A WHITNEY EXTENSION THEOREM FOR CONVEX FUNCTIONS OF CLASS $C^{1,\omega}$.

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Abstract. Let C be a compact subset of $\mathbb{R}^n$ (not necessarily convex), $f : C \to \mathbb{R}$ be a function, and $G : C \to \mathbb{R}^n$ be a continuous function, with modulus of continuity ω. We provide necessary and sufficient conditions on f, G for the existence of a convex function $F \in C^{1,\omega}(\mathbb{R}^n)$ such that $F = f$ on C and $\nabla F = G$ on C. We also give a geometrical application concerning a characterization of compact subsets K of $\mathbb{R}^n$ which can be interpolated by boundaries of $C^{1,1}$ convex bodies with prescribed outer normals on K.

1. Introduction and main results

Throughout this paper, by a modulus of continuity ω we understand a concave, strictly increasing function $\omega : [0, \infty) \to [0, \infty)$ such that $\omega(0^+) = 0$. In particular, ω has an inverse $\omega^{-1} : [0, \beta) \to [0, \infty)$ which is convex and strictly increasing, where $\beta > 0$ may be finite or infinite. Furthermore, ω is subadditive, and satisfies $\omega(\lambda t) \leq \lambda\omega(t)$ for $\lambda \geq 1$, and $\omega(\mu t) \geq \mu\omega(t)$ for $0 \leq \mu \leq 1$. It is well-known that for every uniformly continuous function $f : X \to Y$ between two metric spaces there exists a modulus of continuity ω such that $d_Y(f(x), f(z)) \leq \omega\left(d_X(x, z)\right)$ for every $x, z \in X$.

Let C be a compact subset of $\mathbb{R}^n$, $f : C \to \mathbb{R}$ a function, and $G : C \to \mathbb{R}^n$ a mapping with modulus of continuity ω. Assume that the pair (f, G) satisfies

$$|f(x) - f(y) - \langle G(y), x - y\rangle| \leq M|x - y|\,\omega\left(|x - y|\right) \qquad (W^{1,\omega})$$

for every $x, y \in C$ (in the case that $\omega(t) = t$ we will also denote this condition by $(W^{1,1})$). Then a well-known version of the Whitney extension theorem for the class $C^{1,\omega}$ holds true (see [5, 9]), and we get the existence of a function $F \in C^{1,\omega}(\mathbb{R}^n)$ such that $F = f$ and $\nabla F = G$ on C.

It is natural to ask what further assumptions on f, G would be necessary and sufficient to ensure that F can be taken to be convex. In a recent paper [1], we solved a similar problem for the class of C^1 convex functions on $\mathbb{R}^n$,

Date: July 14, 2015.

2010 *Mathematics Subject Classification.* 54C20, 52A41, 26B05, 53A99, 53C45, 52A20, 58C25, 35J96.

Key words and phrases. extension of functions, convex function, $C^{1,1}$ function.

C. Mudarra was supported by Programa Internacional de Doctorado de la Fundación La Caixa–Severo Ochoa. Both authors were partially supported by Grant MTM2012-34341.

and also, under the additional assumption that C be convex, the corresponding problem for C^∞ smooth convex functions. We refer to the introduction of [1] and the references therein for background. Here we will only mention that results of this nature have interesting applications in differential geometry, PDE theory (such as Monge-Ampère equations), nonlinear dynamics, and quantum computing, see [2, 3, 4, 12] and the references therein.

Let us introduce one condition which, we have found, together with $(W^{1,\omega})$, is necessary and sufficient for a function $f : C \to \mathbb{R}$ to have a convex extension F of class $C^{1,\omega}(\mathbb{R}^n)$ such that $\nabla F = G$ on C. For a mapping $G : C \subset \mathbb{R}^n \to \mathbb{R}^n$ we will denote

$$M(G, C) := \sup_{x,y\in C,\, x\neq y} \frac{|G(x) - G(y)|}{\omega(|x - y|)}.$$

We will say that f and G satisfy the property $(CW^{1,\omega})$ on C if either G is constant, or else there exists a constant $\eta \in (0, 1/2]$ such that

$$f(x) - f(y) - \langle G(y), x - y\rangle \geq \eta|G(x) - G(y)|\omega^{-1}\left(\frac{1}{2M}|G(x) - G(y)|\right), \tag{$CW^{1,\omega}$}$$

for all $x, y \in C$, where

$$M = M(G, C).$$

Throughout this paper we will assume that $M > 0$, or equivalently that G is nonconstant. We may of course do so, because if $M = 0$ then our problem has a trivial solution (namely, the function $x \mapsto f(x_0) + \langle G(x_0), x - x_0\rangle$ defines an affine extension of f to $\mathbb{R}^n$, for any $x_0 \in C$).

In the case that $\omega(t) = t$, we will also denote this condition by $(CW^{1,1})$. That is, (f, G) satisfies $(CW^{1,1})$ if and only if there exists $\delta > 0$ such that

$$f(x) - f(y) - \langle G(y), x - y\rangle \geq \delta\, |G(x) - G(y)|^2, \tag{$CW^{1,1}$}$$

for all $x, y \in C$.

Our main result is the following.

Theorem 1.1. *Let ω be a modulus of continuity. Let C be a compact (not necessarily convex) subset of $\mathbb{R}^n$. Let $f : C \to \mathbb{R}$ be an arbitrary function, and $G : C \to \mathbb{R}^n$ be continuous, with modulus of continuity ω. Then f has a convex, $C^{1,\omega}$ extension F to all of $\mathbb{R}^n$, with $\nabla F = G$ on C, if and only if f and G satisfy the conditions $(W^{1,\omega})$ and $(CW^{1,\omega})$ on C.*

In particular, for the most important case that $\omega(t) = t$, we have the following.

