Smooth extensions of functions on certain Banach spaces

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Abstract. Let $X$ be a Banach space with unconditional basis and which admits a $C^1$-smooth norm. Let $Y \subset X$ be a closed subspace, and $f : Y \to \mathbb{R}$ a $C^1$-smooth function. Then we show there is a $C^1$ extension of $f$ to $X$.

1. Introduction

In this note we address the problem of the extension of smooth functions from subsets of Banach spaces to smooth functions on the whole space. For our results, smoothness is meant in the Fréchet sense, and we shall restrict our attention to real-valued functions. To state the problem more precisely, given a Banach space $X$, a closed subset $Y$, and a $C^p$-smooth function $f : Y \to \mathbb{R}$, when is it possible to find a $C^p$-smooth map $F : X \to \mathbb{R}$ such that $F|_Y = f$? We should note that when $Y$ is a complemented subspace of an arbitrary Banach space $X$, the extension problem can be easily solved. Indeed, let $P : X \to Y$ be a continuous linear projection, and $f : Y \to \mathbb{R}$ a $C^p$-smooth function. Then we define $F(x) = f(Px)$.

When $p = 0$, this question is the problem of the continuous extension of functions from closed subsets. A complete characterization was given by the well known theorem of Tietze (see e.g., [Wi]) which we recall states that $X$ is a normal space iff for every closed subset $Y \subset X$ and continuous function $f : Y \to \mathbb{R}$, there exists a continuous extension $F : X \to \mathbb{R}$ of $f$.

Such characterizations in the differentiable case, where $p \geq 1$, are more delicate. When $X = \mathbb{R}$, $Y \subset X$ is a subset, and $p \geq 1$, necessary and sufficient conditions (in terms of divided differences) for the existence of $C^p$-extensions to $\mathbb{R}$ of $C^p$ functions on $Y$ were given by H. Whitney [W1], [W2]. Apparently, Whitney intended to find such a characterization in the case $X = \mathbb{R}^n$ with $n > 1$, but a sequel to the paper [W2] never appeared.

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Major advances in this area occurred some twenty years later with the fundamental work of Glaeser [G] who solved the problem when \( n \geq 1 \) and \( p = 1 \). Subsequent work included that of Brudnyi and Shvartsman [BS1], [BS2], Bierstone, Milman and Pawlucka [BMP1], [BMP2], and in particular the remarkable results of C. Fefferman [Fe1], [Fe2], [Fe3]. For example, in [Fe2] a complete characterization is given of when a real-valued function defined on a compact subset of \( \mathbb{R}^n \) is the restriction of a \( C^m \)-smooth map on \( \mathbb{R}^n \).

In this paper we consider the case when \( X \) is a (real, infinite dimensional) Banach space which admits an unconditional basis and \( C^1 \)-smooth norm. Then if \( Y \subset X \) is a closed subspace and \( f: Y \to \mathbb{R} \) is \( C^1 \)-smooth, we show there exists a \( C^1 \) extension \( K: X \to \mathbb{R} \). If we require only that \( Y \subset X \) be closed and not necessarily a subspace, then a similar conclusion holds under the stronger assumption that \( f \) is defined on a neighbourhood \( U \supset Y \) and is \( C^1 \)-smooth on \( Y \) as a function on \( X \) (i.e., \( f'(y) \in X^* \) for \( y \in Y \) and \( y \to f'(y) \) is continuous). We observe, however, that in general the smooth extension problem has a negative solution. We give here three examples.

1. In [Z] (see also [DGZ, Theorem II.8.3]) an example is given of a separable Banach space \( Y \subset X = C[0,1] \), and a Gâteaux smooth norm on \( Y \) that cannot be extended to a Gâteaux smooth norm on \( X \).

2. Also in [Z], it is shown that for \( 1 < p < 2 \), there is a subspace \( Y \subset L_p \) isomorphic to Hilbert space, such that the Hilbertian norm of \( Y \) cannot be extended to a function \( \varphi \) on \( L_p \) which is Fréchet smooth on the unit sphere \( S_Y \) of \( Y \) as a function on \( X \) with \( y \to \varphi'(y) \) locally Lipschitz from \( S_Y \) to \( X^* \).

3. As suggested to us by R. Aron [A] (see also Example 2.1 [AB]). Let \( X = C[0,1] \) which has the Dunford-Pettis Property, and hence the polynomial Dunford-Pettis Property (i.e., if \( P: X \to \mathbb{R} \) is a polynomial and \( x_i \overset{w}{\to} 0 \), then \( P(x_i) \to P(0) \)). Now \( Y = l_2 \subset X \) by the Banach-Mazur Theorem, and we consider \( f(x) = \|x\|_{l_2}^2 \). If \( f \) extended to a \( C^2 \)-smooth function \( F \) on an open neighbourhood of \( l_2 \) in \( X \), then

\[
P(h) = (1/2) F''(0)(h,h)
\]

would be a polynomial on \( X \). But \( e_i \overset{w}{\to} 0 \) in \( l_2 \) and so in \( X \), and thus as noted above we would have \( 1 = P(e_j) \to P(0) = 0 \), a contradiction.

One can compare the results of this note with the work of C.J. Atkin [At]. The programme of Atkin is to find smooth extension results for smooth functions \( f \) defined on finite unions of open, convex sets in separable Banach spaces \( X \) that do not admit \( C^p \)-smooth norms, or even \( C^p \)-smooth bump functions. In order to achieve this, however, it is assumed in [At] that...
the function $f$ already possesses smooth extensions to all of $X$ in a neighbourhood of every point in its domain. Finally we mention the result [Z, Proposition VIII.3.8], which states, in particular, that for weakly compactly generated $X$ which admit $C^p$-smooth bump functions, that for any closed subset $Y \subset X$ and continuous function $f : Y \to \mathbb{R}$, there exists a continuous extension of $f$ to $X$ which is $C^p$-smooth on $X \setminus Y$.

