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WHITNEY EXTENSION THEOREMS FOR CONVEX FUNCTIONS OF THE CLASSES C^1 AND $C^{1,\omega}$.

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Abstract. Let C be a compact subset of $\mathbb{R}^n$ (not necessarily convex), $f : C \to \mathbb{R}$ be a function, and $G : C \to \mathbb{R}^n$ be a continuous function, with modulus of continuity ω. We provide a necessary and sufficient condition on f, G for the existence of a *convex* function $F \in C^{1,\omega}(\mathbb{R}^n)$ such that $F = f$ on C and $\nabla F = G$ on C, with a good control of the modulus of continuity of ∇F in terms of that of G. We also solve a similar problem for the class of C^1 convex functions on $\mathbb{R}^n$, with a good control of the Lipschitz constants of the extensions (namely, $\mathrm{Lip}(F) \lesssim \|G\|_\infty$). Finally, we give a geometrical application concerning interpolation of compact subsets K of $\mathbb{R}^n$ by boundaries of C^1 or $C^{1,1}$ convex bodies with prescribed outer normals on K.

1. Introduction and main results

Throughout this paper, by a modulus of continuity ω we understand a concave, strictly increasing function $\omega : [0, \infty) \to [0, \infty)$ such that $\omega(0^+) = 0$. In particular, ω has an inverse $\omega^{-1} : [0, \beta) \to [0, \infty)$ which is convex and strictly increasing, where $\beta > 0$ may be finite or infinite. Furthermore, ω is subadditive, and satisfies $\omega(\lambda t) \leq \lambda\omega(t)$ for $\lambda \geq 1$, and $\omega(\mu t) \geq \mu\omega(t)$ for $0 \leq \mu \leq 1$. It is well-known that for every uniformly continuous function $f : X \to Y$ between two metric spaces there exists a modulus of continuity ω such that $d_Y(f(x), f(z)) \leq \omega(d_X(x, z))$ for every $x, z \in X$. Slightly abusing terminology, we will say that a mapping $G : X \to Y$ has modulus of continuity ω (or that G is ω-continuous) if there exists $M \geq 0$ such that

$$d_Y(G(x), G(y)) \leq M\omega(d_X(x, y))$$

for all $x, y \in X$.

Let C be a compact subset of $\mathbb{R}^n$, $f : C \to \mathbb{R}$ a function, and $G : C \to \mathbb{R}^n$ a mapping with modulus of continuity ω. Assume also that the pair (f, G)

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satisfies

$$|f(x) - f(y) - \langle G(y), x - y\rangle| \leq M|x - y|\,\omega\,(|x - y|) \qquad (W^{1,\omega})$$

for every $x, y \in C$ (in the case that $\omega(t) = t$ we will also denote this condition by $(W^{1,1})$). Then a well-known version of the Whitney extension theorem for the class $C^{1,\omega}$ holds true (see [12, 19]), and we get the existence of a function $F \in C^{1,\omega}(\mathbb{R}^n)$ such that $F = f$ and $\nabla F = G$ on C.

It is natural to ask what further assumptions on f, G would be necessary and sufficient to ensure that F can be taken to be convex. In a recent paper [3], we solved a similar problem for the class of C^∞, under the much more stringent assumption that C be convex. We refer to the introduction of [3] and the references therein for background; see in particular [8, 4] for an account of the spectacular progress made on Whitney extension problems in the last decade. Here we will only mention that results of this nature for the special class of convex functions also have interesting applications in differential geometry, PDE theory (such as Monge-Ampère equations), nonlinear dynamics, and quantum computing, see [9, 10, 11, 22] and the references therein. We should also note here that, in contrast with the classical Whitney extension theorem [21] (concerning jets) and with the solutions [12, 5, 6, 7] to the Whitney extension problem (concerning functions), which are both of a local character, the nature of our problem is global. Indeed, consider the following example: take any four numbers $a, b, c, d \in \mathbb{R}$ with $a < b < 0 < c < d$, and define $C = \{a, b, 0, c, d\}$ and $f(x) = |x|$ for $x \in C$. Since C is a five-point set it is clear that there are infinitely many C^1 functions (even infinitely many polynomials) F with $F = f$ on C. However, none of these F can be convex on $\mathbb{R}$, because, as is easily checked, any convex extension g of f to $\mathbb{R}$ must satisfy $g(x) = |x|$ for every $x \in [a, d]$, and therefore g cannot be differentiable at 0.

Let us introduce one global condition which, we have found, is necessary and sufficient for a function $f : C \to \mathbb{R}$ (and a mapping $G : C \to \mathbb{R}^n$ with modulus of continuty ω) to have a convex extension F of class $C^{1,\omega}(\mathbb{R}^n)$ such that $\nabla F = G$ on C. For a mapping $G : C \subset \mathbb{R}^n \to \mathbb{R}^n$ we will denote

$$M(G, C) := \sup_{x,y\in C,\, x\neq y} \frac{|G(x) - G(y)|}{\omega(|x - y|)}. \tag{1.1}$$

We will say that f and G satisfy the property $(CW^{1,\omega})$ on C if either G is constant, or else $0 < M(G, C) < \infty$ and there exists a constant $\eta \in (0, 1/2]$ such that

$$f(x) - f(y) - \langle G(y), x - y\rangle \geq \eta\,|G(x) - G(y)|\,\omega^{-1}\left(\frac{1}{2M}|G(x) - G(y)|\right)$$
$$\text{for all } x, y \in C, \text{ where } M = M(G, C). \qquad (CW^{1,\omega})$$

Throughout this paper we will assume that $0 < M < \infty$, or equivalently that G is nonconstant and has modulus of continuity ω. We may of course do so, because if $M = 0$ then our problem has a trivial solution (namely, the

function $x \mapsto f(x_0) + \langle G(x_0), x - x_0\rangle$ defines an affine extension of f to $\mathbb{R}^n$, for any $x_0 \in C$).

In the case that $\omega(t) = t$, we will also denote this condition by $(CW^{1,1})$. That is, (f, G) satisfies $(CW^{1,1})$ if and only if there exists $\delta > 0$ such that

$$f(x) - f(y) - \langle G(y), x - y\rangle \geq \delta\, |G(x) - G(y)|^2, \qquad (CW^{1,1})$$

for all $x, y \in C$.