Corollary 1.2. *Let C be a compact (not necessarily convex) subset of $\mathbb{R}^n$. Let $f : C \to \mathbb{R}$ be an arbitrary function, and $G : C \to \mathbb{R}^n$ be a Lipschitz function. Then f has a convex, $C^{1,1}$ extension F to all of $\mathbb{R}^n$, with $\nabla F = G$ on C, if and only if f and G satisfy the conditions $(W^{1,1})$ and $(CW^{1,1})$ on C.*

It is worth noting that our proofs provide good control of the modulus of continuity of the gradients of the extensions F, in terms of that of G. In fact, assuming $\eta = 1/2$ in $(CW^{1,\omega})$, which we can fairly do (see Proposition 2.2 below), there exists a constant $k(n) > 0$, depending only on the dimension n, such that

$$M(\nabla F, \mathbb{R}^n) \leq k(n)\, M(G, C).$$

Since convex functions on $\mathbb{R}^n$ are not bounded (unless they are constant), the most usual definitions of norms in the space $C^{1,\omega}(\mathbb{R}^n)$ are not adequate to estimate convex functions whose gradients are uniformly continuous with modulus of uniform continuity ω. In this paper, for a convex function $F : \mathbb{R}^n \to \mathbb{R}$ we will denote

$$\|F\|_{1,\omega} = |F(0)| + |\nabla F(0)| + \sup_{x,y\in\mathbb{R}^n,\, x\neq y} \frac{|\nabla F(x) - \nabla F(y)|}{\omega(|x-y|)}. \tag{1.1}$$

With this notation, and assuming $0 \in C$ and $\eta = 1/2$ in $(CW^{1,\omega})$, the proof of Theorem 1.1 shows that

$$\|F\|_{1,\omega} \leq k(n)\left(|f(0)| + |G(0)| + M(G, C)\right).$$

Let us conclude this introduction with a geometrical application of Corollary 1.2 concerning a characterization of compact subsets K of $\mathbb{R}^n$ which can be interpolated by boundaries of $C^{1,1}$ convex bodies (with prescribed unit outer normals on K). Namely, if K is a compact subset of $\mathbb{R}^n$ and we are given a Lipschitz map $N : K \to \mathbb{R}^n$ such that $|N(y)| = 1$ for every $y \in K$, it is natural to ask what conditions on K and N are necessary and sufficient for K to be a subset of the boundary of a $C^{1,1}$ convex body V such that $0 \in \mathrm{int}(V)$ and $N(y)$ is outwardly normal to ∂V at y for every $y \in K$. A suitable set of conditions is:

$$(\mathcal{O}) \qquad \langle N(y), y\rangle > 0 \text{ for all } y \in K;$$

$$(\mathcal{KW}^{1,1}) \qquad \langle N(y), y-x\rangle \geq \frac{\eta}{M}|N(y) - N(x)|^2 \text{ for all } x, y \in K;$$

$$(\mathcal{W}^{1,1}) \qquad \langle N(y), y-x\rangle \leq M|x-y|^2, \text{ for all } x, y \in K,$$

for some $M > 0$, $\eta \in (0, \frac{1}{2}]$.

Our main result in this direction is as follows.

Theorem 1.3. *Let K be a compact subset of $\mathbb{R}^n$, and let $N : K \to \mathbb{R}^n$ be a Lipschitz mapping such that $|N(y)| = 1$ for every $y \in K$. Then the following statements are equivalent:*

1. *There exists a $C^{1,1}$ convex body V with $0 \in \mathrm{int}(V)$ and such that $K \subseteq \partial V$ and $N(y)$ is outwardly normal to ∂V at y for every $y \in K$.*
2. *K and N satisfy conditions $(\mathcal{O})$, $(\mathcal{KW}^{1,1})$ and $(\mathcal{W}^{1,1})$.*

This result may be compared to [2], where M. Ghomi showed how to construct C^m smooth convex bodies V with prescribed strongly convex submanifolds and tangent planes. In [1, Theorem 1.11] we obtained a result in this spirit which allowed us to deal with arbitrary compacta instead of

manifolds, and to drop the strong convexity assumption, in the special case $m = 1$. The above Theorem provides a similar result for the class of $C^{1,1}$ bodies.

Let us finally note that in the above results the assumption that C and K be compact cannot be removed in general, see [8, Example 4], [10, Example 3.2], and [1, Example 4.1].

2. Necessity

It is clear that condition $(W^{1,\omega})$ is necessary in Theorem 1.1. Let us prove the necessity of condition $(CW^{1,\omega})$. We will use the following.

Lemma 2.1. *Let $\omega : [0,\infty) \to [0,\infty)$ be a modulus of continuity, let a, b, η be real numbers with $a > 0$, $\eta \in (0, \frac{1}{2}]$, and define $h : [0,\infty) \to \mathbb{R}$ by*

$$h(s) = -as + b + \omega(s)s.$$

Assume that $b < \eta a\, \omega^{-1}(\eta a)$. Then we have

$$h\left(\omega^{-1}(\eta a)\right) < 0.$$

Proof. We can write

$$h\left(\omega^{-1}(\eta a)\right) = -a(1-2\eta)\omega^{-1}(\eta a) + b - \eta a\, \omega^{-1}(\eta a),$$

and the result follows at once. □

Proposition 2.2. *Let $f \in C^{1,\omega}(\mathbb{R}^n)$ be convex. Then*

$$f(x)-f(y)-\langle \nabla f(y), x-y\rangle \geq \frac{1}{2}|\nabla f(x)-\nabla f(y)|\, \omega^{-1}\left(\frac{1}{2M}|\nabla f(x)-\nabla f(y)|\right)$$

for all $x, y \in \mathbb{R}^n$, where

$$M = M(\nabla f, \mathbb{R}^n) = \sup_{x,y\in\mathbb{R}^n,\, x\neq y} \frac{|\nabla f(x)-\nabla f(y)|}{\omega\left(|x-y|\right)}.$$

In particular, the pair $(f, \nabla f)$ satisfies $(CW^{1,\omega})$, with $\eta = 1/2$, on every compact subset $C \subset \mathbb{R}^n$.