We remark that the situation for analytic maps is quite different. Indeed, the paper by R. Aron and P. Berner [AB] characterizes the existence of analytic extensions from subspaces in terms of the existence of a linear extension operator. In particular, in the real case they prove, among many other equivalences, that if $Y$ is a closed subspace of a Banach space $X$, the following are equivalent (recall that $Z$ is a $C$-space if it is complemented in its second dual $Z^{**}$):

1. For any $C$-space $Z$, and (real) analytic map $f : Y \to Z$ which is bounded on bounded sets, there exists an analytic extension $F : X \to Z$ of $f$, also bounded on bounded sets.

2. There exists a continuous, linear extension operator $T : Y^* \to X^*$.

Motivated by the work of Moulis [M], we have combined some of her techniques and the classical method of Tietze to deduce our principal results.

Our notation is standard, with $X$ typically denoting a (real) Banach space. We shall denote an open ball with centre $x \in X$ and radius $r > 0$ either by $B_r(x)$, $B(x;r)$, or $B_r$ if the centre is understood. We write the closed unit ball of $X$ as $B_X$. If $Y \subset X$, we denote the restriction of a function $f : X \to \mathbb{R}$ to $Y$ by $f|_Y$, and we say that a map $K : X \to \mathbb{R}$ is an extension of $f : Y \to \mathbb{R}$ if $K(y) = f(y)$ for all $y \in Y$. As noted above, if $Y \subset X$ and $U \supset Y$ is open, we say that $f : U \to \mathbb{R}$ is $C^1$-smooth on $Y$ as a function on $X$ if $f'(y) \in X^*$ and $y \to f'(y)$ is continuous for $y \in Y$. For any undefined terms we refer the reader to [FHHMPZ], [DGZ].

2. Main Results

An important tool shall be the following $C^1$-smooth version of [F2, Theorem 1]. For the sake of completeness, and as [F2], which is a corrigendum to [F1], does not include details of the proof, we include here an outline of the proof. (See http://www.tru.ca/advtech/faculty/Robb_Fry.html for the full proof).

**Lemma 1.** Let $X$ be a Banach space with unconditional basis and which admits a $C^1$-smooth norm. Let $Y \subset X$ be a subset, and $f : Y \to \mathbb{R}$ an $\eta$-Lipschitz function. Then for each $\varepsilon > 0$ there exists a $C_{0\eta}$-$\eta$-Lipschitz, $C^1$-smooth function $K : X \to \mathbb{R}$ such that for all $y \in Y$,

$$|f(y) - K(y)| < \varepsilon,$$

where $C_0 > 0$ is a constant depending only on $X$. 

Outline of proof. Let \( \{e_j, e_j^*\}_{j=1}^{\infty} \) be an unconditional Schauder basis on \( X \), and \( P_n : X \to X \) the canonical projections given by \( P_n(x) = P_n \left( \sum_{j=1}^{\infty} x_j e_j \right) = \sum_{j=1}^{n} x_j e_j \), and where we set \( P_0 = 0 \). By renorming, we may suppose the unconditional basis constant is 1. In particular, \( \|P_n\| \leq 1 \) for all \( n \). We put \( E_n = P_n(X) \), and \( E^\infty = \cup_n E_n \), noting that \( \dim E_n = n \), \( E_n \subset E_{n+1} \), and \( E^\infty \) is a dense subspace of \( X \). Using an infimal convolution, we extend \( f \) to a Lipschitz map \( F \) on \( X \) with the same constant \( \eta \) by defining, \( F(x) = \inf \{ f(y) + \eta \|x - y\| : y \in Y \} \). Let \( \varepsilon > 0 \). It is proven in [J] that there exists a function \( g : X \to \mathbb{R} \) such that the following hold:

1. \( |g - F| < \varepsilon/3 \) on \( X \),
2. The map \( g \) is \( \eta \)-Lipschitz,
3. The map \( g \) is uniformly Gâteaux differentiable.

Next, following [AFGJL, Lemma 5], let \( \varphi : \mathbb{R} \to [0,1] \) be a \( C^\infty \)-smooth function such that \( \varphi(t) = 1 \) if \( |t| < 1/2 \), \( \varphi(t) = 0 \) if \( |t| > 1 \), \( \varphi([0,\infty)) \subseteq [-3,0] \), \( \varphi(-t) = \varphi(t) \). Since \( X \) admits a \( C^1 \)-smooth norm, it admits a Lipschitz, \( C^1 \)-smooth bump function. Hence, it admits a Lipschitz and \( C^1 \)-smooth function \( \mu \) on \( X \setminus \{0\} \) with the property that there is a number \( M \geq 1 \) such that \( \frac{1}{M}\|x\| \leq \mu_A(x) \leq M\|x\| \) for all \( x \in X \), and \( \|\mu_A'(x)\| \leq M \) for all \( x \in X \setminus \{0\} \) (see e.g., [AFM]). Let us define a function Gâteaux differentiable on \( X \), and \( C^1 \)-smooth on \( E_n \), by

\[
F_n(x) = \frac{(a_n)^n}{c_n} \int_{E_n} g(x - y) \varphi(\mu(y)) \, dy
\]

where

\[
c_n = \int_{E_n} \varphi(\mu(y)) \, dy,
\]

and we have chosen the constants \( a_n \) large enough that

\[
\sup_{x \in E_n} |F_n(x) - g(x)| < \frac{\varepsilon}{6} 2^{-n}.
\]