Remark 1.1. *If (f, G) satisfies condition $(CW^{1,\omega})$ and G has modulus of continuity ω, then (f, G) satisfies condition $(W^{1,\omega})$.*

Proof. Condition $(CW^{1,\omega})$ implies that

$$0 \leq f(x) - f(y) - \langle G(y), x - y\rangle \leq \langle G(y) - G(x), y - x\rangle,$$

for all $x, y \in C$ with $x \neq y$, hence,

$$0 \leq \frac{f(x) - f(y) - \langle G(y), x - y\rangle}{|x - y|\omega(|x - y|)} \leq \frac{\langle G(y) - G(x), y - x\rangle}{|x - y|\omega(|x - y|)} \leq M.$$

□

The first of our main results is the following.

Theorem 1.2. *Let ω be a modulus of continuity. Let C be a compact (not necessarily convex) subset of $\mathbb{R}^n$. Let $f : C \to \mathbb{R}$ be an arbitrary function, and $G : C \to \mathbb{R}^n$ be continuous, with modulus of continuity ω. Then f has a convex, $C^{1,\omega}$ extension F to all of $\mathbb{R}^n$, with $\nabla F = G$ on C, if and only if (f, G) satisfies $(CW^{1,\omega})$ on C.*

In particular, for the most important case that $\omega(t) = t$, we have the following.

Corollary 1.3. *Let C be a compact (not necessarily convex) subset of $\mathbb{R}^n$. Let $f : C \to \mathbb{R}$ be an arbitrary function, and $G : C \to \mathbb{R}^n$ be a Lipschitz function. Then f has a convex, $C^{1,1}$ extension F to all of $\mathbb{R}^n$, with $\nabla F = G$ on C, if and only if (f, G) satisfies $(CW^{1,1})$ on C.*

It is worth noting that our proofs provide good control of the modulus of continuity of the gradients of the extensions F, in terms of that of G. In fact, assuming $\eta = 1/2$ in $(CW^{1,\omega})$, which we can fairly do (see Proposition 3.2 below), there exists a constant $k(n) > 0$, depending only on the dimension n, such that

$$M(\nabla F, \mathbb{R}^n) := \sup_{x,y\in\mathbb{R}^n,\, x\neq y} \frac{|\nabla F(x) - \nabla F(y)|}{\omega(|x - y|)} \leq k(n)\, M(G, C). \tag{1.2}$$

Because convex functions on $\mathbb{R}^n$ are not bounded (unless they are constant), the most usual definitions of norms in the space $C^{1,\omega}(\mathbb{R}^n)$ are not suited to estimate convex functions. In this paper, for a function $F : \mathbb{R}^n \to \mathbb{R}$ we will denote

$$\|F\|_{1,\omega} = |F(0)| + |\nabla F(0)| + \sup_{x,y\in\mathbb{R}^n,\, x\neq y} \frac{|\nabla F(x) - \nabla F(y)|}{\omega(|x - y|)}. \tag{1.3}$$

With this notation, and assuming $0 \in C$ and $\eta = 1/2$ in $(CW^{1,\omega})$, equation (1.2) implies that

$$\|F\|_{1,\omega} \leq k(n) \left(|f(0)| + |G(0)| + M(G,C)\right). \tag{1.4}$$

In particular, the norm of the extension F of f that we construct is nearly optimal, in the sense that

$$\|F\|_{1,\omega} \leq k(n) \inf\{\|\varphi\|_{1,\omega} : \varphi \in C^{1,\omega}(\mathbb{R}^n), \varphi_{|C} = f, (\nabla\varphi)_{|C} = G\} \tag{1.5}$$

for a constant $k(n) \geq 1$ only depending on n.

Since for a function $C^{1,\omega}(\mathbb{R}^n)$ and a compact set $C \subset \mathbb{R}^n$ there always exists a modulus of continuity for the restriction $(\nabla\varphi)_{|C}$, Theorem 1.2 also provides a solution to the following problem: given a continuous mapping $G : C \to \mathbb{R}^n$ and a function $f : C \to \mathbb{R}$, how can we decide whether there is a convex function $F \in C^1(\mathbb{R}^n)$ such that $F_{|C} = f$ and $(\nabla F)_{|C} = G$? However, given such a 1-jet (f, G) on a compact set C, unless $\omega(t) = t$ or one has a clue about what ω might do the job, in practice it may be difficult to find a modulus of continuity ω such that (f, G) satisfies $(CW^{1,\omega})$, and for this reason it is also desirable to have a criterion for C^1 convex extendibility which does not involve dealing with moduli of continuity. We next study this question.

Given a 1-jet (f, G) on C (where $f : C \to \mathbb{R}$ is a function and $G : C \to \mathbb{R}^n$ is a continuous mapping), a necessary condition for the existence of a convex function $F \in C^1(\mathbb{R}^n)$ with $F_{|C} = f$ and $(\nabla F)_{|C} = G$ is given by

$$\lim_{|z-y|\to 0^+} \frac{f(z) - f(y) - \langle G(y), z - y\rangle}{|z - y|} = 0 \text{ uniformly on } C, \tag{W^1}$$

which is equivalent to Whitney's classical condition for C^1 extendibility. If a 1-jet (f, G) satisfies condition (W^1), Whitney's extension theorem [21] provides us with a function $F \in C^1(\mathbb{R}^n)$ such that $F_{|C} = f$ and $(\nabla F)_{|C} = G$. In the special case that C is a convex body, if $f : C \to \mathbb{R}$ is convex and (f, G) satisfies (W^1), we will see that, without any further assumptions on (f, G), f *always* has a convex C^1 extension to all of $\mathbb{R}^n$ with $(\nabla F)_{|C} = G$.