Proof. Suppose that there exist different points $x, y \in \mathbb{R}^n$ such that

$$f(x)-f(y)-\langle \nabla f(y), x-y\rangle < \frac{1}{2}|\nabla f(x)-\nabla f(y)|\, \omega^{-1}\left(\frac{1}{2M}|\nabla f(x)-\nabla f(y)|\right),$$

and we will get a contradiction.

Case 1. Assume further that $M = 1$, $f(y) = 0$, and $\nabla f(y) = 0$. By convexity this implies $f(x) \geq 0$. Then we have

$$0 \leq f(x) < \frac{1}{2}|\nabla f(x)|\, \omega^{-1}\left(\frac{1}{2}|\nabla f(x)|\right).$$

Call $a = |\nabla f(x)| > 0$, $b = f(x)$, set

$$v = -\frac{1}{|\nabla f(x)|}\nabla f(x),$$

and define

$$\varphi(t) = f(x + tv)$$

for every $t \in \mathbb{R}$. We have $\varphi(0) = b$, $\varphi'(0) = -a$, and ω is a modulus of continuity of the derivative φ'. This implies that

$$|\varphi(t) - b + at| \leq t\omega(t)$$

for every $t \in \mathbb{R}^+$, hence also that

$$\varphi(t) \leq h(t) \text{ for all } t \in \mathbb{R}^+,$$

where $h(t) = -at + b + t\omega(t)$. By assumption,

$$b < \frac{1}{2}a\omega^{-1}\left(\frac{1}{2}a\right),$$

and then Lemma 2.1 implies that

$$f\left(x + \omega^{-1}\left(\frac{1}{2}a\right)v\right) = \varphi\left(\omega^{-1}\left(\frac{1}{2}a\right)\right) \leq h\left(\omega^{-1}\left(\frac{1}{2}a\right)\right) < 0,$$

which is in contradiction with the assumptions that f is convex, $f(y) = 0$, and $\nabla f(y) = 0$. This shows that

$$f(x) \geq \frac{1}{2}|\nabla f(x)|\,\omega^{-1}\left(\frac{1}{2}|\nabla f(x)|\right).$$

Case 2. Assume only that $M = 1$. Define

$$g(z) = f(z) - f(y) - \langle \nabla f(y), z - y\rangle$$

for every $z \in \mathbb{R}^n$. Then $g(y) = 0$ and $\nabla g(y) = 0$. By Case 1, we get

$$g(x) \geq \frac{1}{2}|\nabla g(x)|\,\omega^{-1}\left(\frac{1}{2}|\nabla g(x)|\right),$$

and since $\nabla g(x) = \nabla f(x) - \nabla f(y)$ the Proposition is thus proved in the case when $M = 1$.

Case 3. In the general case, we may assume $M > 0$ (the result is trivial for $M = 0$). Consider $\psi = \frac{1}{M}f$, which satisfies the assumption of Case 2. Therefore

$$\psi(x) - \psi(y) - \langle \nabla\psi(y), x - y\rangle \geq \frac{1}{2}|\nabla\psi(x) - \nabla\psi(y)|\,\omega^{-1}\left(\frac{1}{2}|\nabla\psi(x) - \nabla\psi(y)|\right),$$

which is equivalent to the desired inequality. □

3. Sufficiency

Throughout this section C will denote a compact subset of $\mathbb{R}^n$.

Lemma 3.1. *Assume that $f : C \to \mathbb{R}$ and $G : C \to \mathbb{R}^n$ satisfy the conditions $(W^{1,\omega})$ and $(CW^{1,\omega})$ on C. We have the following inequalities:*

(i) *For all $x, y \in C$,*

$$0 \leq f(x) - f(y) - \langle G(y), x - y\rangle \leq \langle G(y) - G(x), y - x\rangle.$$

(ii) *If* $h \in \mathbb{R}^n$, $x_0, y \in C$ *are such that*

$$m_C(f)(x_0+h) = f(y) + \langle G(y), x_0 + h - y\rangle,$$

then

$$f(x_0) - f(y) - \langle G(y), x_0 - y\rangle \leq \langle G(y) - G(x_0), h\rangle.$$

(iii) *For* $h \in \mathbb{R}^n$, $x_0, y \in C$ *such that*

$$m_C(f)(x_0+h) = f(y) + \langle G(y), x_0 + h - y\rangle,$$

we have that

$$|G(x_0) - G(y)| \leq \frac{2M}{\eta}\omega\left(|h|\right),$$

where we denote

$$M = M(G, C) = \sup_{x,y\in C,\, x\neq y} \frac{|G(x) - G(y)|}{\omega\left(|x-y|\right)}. \tag{3.1}$$

Proof.

(i) It is enough to use that $f(y) \geq f(x) + \langle G(x), y - x\rangle$ (which in turn follows from $(CW^{1,\omega})$).