It is easily verified that \( F_n \) is \( \eta \)-Lipschitz since \( g \) is as well. As pointed out to us by P. Hájek, since \( g \) is Lipschitz and uniformly Gâteaux differentiable, by [HJ, Lemma 4] for each \( h \) the map \( x \to D_h g(x) \) is uniformly continuous. From this, the Lipschitzness of \( g \), and compactness of \( B_{E_n} \), we can choose the \( a_n \) larger if need be so that for all \( h \in B_{E_n} \) we have,

\[
\sup_{x \in E_n} |D_h F_n(x) - D_h g(x)| < \frac{\eta a^{-n}}{2}.
\]

Next define a sequence of Gâteaux differentiable functions \( \overline{f}_n : X \to \mathbb{R} \), \( C^1 \)-smooth on \( E_n \), as follows. Put \( \overline{f}_0 = f(0) \), and supposing that \( \overline{f}_0, \ldots, \overline{f}_{n-1} \) have been defined, we set

\[
\overline{f}_n(x) = F_n(x) + \overline{f}_{n-1}(P_{n-1}(x)) - F_n(P_{n-1}(x)).
\]
One can verify by induction using \( \|P_n\| \leq 1 \), and the above estimates that,

(i). The \( f_n \) are Gâteaux differentiable, the restriction of \( f_n \) to \( E_n \) is \( C^1 \)-smooth, and \( f_n \) extends \( \bar{f}_{n-1} \),

(ii). \( \sup_{x \in E_n} |\bar{f}_n(x) - g(x)| < \frac{\varepsilon}{2} \left(1 - \frac{1}{M_2}\right) \),

(iii). \( \sup_{x \in E_n} |D_h \bar{f}_n(x) - D_h g(x)| \leq \eta \left(1 - \frac{1}{M_2}\right) \), for all \( h \in B_{E_n} \).

We now define the map \( \bar{f} : E^\infty \to \mathbb{R} \) by \( \bar{f}(x) = \lim_{n \to \infty} f_n(x) \). For \( x \in E^\infty = \bigcup_n E_n \), define \( n_x \equiv \min \{n : x \in E_n \} \), and note that we have for any \( m \geq n_x \), \( \bar{f}(x) = \lim_{n \to \infty} \bar{f}_n(x) = \bar{f}_m(x) \). In particular, for any \( n \), \( \bar{f} \big|_{E_n} = \bar{f}_n \). One can verify using these facts and (i), (ii) and (iii) above that \( \bar{f} \) has the following properties:

(i)' The restriction of \( \bar{f} \) to every subspace \( E_n \) is \( C^1 \)-smooth,

(ii)' \( \sup_{x \in E^\infty} |\bar{f}(x) - g(x)| \leq \frac{\varepsilon}{2} \)

(iii)' \( \sup_{x \in E_n} |D_h \bar{f}(x) - D_h g(x)| \leq \eta \), for all \( h \in B_{E_n} \).

Let \( r \in (0, \varepsilon/3M\eta) \), and for \( x = \sum_n x_n e_n \in X \) we define the maps \( \chi_n(x) = 1 - \varphi \left[ \frac{\mu_k(x-P_{r-1}(x))}{\epsilon} \right] \), and \( \Psi(x) = \sum_n \chi_n(x) x_n e_n \). For any \( x_0 \), because \( P_n(x_0) \to x_0 \) and the \( \|P_n\| \) are uniformly bounded, there exist a neighbourhood \( N_0 \) of \( x_0 \) and an \( n_0 = n_{x_0} \) so that \( \chi_n(x) = 0 \) for all \( x \in N_0 \) and \( n \geq n_0 \) and so \( \Psi(N_0) \subset E_{n_0} \). Thus, \( \Psi : X \to E^\infty \) is a \( C^1 \)-smooth map whose range is locally contained in the finite dimensional subspaces \( E_n \). Using the fact that \( \{e_n\} \) is unconditional with constant \( C = 1 \), one can show that (see [AFGJL, Fact 7])

\[
(2.1) \quad \|x - \Psi(x)\| < Mr,
\]

and

\[
(2.2) \quad \|\Psi'(x)\| \leq 8M^2.
\]

We define \( K(x) = \bar{f}(\Psi(x)) \). Note that \( K \) is \( C^1 \)-smooth on \( X \), being the composition of \( C^1 \)-smooth maps, and noting that \( \Psi \) maps locally into some \( E_n \). Now, for \( x \in X \), by (ii)', choice of \( r \), using (1), (2.1), and that \( g \) is Lipschitz with constant \( \eta \), one can verify that \( |F(x) - K(x)| < \epsilon \). In particular, since \( F |V = f \), we have for \( y \in Y \) that \( |f(y) - K(y)| < \epsilon \).

To show that \( \|K'(x)\| = \\|\bar{f}'(\Psi(x)) \Psi'(x)\| \) is appropriately bounded, one uses the fact that for \( x \in X \), \( \Psi \) maps a neighbourhood of \( x \) into \( E_{n_x} \), and hence for any \( h \in X \) also \( \Psi'(x)(h) \in E_{n_x} \). Then for any fixed \( h \) with \( \|h\| \leq \frac{1}{M_2} \), which implies by (2.2) that \( \|\Psi'(x)(h)\| < 1 \), by (iii)' one can show that \( |K'(x)(h)| < 16M^2 \eta \equiv C_0 \eta \).