Theorem 1.4. *Let C be a compact convex subset of $\mathbb{R}^n$ with non-empty interior. Let $f : C \to \mathbb{R}$ be a convex function, and $G : C \to \mathbb{R}^n$ be a continuous mapping satisfying Whitney's extension condition (W^1) on C. Then there exists a convex function $F \in C^1(\mathbb{R}^n)$ such that $F_{|C} = f$ and $(\nabla F)_{|C} = G$.*

If C and f are convex but $\mathrm{int}(C)$ is empty then, in order to obtain differentiable convex extensions of f to all of $\mathbb{R}^n$ we will show that it is enough to complement (W^1) with the following global geometrical condition:

$$f(x) - f(y) = \langle G(y), x - y\rangle \implies G(x) = G(y), \text{ for all } x, y \in C. \tag{CW^1}$$

Theorem 1.5. *Let C be a compact convex subset of $\mathbb{R}^n$. Let $f : C \to \mathbb{R}$ be a convex function, and $G : C \to \mathbb{R}^n$ be a continuous mapping. Then f has*

a convex, C^1 extension F to all of $\mathbb{R}^n$, with $\nabla F = G$ on C, if and only if f and G satisfy (W^1) and (CW^1) on C.

In the general case of a non-convex compact set C, we will just have to add another global geometrical condition to (W^1) and (CW^1):

$$f(x) - f(y) \geq \langle G(y), x - y \rangle \text{ for all } x, y \in C. \tag{C}$$

Remark 1.6. *If (f, G) satisfies condition (C) and G is continuous, then (f, G) satisfies Whitney's condition (W^1).*

This is easily shown by an obvious modification of the proof of Remark 1.1.

Theorem 1.7. *Let C be a compact (not necessarily convex) subset of $\mathbb{R}^n$. Let $f : C \to \mathbb{R}$ be an arbitrary function, and $G : C \to \mathbb{R}^n$ be a continuous mapping. Then f has a convex, C^1 extension F to all of $\mathbb{R}^n$, with $\nabla F = G$ on C, if and only if (f, G) satisfies the conditions (C) and (CW^1) on C.*

Similarly to the $C^{1,\omega}$ case, we will see that the proof of the above result provides good control of the Lipschitz constant of the extension F in terms of $\|G\|_\infty$. Namely, we will see that

$$\sup_{x \in \mathbb{R}^n} |\nabla F(x)| \leq k(n) \sup_{y \in C} |G(y)| \tag{1.6}$$

for a constant $k(n) \geq 1$ only depending on n. Interestingly, this kind of control of $\mathrm{Lip}(F)$ in terms of $\|G\|_\infty$ cannot be obtained, in general, for jets (f, G) not satisfying (C), as is easily seen by examples, and the proof of Whitney's extension theorem only permits to obtain extensions $(F, \nabla F)$ (of jets (f, G) on C) which satisfy estimations of the type

$$\sup_{x \in \mathbb{R}^n} |\nabla F(x)| \leq k(n) \left(\sup_{z, y \in C, z \neq y} \frac{|f(z) - f(y)|}{|z - y|} + \sup_{y \in C} |G(y)| \right).$$

or of the type

$$\sup_{x \in \mathbb{R}^n} |\nabla F(x)| \leq k(n) \left(\sup_{y \in C} |f(y)| + \sup_{y \in C} |G(y)| \right).$$

It is condition (C) that allows us to get finer control of $\mathrm{Lip}(F)$ in the convex case; see the proof of Claim 2.3 below. In particular, assuming $0 \in C$ and defining

$$\|F\|_1 := |F(0)| + \sup_{x \in \mathbb{R}^n} |\nabla F(x)|, \tag{1.7}$$

we obtain an extension F of f such that

$$\|F\|_1 \leq k(n) \inf\{\|\varphi\|_1 : \varphi \in C^1(\mathbb{R}^n), \varphi_{|C} = f, (\nabla \varphi)_{|C} = G\}, \tag{1.8}$$

so the norm of our extension is nearly optimal in this case too.

In the particular case when C is finite, Theorem 1.7 provides necessary and sufficient conditions for interpolation of finite sets of data by C^1 convex functions.

Corollary 1.8. *Let S be a finite subset of $\mathbb{R}^n$, and $f : S \to \mathbb{R}$ be a function. Then there exists a convex function $F \in C^1(\mathbb{R}^n)$ with $F = f$ on S if and only if there exists a mapping $G : S \to \mathbb{R}^n$ such that f and G satisfy conditions (C) and (CW^1) on S.*

In [16, Theorem 14] it is proved that, for every finite set of *strictly convex data* in $\mathbb{R}^n$ there always exists a C^∞ convex function (or even a convex polynomial) that interpolates the given data. However, in the case that the data are convex but not strictly convex, the above corollary seems to be new.

Let us conclude this introduction with two geometrical applications of Corollary 1.3 and Theorem 1.7 concerning characterizations of compact subsets K of $\mathbb{R}^n$ which can be interpolated by boundaries of $C^{1,1}$ or C^1 convex bodies (with prescribed unit outer normals on K). Namely, if K is a compact subset of $\mathbb{R}^n$ and we are given an M-Lipschitz (resp. continuous) map $N : K \to \mathbb{R}^n$ such that $|N(y)| = 1$ for every $y \in K$, it is natural to ask what conditions on K and N are necessary and sufficient for K to be a subset of the boundary of a $C^{1,1}$ (resp. C^1) convex body V such that $0 \in \operatorname{int}(V)$ and $N(y)$ is outwardly normal to ∂V at y for every $y \in K$. A suitable set of conditions in the $C^{1,1}$ case is:

$$(\mathcal{O}) \qquad \langle N(y), y\rangle > 0 \text{ for all } y \in K;$$

$$(\mathcal{KW}^{1,1}) \qquad \langle N(y), y - x\rangle \geq \frac{\eta}{2M}|N(y) - N(x)|^2 \text{ for all } x, y \in K,$$

for some $\eta \in (0, \frac{1}{2}]$. Our main result in this direction is as follows.