(ii) By the definition of m_C, we have that

$$f(x_0) + \langle G(x_0), h\rangle \leq f(y) + \langle G(y), x_0 + h - y\rangle.$$

Then, it is easy to deduce that

$$f(x_0) - f(y) - \langle G(y), x_0 - y\rangle \leq \langle G(y) - G(x_0), h\rangle.$$

(iii) By (ii) we have that

$$f(x_0) - f(y) - \langle G(y), x_0 - y\rangle \leq |G(y) - G(x_0)||h|,$$

and from condition $(CW^{1,\omega})$ we immediately deduce that

$$|G(x_0) - G(y)| \leq \frac{2M}{\eta}\omega\left(|h|\right).$$

□

In the sequel we assume that $f : C \to \mathbb{R}$ and $G : C \to \mathbb{R}^n$ satisfy the conditions $(W^{1,\omega})$ and $(CW^{1,\omega})$ on C. By the $C^{1,\omega}$ version of Whitney's Extension Theorem (see [5, 9]), we may further assume that f is extended to a function of class $C^{1,\omega}(\mathbb{R}^n)$ with support contained in $C + \overline{B}(0,1)$, and such that $\nabla f = G$ on C. Moreover, bearing in mind Lemma 3.1(i), it follows from the construction in [9, Chapter VI] that there is a constant $c(n) > 0$, depending only on the dimension n, such that

$$M(\nabla f, \mathbb{R}^n) = \sup_{x,y\in\mathbb{R}^n,\, x\neq y} \frac{|\nabla f(x) - \nabla f(y)|}{\omega\left(|x-y|\right)} \leq c(n)M, \tag{3.2}$$

where M is defined in (3.1). Since the case $M = 0$ is trivial, we also assume $M > 0$. We will denote

$$m_C(f)(x) = \sup_{y\in C}\{f(y) + \langle \nabla f(y), x - y\rangle\}$$

for every $x \in \mathbb{R}^n$. The function $m_C(f) : \mathbb{R}^n \to \mathbb{R}$ is obviously convex and Lipschitz. In the case when C has nonempty interior, $m_C(f)$ is the minimal convex extension of $f_{|_C} : C \to \mathbb{R}$ to all of $\mathbb{R}^n$ (however, if C has empty interior then there is no minimal convex extension operator). See [8, 1] for more information on $m_C(f)$.

Lemma 3.2. *Under the above assumptions, we have:*

$$|f(x) - m_C(f)(x)| \leq \left(c(n) + \frac{2}{\eta}\right) M\omega(d(x,C))d(x,C), \quad \textit{for all} \quad x \in \mathbb{R}^n.$$

Proof. Let $x \in \mathbb{R}^n$, $x_0 \in C$ be such that $|x - x_0| = d(x,C)$, define $h := x - x_0$ (that is, $x = x_0 + h$), and take a point $y \in C$ such that $m_C(f)(x) = f(y) + \langle \nabla f(y), x - y\rangle$. We have

$$\begin{aligned}&|f(x) - m_C(f)(x)| \leq \\ &\qquad |f(x) - f(x_0) - \langle \nabla f(x_0), x - x_0\rangle| + |m_C(f)(x) - f(x_0) - \langle \nabla f(x_0), x - x_0\rangle|.\end{aligned}$$

Using inequality (3.2), the first term satisfies

$$|f(x) - f(x_0) - \langle \nabla f(x_0), h\rangle| \leq M(\nabla f, \mathbb{R}^n)\omega(|h|)|h| \leq c(n)M\omega(d(x,C))d(x,C).$$

The second term can be estimated as follows:

$$\begin{aligned}0 &\leq m_C(f)(x) - f(x_0) - \langle \nabla f(x_0), x - x_0\rangle \\ &= f(y) + \langle \nabla f(y), x - y\rangle - f(x_0) - \langle \nabla f(x_0), x - x_0\rangle \\ &\leq f(y) + \langle \nabla f(y), x - y\rangle - f(y) - \langle \nabla f(y), x_0 - y\rangle - \langle \nabla f(x_0), x - x_0\rangle \\ &= \langle \nabla f(y) - \nabla f(x_0), x - x_0\rangle \leq |\nabla f(y) - \nabla f(x_0)||x - x_0|\end{aligned}$$

By Lemma 3.1 (iii) we have

$$|\nabla f(y) - \nabla f(x_0)| \leq \frac{2M}{\eta}\omega(|h|) = \frac{2M}{\eta}\omega(d(x,C)).$$

This implies that

$$0 \leq m_C(f)(x) - f(x_0) - \langle \nabla f(x_0), x - x_0\rangle \leq \frac{2M}{\eta}\omega(d(x,C))d(x,C).$$

In particular, $m_C(f)$ is ω-differentiable on points $x \in C$, with $\nabla m_C(f)(x) = \nabla f(x)$. [1] Therefore

$$|f(x) - m_C(f)(x)| \leq \left(c(n) + \frac{2}{\eta}\right) M\omega(d(x,C))d(x,C).$$

□

[1]Recall that a function ψ is said to be ω-differentiable on C provided there exists a constant $K > 0$ such that $|\psi(x) - \psi(y) - \langle \nabla\psi(y), x - y\rangle| \leq K|x - y|\,\omega\left(|x - y|\right)$ for every $y \in C$ and every x in a neighbourhood of C.

Our next step is to obtain a function φ of class $C^{1,\omega}(\mathbb{R}^n)$ which vanishes on C and is greater than or equal to the function $|f - m_C(f)|$ on $\mathbb{R}^n \setminus C$. To this purpose we need to use a Whitney descomposition of an open set into cubes and the corresponding partition of unity, see [9, Chapter VI] for an exposition of this technique. So let $\{Q_k\}_k$ be a decomposition of $\mathbb{R}^n \setminus C$ into Whitney cubes, and for a fixed number $0 < \varepsilon_0 < 1/4$, consider the corresponding cubes $\{Q_k^*\}_k$ with the same center as Q_k and dilated by the factor $1+\varepsilon_0$. We next sum up some of the most important properties of this cubes.