We next establish the existence of a continuous and bounded selection of the Hahn-Banach extension operator \( y^* \in Y^* \to G(y^*) \in 2X^* \), \( G(y^*) = \{x^* \in X^* : x^*(y) = y^*(y) \} \) for all \( y \in Y \).
Lemma 2. For every Banach space $X$ and every closed subspace $Y \subset X$ there exist a continuous mapping $H : Y^* \to X^*$ and a number $M \geq 1$ such that

1. $H(y^*)(y) = y^*(y)$ for every $y^* \in Y^*$, $y \in Y$;
2. $\|H(y^*)\|_{X^*} \leq M\|y^*\|_{Y^*}$ for every $y^* \in Y^*$.

Proof. This is an easy consequence of the Bartle-Graves selector theorem (see [DGZ], page 299): Let $W$ and $Z$ be Banach spaces and let $T$ be a bounded linear mapping of $W$ onto $Z$. Then there exists a continuous (non-linear in general) mapping $B$ of $Z$ into $W$ such that $T \circ B(w) = (w)$ for every $w \in W$. Moreover, it follows from the proof of this result that there exists an $M > 1$ such that $\|B(w)\| \leq M\|w\|$.

If we apply this theorem with $W = X^*$, $Z = Y^*$, to the mapping $T : X^* \to Y^*$ defined by $T(x^*) = x^*|_Y$, which is a continuous linear surjection with $\|T\| = 1$ (by the Hahn-Banach theorem), we obtain our continuous map $H = B : Y^* \to X^*$ with the property that the the restriction of $H(y^*)$ to $Y$ is $y^*$, for every $y^* \in Y^*$, and such that $\|H(y^*)\|_{X^*} \leq M\|y^*\|_{Y^*}$. □

Now we are in a situation to deduce an approximation result which is of independent interest and which, combined with some ideas of the Tietze proof, will yield our main results on smooth extension.

Theorem 1. Let $X$ be a Banach space with unconditional basis which admits a $C^1$-smooth norm, and $Y \subset X$ a closed subspace. Let $f : Y \to \mathbb{R}$ be a $C^1$-smooth function, and $F$ a continuous extension of $f$ to $X$. Let $H : Y^* \to X^*$ be any extension operator as in Lemma 2. Then, for every $\varepsilon > 0$, there exists a $C^1$-smooth map $g : X \to \mathbb{R}$ such that

1. $|F(x) - g(x)| < \varepsilon$ on $X$, and
2. $\|H(f^i(y)) - g^i(y)\|_{X^*} < \varepsilon$ on $Y$.

Furthermore, if the given function $f$ is Lipschitz on $Y$ and $F$ is a Lipschitz extension of $f$ to $X$ with $\text{Lip}(F) = \text{Lip}(f)$ (for instance $F(x) = \inf_{y \in Y} \{f(y) + \text{Lip}(f)\|x - y\|\}$), then the function $g$ can be chosen to be Lipschitz on $X$ and with the additional property that

3. $\text{Lip}(g) \leq C\text{Lip}(f)$,

where $C > 1$ is a constant only depending on $X$.

Proof. First note that by the Tietze Theorem, the continuous extension $F$ always exists. We modify the proof of Theorem 4 in [AFGJL] employing Lemma 1. Using the separability of $X$, the closedness of $Y \subset X$, and the continuity of $F$, let $\mathcal{C} = \{B_{r_j}\}_{j=1}^\infty \cup \{B_{s_k}\}_{k=1}^\infty$ be a covering of $X$ by open balls with centres $x_j$ and $y_k$ respectively, such that:

(i) We have $B_{2r_j} \subset X \setminus Y$, and $|F(x) - F(x_j)| < \varepsilon/2C_0$ on $B_{2r_j}$

(ii) The collection $\{B_{s_k}\}_{k} \subset X$ covers $Y$ with centres $y_k \in Y$ and radii $s_k$ chosen using the smoothness of $f$ on $Y$ and the norm-norm continuity of the extension operator $H$, so that $\|T^i_k(y) - f^i(y)\|_{Y^*} < \varepsilon/8C_0$ and
$\|T'_k(y) - H(f'(y))\|_{X^*} < \varepsilon/8C_0$ on $B_{2s_k} \cap Y$, where $T_k$ is the natural $H$-extension of the first order Taylor Polynomial of $f$ at $y_k$; namely, $T_k(x) = f(y_k) + H(f'(y_k))(x - y_k)$. Note in particular that $T_k \in C^\infty(X, \mathbb{R})$, with $T'_k(x) = H(f'(y_k))$ for every $x \in X$, and $T'_k(y) |_{Y} = f'(y_k)$ for all $y \in Y$.

Let $\beta : \mathbb{N} \to \mathcal{C}$ be a bijection where for each $i$, $\beta(i) = B(\beta_1(i); \beta_2(i))$. Let $\varphi_j \in C^1([X, [0, 1]])$ with bounded derivative so that $\varphi_j = 1$ on $B(\beta_1(j); \beta_2(j))$, and $\varphi_j = 0$ outside of $B(\beta_1(j); 2\beta_2(j))$.

By Lemma 1 applied to $T_k(y) - f(y)$ on $B(y_k; 2s_k) \cap Y$, we may choose $C^1$-smooth maps $\delta_k : X \to \mathbb{R}$ so that on each $B(y_k; 2s_k) \cap Y$ we have both $|T_k(y) - f(y) - \delta_k(y)| < 2^{-k-2}\varepsilon M^{-1}_k$, and $\|\delta'_k(y)\|_{X^*} < \varepsilon/8$, where $M_k = \sum_{i=1}^k M_i$ and $\tilde{M}_i = \sup_{x \in Y \cap B(y_j; 2s_i)} \|\varphi'_i(x)\|_{X^*}$.