Theorem 1.9. *Let K be a compact subset of $\mathbb{R}^n$, and let $N : K \to \mathbb{R}^n$ be an M-Lipschitz mapping such that $|N(y)| = 1$ for every $y \in K$. Then the following statements are equivalent:*

1. *There exists a $C^{1,1}$ convex body V with $0 \in \operatorname{int}(V)$ and such that $K \subseteq \partial V$ and $N(y)$ is outwardly normal to ∂V at y for every $y \in K$.*
2. *K and N satisfy conditions $(\mathcal{O})$ and $(\mathcal{KW}^{1,1})$.*

This result may be compared to [9], where M. Ghomi showed how to construct C^m smooth strongly convex bodies V with prescribed strongly convex submanifolds and tangent planes. In the same spirit, the above Theorem allows us to deal with arbitrary compacta instead of manifolds, and to drop the strong convexity assumption, in the particular case of $C^{1,1}$ bodies. Similarly, for interpolation by C^1 bodies, the pertinent conditions are:

$$(\mathcal{O}) \qquad \langle N(y), y\rangle > 0 \text{ for all } y \in K;$$

$$(\mathcal{K}) \qquad \langle N(y), x - y\rangle \leq 0 \text{ for all } x, y \in K;$$

$$(\mathcal{KW}^1) \qquad \langle N(y), x - y\rangle = 0 \implies N(x) = N(y) \text{ for all } x, y \in K,$$

and our result for the class C^1 then reads as follows.

Theorem 1.10. *Let K be a compact subset of $\mathbb{R}^n$, and let $N : K \to \mathbb{R}^n$ be a continuous mapping such that $|N(y)| = 1$ for every $y \in K$. Then the following statements are equivalent:*

(1) *There exists a C^1 convex body V with $0 \in int(V)$ and such that $K \subseteq \partial V$ and $N(y)$ is outwardly normal to ∂V at y for every $y \in K$.*
(2) *K and N satisfy conditions $(\mathcal{O})$, $(\mathcal{K})$, and $(\mathcal{KW}^1)$.*

It is worth noting that, in the above results, the assumption that C and K be compact cannot be removed in general, see [18, Example 4], [20, Example 3.2], and [3, Example 4.1].

The rest of this paper is devoted to the proofs of the above results. The main ideas in the proofs of Theorem 1.2 and 1.7 are similar, but the case $C^{1,1}$ is considerably more technical, so we will begin by proving Theorems 1.4, 1.5, 1.7 and 1.10 in Section 2. The proof of Theorems 1.2 and 1.9 will be provided in Section 3.

2. Proofs of the C^1 results

Theorem 1.4 is a consequence of Theorem 1.5 and of the following result.

Lemma 2.1. *Let $f \in C^1(\mathbb{R}^n)$, $C \subset \mathbb{R}^n$ be a compact convex set with nonempty interior, $x_0, y_0 \in C$. Assume that f is convex on C and*

$$f(x_0) - f(y_0) = \langle \nabla f(y_0), x_0 - y_0 \rangle.$$

Then $\nabla f(x_0) = \nabla f(y_0)$.

Proof. **Case 1.** Suppose first that $f(x_0) = f(y_0) = 0$. We may of course assume that $x_0 \neq y_0$ as well. Then we also have $\langle \nabla f(y_0), x_0 - y_0 \rangle = 0$. If we consider the C^1 function $\varphi(t) = f(y_0 + t(x_0 - y_0))$, we have that φ is convex on the interval $[0,1]$ and $\varphi'(0) = 0$, hence $0 = \varphi(0) = \min_{t\in[0,1]} \varphi(t)$, and because $\varphi(0) = \varphi(1)$ and the set of minima of a convex function on a convex set is convex, we deduce that $\varphi(t) = 0$ for all $t \in [0,1]$. This shows that f is constant on the segment $[x_0, y_0]$ and in particular we have

$$\langle \nabla f(z), z_0 - z_0' \rangle = 0 \text{ for all } z, z_0, z_0' \in [x_0, y_0].$$

Now pick a point a_0 in the interior of C and a number $r_0 > 0$ so that $B(a_0, r_0) \subset \mathrm{int}(C)$. Since C is a compact convex body, every ray emanating from a point $a \in B(a_0, r_0)$ intersects the boundary of C at exactly one point. This implies that (even though the segment $[x_0, y_0]$ might entirely lie on the boundary ∂C), for every $a \in B(a_0, r_0)$, the interior of the triangle Δ_a with vertices x_0, a, y_0, relative to the affine plane spanned by these points, is contained in the interior of C; we will denote $\mathrm{relint}\,(\Delta_a) \subset \mathrm{int}(C)$.

Let p_0 be the unique point in $[x_0, y_0]$ such that $|a_0 - p_0| = d\,(a_0, [x_0, y_0])$ (the distance to the segment $[x_0, y_0]$), set $w_0 = a_0 - p_0$, and denote $v_a := a - p_0$ for each $a \in B(a_0, r_0)$. Thus for every $a \in B(a_0, r_0)$ we can write $v_a = u_a + w_0$, where $u_a := a - a_0 \in B(0, r_0)$, and in particular we have $\{v_a : a \in B(a_0, r_0)\} = B(w_0, r_0)$.

Claim 2.2. *For every z_0, z_0' in the relative interior of the segment $[x_0, y_0]$, we have $\nabla f(z_0) = \nabla f(z_0')$.*

Let us prove our claim. It is enough to show that $\langle \nabla f(z_0) - \nabla f(z_0'), v_a \rangle = 0$ for every $a \in B(a_0, r_0)$ (because if a linear form vanishes on a set with nonempty interior, such as $B(w_0, r_0)$, then it vanishes everywhere). So take $a \in B(a_0, r_0)$. Since z_0 and z_0' are in the relative interior of the segment $[x_0, y_0]$ and $\operatorname{relint}(\Delta_a) \subset \operatorname{int}(C)$, there exists $t_0 > 0$ such that $z_0 + tv_a, z_0' + tv_a \in \operatorname{int}(C)$ for every $t \in (0, t_0]$.

If we had $\langle \nabla f(z_0') - \nabla f(z_0), v_a \rangle > 0$ then, because f is convex on C and $f(z_0) = f(z_0') = 0$, $\langle \nabla f(z_0'), z_0 - z_0' \rangle = 0$, we would get

$$f(z_0 + tv_a) = f(z_0' + z_0 - z_0' + tv_a) \geq \langle \nabla f(z_0'), z_0 - z_0' + tv_a \rangle = \langle \nabla f(z_0'), tv_a \rangle,$$

hence

$$\lim_{t \to 0^+} \frac{f(z_0 + tv_a)}{t} \geq \langle \nabla f(z_0'), v_a \rangle > \langle \nabla f(z_0), v_a \rangle = \lim_{t \to 0^+} \frac{f(z_0 + tv_a)}{t},$$

a contradiction. By interchanging the roles of z_0, z_0', we see that the inequality $\langle \nabla f(z_0') - \nabla f(z_0), v_a \rangle < 0$ also leads to a contradiction. Therefore $\langle \nabla f(z_0') - \nabla f(z_0), v_a \rangle = 0$ and the Claim is proved.