Proposition 3.3. *The families $\{Q_k\}_k$ and $\{Q_k^*\}_k$ are sequences of compact cubes for which:*
(i) $\bigcup_k Q_k = \mathbb{R}^n \setminus C$.
(ii) *The interiors of Q_k are mutually disjoint.*
(iii) $\operatorname{diam}(Q_k) \leq d(Q_k, F) \leq 4 \operatorname{diam}(Q_k)$ *for all k.*
(iv) *If two cubes Q_l and Q_k touch each other, that is $\partial Q_l \cap \partial Q_k \neq \emptyset$, then* $\operatorname{diam}(Q_l) \approx \operatorname{diam}(Q_k)$.
(v) *Every point of $\mathbb{R}^n \setminus C$ is contained in an open neighbourhood which intersects at most $N = (12)^n$ cubes of the family $\{Q_k^*\}_k$.*
(vi) *If $x \in Q_k$, then $d(x, C) \leq 5 \operatorname{diam}(Q_k)$.*
(vii) *If $x \in Q_k^*$, then $\frac{3}{4} \operatorname{diam}(Q_k) \leq d(x, C) \leq (6 + \varepsilon_0) \operatorname{diam}(Q_k)$ and in particular $Q_k^* \subset \mathbb{R}^n \setminus C$.*
(viii) *If two cubes Q_l^* and Q_k^* are not disjoint, then* $\operatorname{diam}(Q_l) \approx \operatorname{diam}(Q_k)$.

Here the notation $A_k \approx B_l$ means that there exist positive constants γ, Γ, depending only on the dimension n, such that $\gamma A_k \leq B_l \leq \Gamma A_k$ for all k, l satisfying the properties specified in each case.

The following Proposition summarizes the basic properties of the Whitney partition of unity associated to these cubes.

Proposition 3.4. *There exists a sequence of functions $\{\varphi_k\}_k$ defined on $\mathbb{R}^n \setminus C$ such that*

(i) $\varphi_k \in C^\infty(\mathbb{R}^n \setminus C)$.
(ii) $0 \leq \varphi_k \leq 1$ *on* $\mathbb{R}^n \setminus C$ *and* $\operatorname{supp}(\varphi_k) \subseteq Q_k^*$.
(iii) $\sum_k \varphi_k = 1$ *on* $\mathbb{R}^n \setminus C$.
(vi) *For all multiindex α there exists a constant $A_\alpha > 0$, depending only on α and on the dimension n, such that*

$$|D^\alpha \varphi_k(x)| \leq A_\alpha \operatorname{diam}(Q_k)^{-|\alpha|},$$

for all $x \in \mathbb{R}^n \setminus C$ and for all k.

The statements contained in the following Proposition must look fairly obvious to those readers well acquainted with Whitney's techniques, but we have not been able to find an explicit reference, so we include a proof for the general reader's convenience.

Proposition 3.5. *Consider the family of cubes $\{Q_k\}_k$ asociated to $\mathbb{R}^n \setminus C$ and its partition of unity $\{\varphi_k\}_k$ as in the preceding Propositions.*

(i) *Suppose that there is a sequence of nonnegative numbers* $\{p_k\}_k$ *and a positive constant* $\lambda > 0$ *such that* $p_k \leq \lambda\, \omega(\mathrm{diam}(Q_k))\, \mathrm{diam}(Q_k)$, *for every* k. *Then, the function defined by*

$$\varphi(x) = \begin{cases} \sum_k p_k\, \varphi_k(x) & \text{if} \quad x \in \mathbb{R}^n \setminus C, \\ 0 & \text{if} \quad x \in C \end{cases}$$

is of class $C^{1,\omega}(\mathbb{R}^n)$ *and there is a constant* $\gamma(n) > 0$, *depending only on the dimension* n, *such that*

$$M(\nabla\varphi, \mathbb{R}^n) := \sup_{x,y\in\mathbb{R}^n,\, x\neq y} \frac{|\nabla\varphi(x) - \nabla\varphi(y)|}{\omega\left(|x-y|\right)} \leq \gamma(n)\lambda. \tag{3.3}$$

(ii) *There exists a function* $\Delta \in C^{1,\omega}(\mathbb{R}^n)$ *such that* $\Delta = 0$ *on* C *and*

$$\frac{\omega(d(x,C))d(x,C)}{6+\varepsilon_0} \leq \Delta(x) \leq \frac{(6+\varepsilon_0)16}{9}\omega(d(x,C))d(x,C), \quad \text{for all} \quad x \in \mathbb{R}^n.$$

In addition, we have:

$$M(\nabla\Delta, \mathbb{R}^n) = \sup_{x,y\in\mathbb{R}^n,\, x\neq y} \frac{|\nabla\Delta(x) - \nabla\Delta(y)|}{\omega\left(|x-y|\right)} \leq (6+\varepsilon_0)\gamma(n).$$

Proof. Let us start with the proof of (i). Since every point in $\mathbb{R}^n \setminus C$ has an open neighbourhood which intersects at most $N = (12)^n$ cubes, and all the functions φ_k are of class C^∞, it is clear that $\varphi \in C^\infty(\mathbb{R}^n \setminus C)$. Given a point $x \in \mathbb{R}^n$, by our assumptions on the sequence $\{p_k\}_k$, we easily have

$$\varphi(x) = \sum_{Q_k^* \ni x} p_k \varphi_k(x) \leq \lambda \sum_{Q_k^* \ni x} \omega(\mathrm{diam}(Q_k))\, \mathrm{diam}(Q_k)\varphi_k(x).$$

Using Proposition 3.3 we have that $\mathrm{diam}(Q_k) \leq \frac{4}{3}d(x,C)$ for those k such that $x \in Q_k^*$. Then, we can write

$$\varphi(x) \leq \lambda \sum_{Q_k^* \ni x} \omega\left(\frac{4}{3}d(x,C)\right)\frac{4}{3}d(x,C)\varphi_k(x) \leq$$
$$\lambda\left(\frac{4}{3}\right)^2 \omega(d(x,C))d(x,C) \sum_{Q_k^* \ni x} \varphi_k(x) = \left(\frac{4}{3}\right)^2 \lambda\omega(d(x,C))d(x,C).$$