Then we also have, for $y \in B(y_k; 2s_k) \cap Y$ using our estimate above,

$$\|T'_k(y) - H(f'(y)) - \delta'_k(y)\|_{X^*} \leq \|T'_k(y) - H(f'(y))\|_{X^*} + \|\delta'_k(y)\|_{X^*} < \varepsilon/8C_0 + \varepsilon/8 \leq \varepsilon/4.$$

Set $\Delta_i(x) = T_k(x) - \delta_k(x)$ if $\beta(i) = B_{s_k}$ is a ball from the subcollection $\{B_{s_i}\}_{i=1}^\infty$ covering $Y$, and $\Delta_i = F(x_j)$ if $\beta(i) = B_{r_j}$ belongs to the subcollection $\{B_{r_i}\}_{i=1}^\infty$ covering $X \setminus Y$.

Next, we define

$$h_i = \varphi_i \prod_{k < i} (1 - \varphi_k),$$

and

$$g(x) = \sum_i h_i(x) \Delta_i(x).$$

Note that for each $x$, if $n := n(x) := \min \{m : x \in \beta(m)\}$, then because $1 - \varphi_n(x) = 0$ and $\beta(n)$ is open, it follows from the definition of the $h_j$ that there is a neighbourhood $N \subset \beta(n)$ of $x$ so that for $z \in N$, $g(z) = \sum_{j \leq n} h_j(z) \Delta_j(z)$, and $\sum_j h_j(z) = \sum_{j \leq n} h_j(z)$. Also, by a straightforward calculation, again using the fact that $\varphi_n = 1$ on $\beta(n)$, we have that $\sum_j h_j(z) = 1$ for $z \in \beta(n)$, and so for all $z \in X$.

Now, fix any $x_0 \in X$, and let $n_0 = n(x_0)$ and a neighborhood $N_0$ of $x_0$ be as above. For each $j \leq n_0$ define the functions $V_j : N_0 \to \mathbb{R}$ and $W_j : N_0 \to \mathbb{R}$ by

$$V_j(x) = \begin{cases} 0 \text{ if } \beta_1(j) \notin Y \\ |T_k(x) - F(x) - \delta_k(x)| \text{ if } \beta_1(j) = y_k \end{cases}$$

and

$$W_j(x) = \begin{cases} 0 \text{ if } \beta_1(j) \in Y \\ |F(x_i) - F(x)| \text{ if } \beta_1(j) = x_i \end{cases}$$
Then for any $x \in N_0$ we have that
\[ |g(x) - F(x)| \leq \sum_{j \leq n_0} h_j(x) \max\{V_j(x), W_j(x)\} \]
\[ \leq \sum_{j \leq n_0} h_j(x) \frac{\varepsilon}{2} < \varepsilon. \]

Now define the function $\alpha$ so that when $\beta_1(j) \in Y$, $\beta_1(j) = y_{\alpha(j)}$ for $j \leq n(y)$. Recall that $B(x_j; 2r_j) \cap Y = \emptyset$ for every $j$, and that $\varphi_j = 0$ off of $B(x_j; 2r_j)$. Hence, if $y \in Y$, then in the sum $g(y)$ only those indices $j$ such that $\beta_1(j) \in Y$ are non-zero.

Recall also that $\sum_j h_j(x) = 1$ for all $x \in X$ (and hence $\sum_j h_j(x) = 0$), and so for $y \in Y$ we have, $H(f'(y)) = \sum_j h_j'(y) f(y) + \sum_j h_j(y) H(f'(y))$.

And, $g'(y) = \sum_j h_j'(y) \left( T_{\alpha(j)}(y) - \delta_{\alpha(j)}(y) \right) + \sum_j h_j(y) \left( T_{\alpha(j)}'(y) - \delta_{\alpha(j)}'(y) \right)$.

Finally, a straightforward calculation shows that $\|h_j'(x)\| \leq M_{\alpha(j)}$ for $\beta_1(j) \in Y$. With these observations in mind, we have,
\[ \|g'(y) - H(f'(y))\|_{X^*} \]
\[ \leq \sum_{j \leq n(y) \atop \beta_1(j) \in Y} \left( \|h_j'(y)\| \|T_{\alpha(j)}(y) - f(y) - \delta_{\alpha(j)}(y)\| \right) 
+ h_j(y) \left\| T_{\alpha(j)}'(y) - H(f'(y)) - \delta_{\alpha(j)}'(y) \right\|_{X^*}} 
\[ < \sum_{j \leq n(y) \atop \beta_1(j) \in Y} \|h_j'(y)\| \left( 2^{-\alpha(j) - 2\varepsilon M_{\alpha(j)}^{-1}} \right) + \sum_{j \leq n(y) \atop \beta_1(j) \in Y} h_j(y) \frac{\varepsilon}{4} \]
\[ < \sum_{j \leq n(y) \atop \beta_1(j) \in Y} M_{\alpha(j)} \left( 2^{-\alpha(j) - 2\varepsilon M_{\alpha(j)}^{-1}} \right) + \frac{\varepsilon}{4} < \varepsilon. \]

As $H(f'(y)) \mid_Y = f'(y)$, we also have the estimate $\|g'(y) - f'(y)\|_{Y^*} < \varepsilon$.