Now, by using the continuity of ∇f, we easily conclude the proof of the Lemma in Case 1.

Case 2. In the general situation, let us consider the function h defined by

$$h(x) = f(x) - f(y_0) - \langle \nabla f(y_0), x - y_0 \rangle, \quad x \in \mathbb{R}^n.$$

It is clear that h is convex on C, and $h \in C^1(\mathbb{R}^n)$. We also have

$$\nabla h(x) = \nabla f(x) - \nabla f(y_0),$$

and in particular $\nabla h(y_0) = 0$. Besides, using the assumption that $f(x_0) - f(y_0) = \langle \nabla f(y_0), x_0 - y_0 \rangle$, we have $h(x_0) = 0 = h(y_0)$, and $h(x_0) - h(y_0) = \langle \nabla h(y_0), x_0 - y_0 \rangle$. Therefore we can apply Case 1 with h instead of f and we get that $\nabla h(x_0) = \nabla h(y_0) = 0$, which implies that $\nabla f(x_0) = \nabla f(y_0)$. □

From the above Lemma it is clear that (CW^1) is a necessary condition for a convex function $f : C \to \mathbb{R}$ (and a mapping $G : C \to \mathbb{R}^n$) to have a convex, C^1 extension F to all of $\mathbb{R}^n$ with $\nabla F = G$ on C, and also that if the jet (f, G) satisfies (W^1) and $\operatorname{int}(C) \neq \emptyset$ then (f, G) automatically satisfies (CW^1) on C as well. It is also obvious that Theorem 1.5 is an immediate consequence of Theorem 1.7, and that the condition (C) is also necessary in Theorem 1.7. Thus, in order to prove Theorems 1.4, 1.5, and 1.7 it will be sufficient to establish the *if* part of Theorem 1.7.

2.1. **Proof of Theorem 1.7.** Because f satisfies (C) and G is continuous, by Remark 1.6 we know that (f, G) satisfies (W^1). Then, according to Whitney's Extension Theorem, there exists $\widetilde{f} \in C^1(\mathbb{R}^n)$ such that, on C, we have $\widetilde{f} = f$ and $\nabla \widetilde{f} = G$.

Claim 2.3. *If f satisfies (C) then we can further assume that there exists a constant $k(n)$, only depending on n, such that*

$$Lip(\widetilde{f}) = \sup_{x\in\mathbb{R}^n} |\nabla \widetilde{f}(x)| \leq k(n) \sup_{y\in C} |G(y)|. \tag{2.1}$$

Proof. Let us recall the construction of the function $\tilde{f}$. Consider a Whitney's partition of unity $\{\varphi_k\}_k$ associated to the family of Whitney's cubes $\{Q_k,\ Q_k^*\}_k$ decomposing $\mathbb{R}^n \setminus C$ (see the following section or [19, Chapter VI] for notation). Define polynomials $P_z(x) = f(z) + \langle G(z), x - z\rangle$ for all $x \in \mathbb{R}^n$ and all $z \in C$. For every k, find a point $p_k \in C$ such that $d(C, Q_k) = d(p_k, Q_k)$. Then define the function $\tilde{f}$ by

$$\tilde{f}(x) = \begin{cases} \sum_k P_{p_k}(x)\varphi_k(x) & \text{if } x \in \mathbb{R}^n \setminus C \\ f(x) & \text{if } x \in C. \end{cases} \tag{2.2}$$

As a particular case of the proof of the Whitney extension theorem, we know that this function $\tilde{f}$ is of class $C^1(\mathbb{R}^n)$, extends f to $\mathbb{R}^n$ and satisfies $\nabla\tilde{f} = G$ on C. By the definition of $\tilde{f}$ we can write, for $x \in \mathbb{R}^n \setminus C$,

$$\nabla\tilde{f}(x) = \sum_k \nabla P_{p_k}(x)\varphi_k(x) + \sum_k P_{p_k}(x)\nabla\varphi_k(x). \tag{2.3}$$

Since $\nabla P_{p_k} = G(p_k)$ for all k, the first sum is bounded above by $\sup\{|G(y)| : y \in C\} := \|G\|_\infty$. In order to estimate the second sum, recall that $\sum_k \nabla\varphi_k = 0$ and find a point $b \in C$ such that $|b - x| = d(x, C)$. Then we write

$$\sum_k P_{p_k}(x)\nabla\varphi_k(x) = \sum_k \left(P_{p_k}(x) - P_b(x)\right)\nabla\varphi_k(x), \tag{2.4}$$

and observe that

$$\begin{aligned} P_{p_k}(x) - P_b(x) &= f(p_k) + \langle G(p_k), x - p_k\rangle - f(b) - \langle G(b), x - b\rangle \\ &= f(p_k) - f(b) - \langle G(b), p_k - b\rangle + \langle G(p_k) - G(b), x - p_k\rangle. \end{aligned}$$

By the same argument used in Remark 1.6 involving condition (C), we have that

$$\begin{aligned} 0 &\leq f(p_k) - f(b) - \langle G(b), p_k - b\rangle \\ &\leq \langle G(b) - G(p_k), b - p_k\rangle \leq 2\|G\|_\infty |b - p_k|. \end{aligned}$$

On the other hand,

$$|\langle G(p_k) - G(b), x - p_k\rangle| \leq 2\|G\|_\infty |x - p_k|.$$

These inequalities lead us to

$$|P_{p_k}(x) - P_b(x)| \leq 2\|G\|_\infty (|b - p_k| + |x - p_k|). \tag{2.5}$$

For those integers k such that $x \in Q_k^*$, the results exposed in [19, Chapter VI] show that $|b - p_k| \leq 8|x - p_k|$, and that $|x - p_k|$ is of the same order as $\operatorname{diam}(Q_k)$ (with constants not even depending on n). Hence

$$|P_{p_k}(x) - P_b(x)| \lesssim \|G\|_\infty \operatorname{diam}(Q_k) \quad \text{if} \quad x \in Q_k^*$$