In particular, due to the fact that $\omega(0^+) = 0$, the above estimation shows that φ is differentiable on C with $\nabla\varphi = 0$ on C. We next give an estimation for $\nabla\varphi$. Bearing in mind Proposition 3.4, we set

$$A = A(n) := \max\{\sqrt{n}A_\alpha,\ |\alpha| \leq 2\}. \tag{3.4}$$

Given a point $x \in \mathbb{R}^n$, we have that

$$|\nabla\varphi(x)| \leq \sum_{Q_k^* \ni x} p_k |\nabla\varphi_k(x)| \leq A \sum_{Q_k^* \ni x} p_k\, \mathrm{diam}(Q_k)^{-1} \leq A\lambda \sum_{Q_k^* \ni x} \omega(\mathrm{diam}(Q_k))$$
$$\leq A\,\lambda \sum_{Q_k^* \ni x} \omega\left(\frac{4}{3}d(x,C)\right) \leq \frac{4AN}{3}\lambda\,\omega(d(x,C)).$$

Summing up,

$$|\nabla\varphi(x)| \leq \frac{4AN}{3}\lambda\,\omega(d(x,C)) \quad \text{for every} \quad x\in\mathbb{R}^n. \tag{3.5}$$

Note that the above shows inequality (3.3) when $x\in\mathbb{R}^n$ and $y\in C$. Hence, in the rest of the proof we only have consider the situation where x,y are such that $x\neq y$ and $x,y\in\mathbb{R}^n\setminus C$. However, we will still have to separately consider two cases. Let us denote $L:=[x,y]$, the straight line connecting x to y.

Case 1: $d(L,C)\geq|x-y|$. Take a multiindex α with $|\alpha|=1$. Because in this case the segment L is necessarily contained in $\mathbb{R}^n\setminus C$, and the function φ is of class C^2 on $\mathbb{R}^n\setminus C$, we can write

$$|D^\alpha\varphi(x)-D^\alpha\varphi(y)| \leq \left(\sup_{z\in L}|\nabla(D^\alpha\varphi)(z)|\right)|x-y|. \tag{3.6}$$

Using Proposition 3.4 and the definition of A in (3.4), we may write

$$|\nabla(D^\alpha\varphi)(z)| \leq \sum_{Q_k^*\ni z} p_k|\nabla(D^\alpha\varphi_k)(z)| \leq A\sum_{Q_k^*\ni z} p_k\,\mathrm{diam}(Q_k)^{-2} \leq A\lambda\sum_{Q_k^*\ni z}\frac{\omega(\mathrm{diam}(Q_k))}{\mathrm{diam}(Q_k)}.$$

By Proposition 3.3, we have that $(6+\varepsilon_0)\,\mathrm{diam}(Q_k)\geq d(z,C)\geq d(L,C)\geq|x-y|$ for those k with $Q_k^*\ni z$ and by the properties of ω, we have that

$$A\lambda\sum_{Q_k^*\ni z}\frac{\omega(\mathrm{diam}(Q_k))}{\mathrm{diam}(Q_k)} \leq A\lambda\sum_{Q_k^*\ni z}\frac{\omega\left(\frac{|x-y|}{6+\varepsilon_0}\right)}{\frac{|x-y|}{6+\varepsilon_0}} \leq (6+\varepsilon_0)AN\lambda\frac{\omega(|x-y|)}{|x-y|}.$$

Therefore, by substituting in (3.6) we find that

$$|\nabla\varphi(x)-\nabla\varphi(y)| \leq (6+\varepsilon_0)\sqrt{n}AN\lambda\omega(|x-y|).$$

Case 2: $d(L,C)\leq|x-y|$. Take points $x'\in L$ and $y'\in C$ such that $d(L,C)=|x'-y'|\leq|x-y|$. We have that

$$|x-y'|\leq|x-x'|+|y'-x'|\leq|x-y|+|x'-y'|\leq 2|x-y|,$$

and similarly we obtain $|y-y'|\leq 2|x-y|$. Hence, if we use (3.5) we obtain

$$|\nabla\varphi(x)-\nabla\varphi(y)| \leq |\nabla\varphi(x)|+|\nabla\varphi(y)| \leq \frac{4AN}{3}\lambda\left(\omega(d(x,C))+\omega(d(y,C))\right)$$
$$\leq \frac{4AN}{3}\lambda\left(\omega(|x-y'|)+\omega(|y-y'|)\right) \leq \frac{8AN}{3}\lambda\omega(2|x-y|) \leq \frac{16AN}{3}\lambda\omega(|x-y|).$$

If we call

$$\gamma(n)=\max\left\{\frac{4AN}{3},(6+\varepsilon_0)\sqrt{n}AN,\frac{16AN}{3}\right\}=(6+\varepsilon_0)\sqrt{n}AN,$$

we get (3.3).
Now let us prove the part (ii). Define the sequence

$$p_k = \omega((6+\varepsilon_0)\operatorname{diam}(Q_k))\operatorname{diam}(Q_k), \quad \text{for all} \quad k.$$

We have that $p_k \le (6+\varepsilon_0)\omega(\operatorname{diam}(Q_k))\operatorname{diam}(Q_k)$ and, by (i), the function defined by

$$\Delta(x) = \begin{cases} \sum_k p_k\,\varphi_k(x) & \text{if} \quad x \in \mathbb{R}^n \setminus C, \\ 0 & \text{if} \quad x \in C \end{cases}$$

is of class $C^{1,\omega}(\mathbb{R}^n)$, vanishes on C, and satisfies $M(\nabla\Delta, \mathbb{R}^n) \le (6+\varepsilon_0)\gamma(n)$ (that is, the second part of (ii)). Given a point $x \in \mathbb{R}^n \setminus C$, we know that $(6+\varepsilon_0)\operatorname{diam}(Q_k) \ge d(x,C)$ if $x \in Q_k^*$. Then

$$\Delta(x) = \sum_{Q_k^* \ni x} p_k\,\varphi_k(x) \ge \frac{\omega(d(x,C))d(x,C)}{6+\varepsilon_0} \sum_{Q_k^* \ni x} \varphi_k(x) = \frac{\omega(d(x,C))d(x,C)}{6+\varepsilon_0}.$$