Let us now consider the case when $f$ is $C^1$ and Lipschitz on $Y$ and $F$ is any Lipschitz extension of $f$ to $X$ with $\text{Lip}(F) = \text{Lip}(f)$. In this case we have to modify the definition of the functions $\Delta_i$ as follows. We let $\Delta_i(x) = T_k(x) - \delta_k(x)$ if $\beta(i) = B_{\delta_k}$ is a ball from the subcollection $\{B_{\delta_k}\}_{i=1}^\infty$ covering $Y$ where $\delta_k$ is chosen (by using Lemma 1) so that $|T_k(y) - f(y) - \delta_k(y)| < 2^{-i - 2\varepsilon M_i^{-1}}$, and $\|\delta_k'(y)\|_{X^*} < \varepsilon/8$, where now the $M_i$ are defined by $M_i = \ldots$
\[ \sum_{j=1}^{i} \tilde{M}_j \text{ and } \tilde{M}_j = \sup_{x \in B(\beta_1(j) ; \beta_2(j))} \| \varphi'_j (x) \|_{X^*} ; \] and we let \( \Delta_i(x) = F_i(x) \)
if \( \beta(i) = B_{r_i} \) belongs to the subcollection \( \{ B_{r_j} \}_{j=1}^{\infty} \) covering \( X \setminus Y \), where
the function \( F_i \) is again chosen by using Lemma 1 so that \( |F_i(x) - F(x)| < 2^{-i-2} \varepsilon M_i^{-1} \)
on \( B_{2r_i} \) and \( \text{Lip}(F_i) \leq C_0 \text{Lip}(F) = C_0 \text{Lip}(f) \). Note that, with
these choices, we have
\[ |\Delta_i(x) - F(x)| < 2^{-i-2} \varepsilon M_i^{-1} \text{ on } B(\beta_1(i), \beta_2(i)) , \]
and \( \text{Lip}(\Delta_i) \leq \max\{ MLip(f) + \frac{\varepsilon}{8}, C_0 \text{Lip}(f) \} \).

We can obviously assume \( f \) is not constant, so that \( \text{Lip}(f) > 0 \), and if \( \varepsilon \) is
chosen small enough the last inequality implies
\[ \text{Lip}(\Delta_i) \leq C_1 \text{Lip}(f) , \]
where \( C_1 \) is a constant depending only on \( X \).

Now define the \( C^1 \) function \( g : X \to \mathbb{R} \) by
\[ g(x) = \sum_i \Delta_i(x) h_i(x) . \]
As above, one can check that \( |g(x) - F(x)| < \varepsilon \) for all \( x \in X \), and \( \|g'(y) - H(f'(y))\|_{X^*} < \varepsilon \) for all \( y \in Y \), that is \( g \) satisfies properties (1) and (2) of
the statement. Let us see that \( g \) satisfies (3) as well. Noting that \( \text{Lip}(h_j) \leq M_j \),
we can estimate, for every \( x, z \in X \),
\[ g(x) - g(z) = \sum_j (\Delta_j(x) - F(x))(h_j(x) - h_j(z)) + \sum_j (\Delta_j(x) - \Delta_j(z)) h_j(z) \leq \]
\[ \sum_j \frac{\varepsilon}{2^{j+2} M_j} \text{Lip}(h_j) \| x - z \| + \sum_j \text{Lip}(\Delta_j) \| x - z \| \leq \]
\[ \left( \frac{\varepsilon}{4} + C_1 \text{Lip}(f) \right) \| x - z \| \leq \text{C Lip}(f) \| x - z \| , \]
provided that \( \varepsilon > 0 \) is chosen small enough (recall that we are assuming
\( \text{Lip}(f) > 0 \)), and where \( C = 2C_1 > 1 \), a constant only depending on \( X \).
This shows that \( \text{Lip}(g) \leq \text{C Lip}(f) \). \( \blacksquare \)

**Theorem 2.** Let \( X \) be a Banach space with unconditional basis which
admits a \( C^1 \)-smooth norm. Let \( Y \subset X \) be a closed subspace, and \( f : Y \to \mathbb{R} \) a \( C^1 \)-smooth and Lipschitz function. Then there is a \( C^1 \) and Lipschitz
extension \( g : X \to \mathbb{R} \) of \( f \) such that \( \text{Lip}(g) \leq C \text{Lip}(f) \), where \( C \) is a constant
depending only on \( X \).

**Proof.** First note that if \( h \) is a bounded, Lipschitz function defined on \( Y \),
there always exists a bounded, Lipschitz extension of \( h \) to \( X \), with the same
Lipschitz constant, and bounded by the same constant (defined for instance
by $x \mapsto \max\{-\|h\|_{\infty}, \min\{\|h\|_{\infty}, \inf_{y \in Y}\{h(y) + \text{Lip}(h)\|x-y\|}\}\}$. For the purposes of the proof, we denote such an extension by $\overline{h}$.

We are going to define our function $g$ by means of a series constructed by induction (the first step of the inductive argument being somewhat different from the rest).

By Theorem 1 there exists a $C^1$ function $g_1 : X \to \mathbb{R}$ such that

- $|f - g_1| < 2^{-1}\varepsilon$ on $X$,
- $\|H(f'(y)) - g'_1(y)\|_{X^*} < 2^{-1}\varepsilon$,
- $\|f'(y) - g'_1(y)\|_{Y^*} < 2^{-1}\varepsilon/M$ (note in particular that this implies $\text{Lip}(f - g_1) \leq 2^{-1}\varepsilon/M$), and
- $\text{Lip}(g_1) \leq M \text{Lip}(f)$

Now, for $n \geq 2$, suppose that we have chosen $g_1, \ldots, g_n$, real-valued and $C^1$-smooth on $X$ such that for all $x \in X$ and $y \in Y$,

$$\left|\left(f - \sum_{i=1}^{n} g_i\right)(x)\right| < 2^{-n}\varepsilon,$$

$$\text{Lip}(g_n) \leq M \text{Lip}\left(f - \sum_{j=1}^{n-1} g_j\right),$$

$$\left\|H\left(f'(y) - \sum_{i=1}^{n} g_i(y)\right)\right\|_{X^*} < 2^{-n}\varepsilon,$$

and

$$\text{Lip}\left(f - \sum_{i=1}^{n} g_i\right) \leq 2^{-n}\varepsilon/M$$

(it is clear that an application of Theorem 1 to the function $f - (g_1)_{|Y}$ provides us with a function $g_2$ which, together with $g_1$, makes the above properties true for $n = 2$, hence we can proceed to the general step of our inductive construction).