(by $A \lesssim B$ we mean that $A \leq KB$, where K is a constant only depending on the dimension n). Also, by the properties of the Whitney's partition of unity $\{\varphi_k\}_k$ we know that

$$\big|\nabla\varphi_k(x)\big| \lesssim \operatorname{diam}(Q_k)^{-1},$$

and because all these sums has at most $N = (12)^n$ nonzero terms, we obtain

$$\sum_{Q_k^* \ni x} |P_{p_k}(x) - P_b(x)|\big|\nabla\varphi_k(x)\big| \lesssim \|G\|_\infty \sum_{Q_k^* \ni x} \operatorname{diam}(Q_k)\operatorname{diam}(Q_k)^{-1} \lesssim \|G\|_\infty,$$

which together with (2.4) allows us to control the second sum in (2.3) as required. □

Thus we may and do assume in what follows, for simplicity of notation, that f is of class $C^1(\mathbb{R}^n)$, with $\nabla f = G$ on C, and that f satisfies conditions (C) and (CW^1) on C. Occasionally, if the distinction between $\widetilde{f}$ and f matters (e.g. in the estimations of Lipschitz constants involving $\widetilde{f}$), we will nevertheless write $\widetilde{f}$ instead of f in order to prevent any misinterpretation. Let us consider the function $m(f) : \mathbb{R}^n \to \mathbb{R}$ defined by

$$m(f)(x) = \sup_{y\in C}\{f(y) + \langle\nabla f(y), x - y\rangle\}. \tag{2.6}$$

Since C is compact and the function $y \mapsto f(y) + \langle\nabla f(y), x - y\rangle$ is continuous, it is obvious that $m(f)(x)$ is well defined, and in fact the sup is attained, for every $x \in \mathbb{R}^n$. Furthermore, if we set

$$K := \max_{y\in C} |\nabla f(y)| = \max_{y\in C} |G(y)| \tag{2.7}$$

then each affine function $x \mapsto f(y) + \langle\nabla f(y), x - y\rangle$ is K-Lipschitz, and therefore $m(f)$, being a sup of a family of convex and K-Lipschitz functions, is convex and K-Lipschitz on $\mathbb{R}^n$. Note also that

$$K \leq \operatorname{Lip}(\widetilde{f}). \tag{2.8}$$

Moreover, we have

$$m(f) = f \text{ on } C. \tag{2.9}$$

Indeed, if $x \in C$ then, because f satisfies (C) on C, we have $f(x) \geq f(y) + \langle\nabla f(y), x - y\rangle$ for every $y \in C$, hence $m(f)(x) \leq f(x)$. On the other hand, we also have $f(x) \leq m(f)(x)$ because of the definition of $m(f)(x)$ and the fact that $x \in C$.

(In the case when C is convex and has nonempty interior, it is easy to see that if $h : \mathbb{R}^n \to \mathbb{R}$ is convex and $h = f$ on C, then $m(f) \leq h$. Thus, in this case, $m(f)$ is the minimal convex extension of f to all of $\mathbb{R}^n$, which accounts for our choice of notation. However, if C is convex but has empty interior then there is no minimal convex extension operator. We refer the interested reader to [18] for necessary and sufficient conditions for $m(f)$ to be finite everywhere, in the situation when $f : C \to \mathbb{R}$ is convex but not necessarily everywhere differentiable.)

If the function $m(f)$ were differentiable on $\mathbb{R}^n$, there would be nothing else to say. Unfortunately, it is not difficult to construct examples showing that $m(f)$ need not be differentiable outside C (even when C is convex and f satisfies (CW^1), see Example 2.9 at the end of this section). Nevertheless, a crucial step in our proof is the following fact: $m(f)$ is differentiable on C, provided that f satisfies conditions (C) and (CW^1) on C.

Lemma 2.4. *Let $f \in C^1(\mathbb{R}^n)$, let C be a compact subset of $\mathbb{R}^n$ (not necessarily convex), and assume that f satisfies (C) and (CW^1) on C. Then, for each $x_0 \in C$, the function $m(f)$ is differentiable at x_0, with $\nabla m(f)(x_0) = \nabla f(x_0)$.*

Proof. Notice that, by definition of $m(f)$ we have, for every $x \in \mathbb{R}^n$,

$$\langle \nabla f(x_0), x - x_0\rangle + m(f)(x_0) = \langle \nabla f(x_0), x - x_0\rangle + f(x_0) \leq m(f)(x).$$

Since $m(f)$ is convex, this means that $\nabla f(x_0)$ belongs to $\partial m(f)(x_0)$ (the subdifferential of $m(f)$ at x_0). If $m(f)$ were not differentiable at x_0 then there would exist a number $\varepsilon > 0$ and a sequence (h_k) converging to 0 in $\mathbb{R}^n$ such that

$$\frac{m(f)(x_0 + h_k) - m(f)(x_0) - \langle \nabla f(x_0), h_k\rangle}{|h_k|} \geq \varepsilon \quad \text{for every } k \in \mathbb{N}. \tag{2.10}$$

