and property (DS3) has already been checked above (in the proof of Theorem 1.5 that we included at the beginning of Section 5). The only delicate point is thus checking property (DS2). This is implicitly shown, by combining Morrey's and Hölder's inequalities with a clever use of the upper integral, in [14, Claim 3.1, Step 1], to where we refer the reader for details.

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ICMAT (CSIC-UAM-UC3-UCM), Departamento de Análisis Matemático, Facultad Ciencias Matemáticas, Universidad Complutense, 28040, Madrid, Spain
E-mail address: azagra@mat.ucm.es

IMI, Departamento de Análisis Matemático, Facultad Ciencias Matemáticas, Universidad Complutense, 28040, Madrid, Spain
E-mail address: ferrera@mat.ucm.es

Departamento de Análisis Matemático, Facultad Ciencias Matemáticas, Universidad Complutense, 28040, Madrid, Spain
E-mail address: gomezgil@mat.ucm.es