Consider the function $l = f - \sum_{i=1}^{n} g_i$, which is $C^1$-smooth on $Y$. By Theorem 1, we can find a $C^1$-smooth map $g_{n+1}$ on $X$ such that we have,

(2.3) $$|l(x) - g_{n+1}(x)| < 2^{-n-1}\varepsilon \quad \text{on } X,$$

and also, for $y \in Y$,

(2.4) $$\left\|H\left(l'(y)\right) - g'_{n+1}(y)\right\|_{X^*} < 2^{-n-1}\varepsilon$$

(2.5) $$\left\|l'(y) - g'_{n+1}(y)\right\|_{Y^*} < 2^{-n-1}\varepsilon/M,$$

and

(2.6) $$\text{Lip}\left(g_{n+1}\right) \leq M \text{Lip}(l).$$
From (2.3), we have in particular, 

\[ |f(y) - \sum_{i=1}^{n+1} g_i(y)| = |l(y) - g_{n+1}(y)| < 2^{-n-1}\varepsilon \]
on Y, and so 

\[ \left| f - \sum_{i=1}^{n+1} g_i \right|(x) < 2^{-n-1}\varepsilon \]
on X.

We also have, from (2.5) and Lemma 2 (2),

\[ \left\| H \left( f' - \sum_{i=1}^{n+1} g'_i(y) \right) \right\|_{X^*} \leq M \left\| f' - \sum_{i=1}^{n+1} g'_i(y) \right\|_{Y^*} < 2^{-n-1}\varepsilon. \]

Finally, from (2.5) we get 

\[ \text{Lip} \left( f - \left( \sum_{i=1}^{n+1} g_i \right)_Y \right) \leq 2^{-n}\varepsilon/M, \]

and this completes the inductive step.

From the construction of the functions \( g_n \), we have that, for all \( x \in X \) and \( n \geq 2 \),

\[ \| g'_{n+1}(x) \| \leq \text{Lip}(g_{n+1}) \leq M \text{Lip} \left( f - \left( \sum_{j=1}^{n} g_j \right)_Y \right) \leq M 2^{-n} \varepsilon/M = 2^{-n} \varepsilon, \]

hence the series \( \sum_j g_j(x) \) is absolutely and uniformly convergent on \( X \). Similarly, we have the estimate \( |g_{n+1}(x)| \leq 2^{-n+1} \varepsilon \). Therefore the series

\[ g(x) = \sum_{n=1}^{\infty} g_n(x) \]

defines a \( C^1 \) function on \( X \), which coincides with \( f \) on \( Y \) because of the first inequality in the inductive assumptions. Finally, we have

\[ \text{Lip}(g) \leq \text{Lip}(g_1) + \sum_{n=2}^{\infty} \text{Lip}(g_n) \leq M \text{Lip}(f) + \varepsilon 2^{-1} M + \sum_{n=2}^{\infty} 2^{-n} \varepsilon \leq 2M \text{Lip}(f), \]

provided that \( \text{Lip}(f) > 0 \) (which we can always assume) and \( \varepsilon \) is small enough. ■

**Theorem 3.** Let \( X \) be a Banach space with unconditional basis which admits a \( C^1 \)-smooth norm. Let \( Y \subset X \) be a closed subspace, and \( f: Y \to \mathbb{R} \) a \( C^1 \)-smooth function. Then there is a \( C^1 \) extension of \( f \) to \( X \).

**Proof.** Since \( f \) is \( C^1 \) on \( Y \), there exists \( \{B_j\} := \{B(y_j, r_j)\}_{j=1}^{\infty} \), a countable covering of \( Y \) by open balls in \( X \) such that \( f \) is Lipschitz on \( B_j \cap Y \) for each \( j \in \mathbb{N} \). Let \( U = \bigcup_{j=1}^{\infty} B_j \), and \( W = X \setminus Y \).

Consider the mapping \( h: X \to X \) defined by

\[ h(x) = \frac{1}{1 + \|x\|} x. \]

It is easily checked that \( h \) is a \( C^1 \) diffeomorphism from \( X \) onto its open unit ball \( B_X \) (with inverse \( h^{-1}(y) = (1/(1 - \|y\|)) y \)), that \( h \) has a bounded derivative, and that \( h \) preserves lines and in particular leaves the subspace \( Y \) invariant. By composing \( h \) with suitable dilations and translations we get \( C^1 \) diffeomorphisms \( h_j: X \to B_j \) such that \( h_j \) is Lipschitz for each \( j \).
And, by composing the restrictions to $Y$ of these $h_j$ with our function $f$, we get $C^1$ and Lipschitz functions $f_j := f \circ (h_j)|_Y : Y \to \mathbb{R}$. According to the preceding result there exist $C^1$ (and Lipschitz) extensions $G_j : X \to \mathbb{R}$ of $f_j$. Then the composition

$$g_j = G_j \circ h_j^{-1}$$

defines a $C^1$ extension of $f_{|B_j \cap Y}$ to $B_j$.