Because the sup defining $m(f)(x_0 + h_k)$ is attained, we obtain a sequence $(y_k) \subset C$ such that

$$m(f)(x_0 + h_k) = f(y_k) + \langle \nabla f(y_k), x_0 + h_k - y_k\rangle,$$

and by compactness of C we may assume, up to passing to a subsequence, that (y_k) converges to some point $y_0 \in C$. Because $f = m(f)$ on C, and by continuity of f, ∇f, and $m(f)$ we then have

$$f(x_0) = m(f)(x_0) = \lim_{k\to\infty} m(f)(x_0 + h_k) =$$
$$\lim_{k\to\infty} (f(y_k) + \langle \nabla f(y_k), x_0 + h_k - y_k\rangle) = f(y_0) + \langle \nabla f(y_0), x_0 - y_0\rangle,$$

that is, $f(x_0) - f(y_0) = \langle \nabla f(y_0), x_0 - y_0\rangle$. Since $x_0, y_0 \in C$ and f satisfies (CW^1), this implies that $\nabla f(x_0) = \nabla f(y_0)$. And because $m(f)(x_0) \geq f(y_k) + \langle \nabla f(y_k), x_0 - y_k\rangle$ by definition of $m(f)$, we then have

$$\frac{m(f)(x_0 + h_k) - m(f)(x_0) - \langle \nabla f(x_0), h_k\rangle}{|h_k|} \leq$$
$$\frac{f(y_k) + \langle \nabla f(y_k), x_0 + h_k - y_k\rangle - f(y_k) - \langle \nabla f(y_k), x_0 - y_k\rangle - \langle \nabla f(x_0), h_k\rangle}{|h_k|} =$$
$$\frac{\langle \nabla f(y_k) - \nabla f(x_0), h_k\rangle}{|h_k|} \leq |\nabla f(y_k) - \nabla f(x_0)| = |\nabla f(y_k) - \nabla f(y_0)|,$$

from which we deduce, using the continuity of ∇f, that

$$\limsup_{k\to\infty} \frac{m(f)(x_0 + h_k) - m(f)(x_0) - \langle \nabla f(x_0), h_k\rangle}{|h_k|} \leq 0,$$

in contradiction with (2.10). □

Now we proceed with the rest of the proof of Theorem 1.7. Our strategy will be to use the differentiability of $m(f)$ on ∂C in order to construct a (not necessarily convex) differentiable function g such that $g = f$ on C, $g \geq m(f)$ on $\mathbb{R}^n$, and $\lim_{|x|\to\infty} g(x) = \infty$. Then we will define F as the convex envelope of g, which will be of class $C^1(\mathbb{R}^n)$ and will coincide with f on C.

For each $\varepsilon > 0$, let $\theta_\varepsilon : \mathbb{R} \to \mathbb{R}$ be defined by

$$\theta_\varepsilon(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ t^2 & \text{if } t \leq \frac{K+\varepsilon}{2} \\ (K+\varepsilon)\left(t - \frac{K+\varepsilon}{2}\right) + \left(\frac{K+\varepsilon}{2}\right)^2 & \text{if } t > \frac{K+\varepsilon}{2} \end{cases}$$

(recall that $K = \max_{y\in C} \|\nabla f(y)\| \leq \operatorname{Lip}(\widetilde{f})$). Observe that $\theta_\varepsilon \in C^1(\mathbb{R})$, $\operatorname{Lip}(\theta_\varepsilon) = K + \varepsilon$. Now set

$$\Phi_\varepsilon(x) = \theta_\varepsilon\left(d(x, C)\right),$$

where $d(x, C)$ stands for the distance from x to C, notice that $\Phi_\varepsilon(x) = d(x, C)^2$ on an open neighborhood of C, and define

$$H_\varepsilon(x) = |f(x) - m(f)(x)| + 2\Phi_\varepsilon(x).$$

Note that $\operatorname{Lip}(\Phi_\varepsilon) = \operatorname{Lip}(\theta_\varepsilon)$ because $d(\cdot, C)$ is 1-Lipschitz, and therefore

$$\operatorname{Lip}(H_\varepsilon) \leq \operatorname{Lip}(\widetilde{f}) + K + 2(K+\varepsilon) \leq 4\operatorname{Lip}(\widetilde{f}) + 2\varepsilon. \tag{2.11}$$

Claim 2.5. *H_ε is differentiable on C, with $\nabla H_\varepsilon(x_0) = 0$ for every $x_0 \in C$.*

Proof. The function $d(\cdot, C)^2$ is obviously differentiable, with a null gradient, at x_0, hence we only have to see that $|f - m(f)|$ is differentiable, with a null gradient, at x_0. Since $\nabla m(f)(x_0) = \nabla f(x_0)$ by Lemma 2.4, the Claim boils down to the following easy exercise: if two functions h_1, h_2 are differentiable at x_0, with $\nabla h_1(x_0) = \nabla h_2(x_0)$, then $|h_1 - h_2|$ is differentiable, with a null gradient, at x_0. □

Now, because Φ_ε is continuous and positive on $\mathbb{R}^n \setminus C$, using mollifiers and a partition of unity, one can construct a function $\varphi_\varepsilon \in C^\infty(\mathbb{R}^n \setminus C)$ such that

$$|\varphi_\varepsilon(x) - H_\varepsilon(x)| \leq \Phi_\varepsilon(x) \quad \text{for every } x \in \mathbb{R}^n \setminus C, \tag{2.12}$$

and

$$\operatorname{Lip}(\varphi_\varepsilon) \leq \operatorname{Lip}(H_\varepsilon) + \varepsilon \tag{2.13}$$

(see for instance [13, Proposition 2.1] for a proof in the more general setting of Riemannian manifolds, or [2] even for possibly infinite-dimensional Riemannian manifolds). Let us define $\widetilde{\varphi} = \widetilde{\varphi_\varepsilon} : \mathbb{R}^n \to \mathbb{R}$ by

$$\widetilde{\varphi} = \begin{cases} \varphi_\varepsilon(x) & \text{if } x \in \mathbb{R}^n \setminus C \\ 0 & \text{if } x \in C. \end{cases}$$

Claim 2.6. *The function $\widetilde{\varphi}$ is differentiable on $\mathbb{R}^n$, and it satisfies $\nabla\widetilde{\varphi}(x_0) = 0$ for every $x_0 \in C$.*

Proof. It is obvious that $\widetilde{\varphi}$ is differentiable on $\mathrm{int}(C)\cup(\mathbb{R}^n \setminus C)$. We also have $\nabla\widetilde{\varphi} = 0$ on $\mathrm{int}(C)$, trivially. We only have to check that $\widetilde{\varphi}$ is differentiable, with a null gradient, on ∂C. If $x_0 \in \partial C$ we have (recalling that $\Phi_\varepsilon(x) = d(x, C)^2$ on a neighborhood of C) that

$$\frac{|\widetilde{\varphi}(x) - \widetilde{\varphi}(x_0)|}{|x - x_0|} = \frac{|\widetilde{\varphi}(x)|}{|x - x_0|} \leq \frac{|H_\varepsilon(x)| + d(x, C)^2}{|x - x_0|} \to 0$$

as $|x - x_0| \to 0^+$, because both H_ε and $d(\cdot, C)^2$ vanish at x_0 and are differentiable, with null gradients, at x_0. Therefore $\widetilde{\varphi}$ is differentiable at x_0, with $\nabla\widetilde{\varphi}(x_0) = 0$. □