On the other hand, from Proposition 3.3, we have that $\operatorname{diam}(Q_k) \le \frac{4}{3}d(x,C)$ if $x \in Q_k^*$. This yields

$$\begin{aligned} \Delta(x) = \sum_{Q_k^* \ni x} p_k\,\varphi_k(x) &\le (6+\varepsilon_0)\omega\left(\frac{4}{3}d(x,C)\right)\left(\frac{4}{3}d(x,C)\right) \\ &\le \frac{(6+\varepsilon_0)16}{9}\omega(d(x,C))d(x,C), \end{aligned}$$

hence the first inequality in (ii) is proved. □

Let us continue with our proof. Consider the numbers

$$p_k := \sup_{x \in Q_k^*} \{|f(x) - m_C(f)(x)|\}, \quad \text{for every} \quad k.$$

By Lemma 3.2, $|f(x) - m_C(f)(x)| \le M^*\omega(d(x,C))d(x,C)$, where $M^* = (c(n) + 2\eta^{-1})M$, and if k is such that $x \in Q_k^*$, the last term is smaller than or equal to

$$M^*\omega((6+\varepsilon_0)\operatorname{diam}(Q_k))((6+\varepsilon_0)\operatorname{diam}(Q_k)) \le (6+\varepsilon_0)^2 M^*\omega(\operatorname{diam}(Q_k))\operatorname{diam}(Q_k).$$

This shows that

$$p_k \le (6+\varepsilon_0)^2 M^*\omega(\operatorname{diam}(Q_k))\operatorname{diam}(Q_k) \quad \text{for every} \quad k,$$

and, by Theorem 3.5, the function defined by

$$\varphi(x) = \begin{cases} \sum_k p_k\,\varphi_k(x) & \text{if} \quad x \in \mathbb{R}^n \setminus C, \\ 0 & \text{if} \quad x \in C \end{cases}$$

is of class $C^{1,\omega}(\mathbb{R}^n)$ and

$$M(\nabla\varphi, \mathbb{R}^n) \le \gamma(n)(6+\varepsilon_0)^2 M^* = \gamma(n)(6+\varepsilon_0)^2(c(n) + 2\eta^{-1})M.$$

We claim that $\varphi \ge |f - m_C(f)|$. Indeed, if $x \in C$, there is nothing to say. If $x \in \mathbb{R}^n \setminus C$, by the definition of the p_k's we easily have

$$\varphi(x) = \sum_{Q_k^* \ni x} p_k\,\varphi_k(x) \ge \sum_{Q_k^* \ni x} |f(x) - m_C(f)(x)|\varphi_k(x) = |f(x) - m_C(f)(x)|.$$

Now, making use of Proposition 3.5(ii), we obtain a function $\Delta \in C^{1,\omega}(\mathbb{R}^n)$ such that Δ vanishes on C,

$$\Delta(x) \geq \frac{\omega(d(x,C))d(x,C)}{6+\varepsilon_0} \quad \text{for all} \quad x \in \mathbb{R}^n,$$

and $M(\nabla\Delta, \mathbb{R}^n) \leq (6+\varepsilon_0)\gamma(n)$. Let us consider the function $g = f+\varphi+M\Delta$. Our function g is clearly of class $C^{1,\omega}(\mathbb{R}^n)$ and satisfies $g = f$, $\nabla g = \nabla f$ on C. Using that $\varphi \geq |f - m_C(f)|$, we easily obtain that

$$g(x) \geq f(x) + |m_C(f)(x) - f(x)| + M\Delta(x) \geq m_C(f)(x),$$

that is, $g \geq m_C(f)$ on $\mathbb{R}^n$. Since f has compact support and $\varphi \geq 0$, we also have that

$$g(x) \geq -\|f\|_\infty + M\Delta(x) \geq -\|f\|_\infty + M\frac{\omega(d(x,C))d(x,C)}{6+\varepsilon_0},$$

which tends to $+\infty$ as $|x|$ goes to $+\infty$. Hence $\lim_{|x|\to+\infty} g(x) = +\infty$. We can also estimate $M(\nabla g, \mathbb{R}^n)$ as follows,

$$\begin{aligned} M(\nabla g, \mathbb{R}^n) &\leq M(\nabla f, \mathbb{R}^n) + M(\nabla\varphi, \mathbb{R}^n) + M \cdot M(\nabla\Delta, \mathbb{R}^n) \\ &\leq c(n)M + \gamma(n)(6+\varepsilon_0)^2(c(n)+2\eta^{-1})M + (6+\varepsilon_0)\gamma(n)M \\ &= \mu(n,\eta)M, \end{aligned}$$

where

$$\mu(n,\eta) := c(n) + \gamma(n)(6+\varepsilon_0)^2(c(n)+2\eta^{-1}) + (6+\varepsilon_0)\gamma(n). \tag{3.7}$$

Finally, let us define $F := \mathrm{conv}(g)$, the convex envelope of the function g. Namely,

$$\mathrm{conv}(g)(x) = \sup\{h(x) \,:\, h \text{ is convex }, h \leq g\}, \quad x \in \mathbb{R}^n.$$

In order to finish our proof we need to use the following result of Kirchheim and Kristensen on differentiability of convex envelopes (see [6] for the proof).

Theorem 3.6 (Kirchheim-Kristensen)**.** *If $\psi : \mathbb{R}^n \to \mathbb{R}$ is an ω-differentiable function on $\mathbb{R}^n$ such that $\lim_{|x|\to+\infty} \psi(x) = +\infty$, then conv$(\psi)$ is a convex function on class $C^{1,\omega}(\mathbb{R}^n)$, and*

$$M(\nabla\, conv(\psi), \mathbb{R}^n) \leq 4(n+1)M(\nabla\psi, \mathbb{R}^n).$$

As a matter of fact, Theorem 3.6 is a restatement of a particular case of Kirchheim and Kristensen's result in [6], and the estimation on the modulus of continuity of the gradient of the convex envelope does not appear in their original statement, but it does follow, with some extra work, from their proof.