Now let $\{\varphi\} \cup \{\varphi_j\}_{j=1}^\infty$ be a $C^1$ partition of unity subordinated to the open covering $\{W\} \cup \{B_j\}_{j=1}^\infty$ of $X$ (such partitions of unity always exist for separable spaces with $C^1$ norms, see [DGZ]). Define

$$g(x) = \sum_j \varphi_j(x) g_j(x).$$

Then it is clear that $g$ is a $C^1$ extension of $f$ to $X$. ■

**Corollary 1.** Let $M$ be a separable Banach manifold modelled on a Banach space $X$ with unconditional basis which admits a $C^1$ norm, and let $N$ be a closed $C^1$ submanifold of $M$. Then every $C^1$ function $f : N \to \mathbb{R}$ has a $C^1$ extension to $M$.

**Proof.** Let $\{V_j\}_{j=1}^\infty$ be a covering of $N$ by open sets in $M$ so that there are $C^1$ diffeomorphisms $\psi_j : V_j \to X$ such that $\psi_j(N \cap V_j) = Y$, where $Y$ is a closed subspace of $X$.

The functions $f \circ \psi_j^{-1} : Y \to \mathbb{R}$ are $C^1$ and (by the preceding theorem) there are $C^1$ extensions $G_j : X \to \mathbb{R}$, which in turn give, by composition, $C^1$ extensions $g_j := G_j \circ \psi_j$ of $f_{|V_j \cap N}$ to $V_j$.

Then, if $\{\theta\} \cup \{\theta_j\}_{j=1}^\infty$ is a $C^1$ partition of unity subordinated to the open covering $\{M \setminus N\} \cup \{V_j\}_{j=1}^\infty$ of $M$ (note that a separable Banach manifold modelled on a Banach space $X$ admits $C^1$ partitions of unity if and only if $X$ does), the function

$$g(x) = \sum_j \theta_j(x) g_j(x)$$

is a $C^1$ extension of $f$ to $M$. ■

If $Y$ is not required to be a closed submanifold of $X$ but is merely closed, results similar to Theorem 1 and Theorem 2 can be obtained. However, the differentiability requirements on $f$ must be strengthened. The proofs, which we omit, closely parallel those for Theorem 1 and Theorem 2 (replacing the space $X$ with an open subset $U$ of $X$), the essential difference being that $f'(y)$ is extended to directions off of $Y$, not by the Bartle-Graves generated $H(f'(y))$, but by explicit hypothesis.
THEOREM 4. Let $X$ be a Banach space with unconditional basis which admits a $C^1$-smooth norm, $Y \subset X$ a closed subset, and $U \supset Y$ a neighbourhood of $Y$. Let $\varepsilon > 0$, and $f : U \to \mathbb{R}$ be a map which is $C^1$-smooth on $Y$ as a function on $X$. Then there exists a $C^1$-smooth map $g : X \to \mathbb{R}$ such that,

1. $|f(y) - g(y)| < \varepsilon$ on $Y$,
2. $\|f'(y) - g'(y)\|_{Y^*} < \varepsilon$ on $Y$.

THEOREM 5. Let $X$ be a Banach space with unconditional basis which admits a $C^1$-smooth norm. Let $Y \subset X$ be a closed subset, and $U$ an open set containing $Y$. Let $f : U \to \mathbb{R}$ be a map $C^1$-smooth on $Y$ as a function on $X$. Then there is a $C^1$ extension of $f|_Y$ to $X$.

We have the following easy corollary,

**Corollary 2.** Let $X$ be a Banach space with unconditional basis which admits a $C^1$-smooth norm. Let $U \subset X$ be open and $f : U \to \mathbb{R}$ a $C^1$-smooth function. Then for any open set $V \subset U$ with $V \subset U$, there is a $C^1$ extension of $f|_V$ to $X$.

Somewhat in contrast to the previous results, by adopting an even stronger differentiability hypothesis, we may obtain the following proposition that does not use any of the foregoing machinery.

**Proposition 1.** Let $X$ be a weakly compactly generated Banach space that admits a $C^p$-smooth bump function for some $p \geq 1$. Let $Y \neq \emptyset$ be a subset of $X$ and let $U \neq X$ be an open set that contains $Y$ and such that $\delta = \text{dist} (Y, X - U) > 0$. Suppose that $f : U \to \mathbb{R}$ is $C^p$-smooth. Then there is a $C^p$-smooth function $F : X \to \mathbb{R}$ that agrees with $f$ on $Y$.

**Proof.** For $x$ in $X$ let $h(x) = \text{dist}(x, X - U)$. It is known [DGZ] that under the hypotheses above, the continuous function $h$ may be approximated arbitrarily closely on $X$ by a $C^p$-smooth function. As above, it follows that there is a $C^p$-smooth function $g : X \to [0, 1]$ such that $g(x) = 1$ if $\text{dist}(x, X - U) \geq \delta$ and $g(x) = 0$ if $\text{dist}(x, X - U) \leq 2\delta/3$. Extend $f$ arbitrarily to $\tilde{f} : X \to \mathbb{R}$ and define $F = \tilde{f}g$. If $x \in Y$ then $g(x) = 1$ so $f(x) = F(x)$. If $x$ is in the open set $V_1 = \{ y \in X : \text{dist}(y, X - U) > \delta/3 \}$ then $x \in U$, implying that $F$ is $C^p$-smooth on $V_1$. If $x$ is in the open set $V_2 = \{ y \in X : \text{dist}(y, X - U) < 2\delta/3 \}$ then $F(x) = 0$. But since $X = V_1 \cup V_2$, we conclude that $F$ is $C^p$-smooth on $X$. \hfill \Box

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**References**


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