According to Theorem 3.6, our function F is a convex function of class $C^{1,\omega}(\mathbb{R}^n)$. It only remains to see that $F = f$ on C and $\nabla F = \nabla f$ on C. Using the fact that $g \geq m_C(f)$ and $m_C(f)$ is convex, we must have $F \geq m_C(f)$. On the other hand, because $g = f$ on C, for every $x \in C$ we have $F(x) \leq g(x) = f(x) = m_C(f)(x)$. Hence $F = m_C(f) = f$ on C. In order to show that $\nabla F(y) = \nabla f(y)$ for every $y \in C$, we use the following

well-known criterion for differentiability of convex functions, whose proof is straightforward and can be left to the interested reader.

Lemma 3.7. *If ϕ is convex, ψ is differentiable at y, $\phi \leq \psi$, and $\phi(y) = \psi(y)$, then ϕ is differentiable at y, with $\nabla\phi(y) = \nabla\psi(y)$.*

Since we know that $m_C(f) \leq F$, $m_C(f)(y) = f(y) = F(y)$ for all $y \in C$, and $F \in C^1(\mathbb{R}^n)$, it follows from this criterion that

$$\nabla f(y) = \nabla m_C(f)(y) = \nabla F(y) \text{ for all } y \in C.$$

As for the control of the modulus of continuity of the extension, since $M(\nabla g, \mathbb{R}^n) \leq \mu(n,\eta)M$ (see the inequalities before (3.7)), Theorem 3.6 gives us

$$M(\nabla F, \mathbb{R}^n) \leq 4(n+1)\mu(n,\eta)M.$$

Assuming $\eta = 1/2$, let us denote $k(n) = 4(n+1)\mu(n, 1/2)$. We have proved that

$$M(\nabla F, \mathbb{R}^n) \leq k(n)M(G, C),$$

where $k(n) > 0$ only depends on the dimension n. In terms of the norm defined in (1.1), assuming $0 \in C$ and $\eta = 1/2$, we have that $F(0) = f(0)$ and $\nabla F(0) = G(0)$, and therefore

$$\|F\|_{C^{1,\omega}} = |F(0)| + |\nabla F(0)| + M(\nabla F, \mathbb{R}^n) \leq k(n)\left(|f(0)| + |\nabla f(0)| + M(G, C)\right).$$

4. Proof of Theorem 1.2.

(2) $\implies$ (1): Set $C = K \cup \{0\}$. We may assume $\eta > 0$ small enough, and choose $\alpha > 0$ sufficiently close to 1, so that

$$0 < 1 - \alpha + \frac{\eta}{M} < \min_{y \in K} \langle N(y), y \rangle$$

(this is possible thanks to condition $(\mathcal{O})$, Lipschitzness of N and compactness of K; notice in particular that $0 \notin K$). Now define $f : C \to \mathbb{R}$ and $G : C \to \mathbb{R}^n$ by

$$f(y) = \begin{cases} 1 & \text{if } y \in K \\ \alpha & \text{if } y = 0, \end{cases} \quad \text{and } G(y) = \begin{cases} N(y) & \text{if } y \in K \\ 0 & \text{if } y = 0. \end{cases}$$

By using conditions $(\mathcal{KW}^{1,1})$ and $(\mathcal{W}^{1,1})$, it is straightforward to check that f and G satisfy conditions $(CW^{1,1})$ and $(W^{1,1})$. Therefore, according to Theorem 1.2, there exists a convex function $F \in C^{1,1}(\mathbb{R}^n)$ such that $F = f$ and $\nabla F = G$ on C. Moreover, the proof of Theorem 1.2 indicates that F can be taken so as to satisfy $\lim_{|x|\to\infty} F(x) = \infty$. If we define $V = \{x \in \mathbb{R}^n : F(x) \leq 1\}$ we then have that V is a compact convex body of class $C^{1,1}$ with $0 \in \operatorname{int}(V)$ (because $F(0) = \alpha < 1$), and $\nabla F(y) = N(y)$ is outwardly normal to $\{x \in \mathbb{R}^n : F(x) = 1\} = \partial V$ at each $y \in K$. Moreover, $F(y) = f(y) = 1$ for each $y \in K$, hence $K \subseteq \partial V$.

(1) $\implies$ (2): Let μ_V be the Minkowski functional of V. By composing μ_V with a C^∞ convex function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(t) = |t|$ if and only if

$|t| \geq 1/2$, we obtain a function $F(x) := \phi(\mu_V(x))$ which is of class $C^{1,1}$ and convex on $\mathbb{R}^n$ and coincides with μ_V on a neighborhood of ∂V. By Theorem 1.2 we then have that F satisfies conditions $(CW^{1,1})$ and $(W^{1,1})$ on ∂V. On the other hand $\nabla\mu_V(y)$ is outwardly normal to ∂V at every $y \in \partial V$ and $\nabla F = \nabla\mu_V$ on ∂V, hence we have $\nabla F(y) = N(y)$ for every $y \in \partial V$. These facts, together with the assumption $K \subseteq \partial V$, are easily checked to imply that N and K satisfy conditions $(\mathcal{KW}^{1,1})$ and $(\mathcal{W}^{1,1})$. Finally, because $0 \in \mathrm{int}(V)$ and $\nabla\mu_V(x)$ is outwardly normal to ∂V for every $x \in \partial V$, we have that $\langle \nabla\mu_V(x), x\rangle > 0$ for every $x \in \partial V$, and in particular condition $(\mathcal{O})$ is satisfied as well. $\square$

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