Note also that

$$\mathrm{Lip}(\widetilde{\varphi}) = \mathrm{Lip}(\varphi_\varepsilon) \leq \mathrm{Lip}(H_\varepsilon) + \varepsilon \leq 4\,\mathrm{Lip}(\widetilde{f}) + 3\varepsilon. \tag{2.14}$$

Next we define

$$g = g_\varepsilon := f + \widetilde{\varphi}. \tag{2.15}$$

The function g is differentiable on $\mathbb{R}^n$, and coincides with f on C. Moreover, we also have $\nabla g = \nabla f$ on C (because $\nabla\widetilde{\varphi} = 0$ on C). And, for $x \in \mathbb{R}^n \setminus C$, we have

$$g(x) \geq f(x) + H(x) - \Phi_\varepsilon(x) = f(x) + |f(x) - m(f)(x)| + \Phi_\varepsilon(x) \geq m(f)(x) + \Phi_\varepsilon(x).$$

This shows that $g \geq m(f)$. On the other hand, because $m(f)$ is K-Lipschitz, we have

$$m(f)(x) \geq m(f)(0) - K|x|,$$

and because C is bounded, say $C \subset B(0, R)$ for some $R > 0$, also

$$\begin{aligned}\Phi_\varepsilon(x) &= (K + \varepsilon)d(x, C) - \frac{(K + \varepsilon)^2}{4} \\ &\geq (K + \varepsilon)d(x, B(0, R)) - \frac{(K + \varepsilon)^2}{4} = (K + \varepsilon)\left(|x| - R - \frac{K + \varepsilon}{4}\right)\end{aligned}$$

for $|x| \geq R + \frac{K+\varepsilon}{2}$. Hence

$$g(x) \geq m(f)(x) + \Phi_\varepsilon(x) \geq m(f)(0) - K|x| + (K + \varepsilon)\left(|x| - R - \frac{K + \varepsilon}{4}\right),$$

for $|x|$ large enough, which implies

$$\lim_{|x|\to\infty} g(x) = \infty. \tag{2.16}$$

Also, notice that according to (2.14) and the definition of g, we have

$$\mathrm{Lip}(g) \leq \mathrm{Lip}(\widetilde{f}) + \mathrm{Lip}(\widetilde{\varphi}) \leq 5\,\mathrm{Lip}(\widetilde{f}) + 3\varepsilon. \tag{2.17}$$

Now we will use a differentiability property of the convex envelope of a function $\psi : \mathbb{R}^n \to \mathbb{R}$, defined by

$$\mathrm{conv}(\psi)(x) = \sup\{h(x) \,:\, h \text{ is convex }, h \leq \psi\}$$

(another expression for $\mathrm{conv}(\psi)$, which follows form Carathéodory's Theorem, is

$$\mathrm{conv}(\psi)(x) = \inf\left\{\sum_{j=1}^{n+1} \lambda_j \psi(x_j) \,:\, \lambda_j \geq 0, \sum_{j=1}^{n+1} \lambda_j = 1, x = \sum_{j=1}^{n+1} \lambda_j x_j\right\},$$

see [17, Corollary 17.1.5] for instance). The following result is a restatement of a particular case of the main theorem in [15]; see also [14].

Theorem 2.7 (Kirchheim-Kristensen)**.** *If $\psi : \mathbb{R}^n \to \mathbb{R}$ is differentiable and $\lim_{|x|\to\infty} \psi(x) = \infty$, then conv$(\psi) \in C^1(\mathbb{R}^n)$.*

Although not explicitly stated in that paper, the proof of [15] also shows that

$$\mathrm{Lip}\,(\mathrm{conv}(\psi)) \leq \mathrm{Lip}(\psi).$$

If we define $F = \mathrm{conv}(g)$ we thus get that F is convex on $\mathbb{R}^n$ and $F \in C^1(\mathbb{R}^n)$, with

$$\mathrm{Lip}(F) \leq \mathrm{Lip}(g) \leq 5\mathrm{Lip}(\widetilde{f}) + 3\varepsilon \leq 5\,k(n) \sup_{y\in C} |G(y)| + 3\varepsilon. \tag{2.18}$$

Let us now check that $F = f$ on C. Since $m(f)$ is convex on $\mathbb{R}^n$ and $m(f) \leq g$, we have that $m(f) \leq F$ on $\mathbb{R}^n$ by definition of $\mathrm{conv}(g)$. On the other hand, since $g = f$ on C we have, for every convex function h with $h \leq g$, that $h \leq f$ on C, and therefore, for every $y \in C$,

$$F(y) = \sup\{h(y) \,:\, h \text{ is convex }, h \leq g\} \leq f(y) = m(f)(y).$$

This shows that $F(y) = f(y)$ for every $y \in C$.

Next let us see that we also have $\nabla F(y) = \nabla f(y)$ for every $y \in C$. In order to do so we use the following well known criterion for differentiability of convex functions, whose proof is straightforward and can be left to the interested reader.

Lemma 2.8. *If ϕ is convex, ψ is differentiable at y, $\phi \leq \psi$, and $\phi(y) = \psi(y)$, then ϕ is differentiable at y, with $\nabla\phi(y) = \nabla\psi(y)$.*

(This fact can also be phrased as: a convex function ϕ is differentiable at y if and only if ϕ is superdifferentiable at y.)

Since we know that $m(f) \leq F$, $m(f)(y) = f(y) = F(y)$ for all $y \in C$, and $F \in C^1(\mathbb{R}^n)$, it follows from this criterion (by taking $\phi = m(f)$ and $\psi = F$), and from Lemma 2.4, that

$$G(y) = \nabla f(y) = \nabla m(f)(y) = \nabla F(y) \text{ for all } y \in C.$$

Finally, note that, equation (2.18) implies (by assuming that $\varepsilon \leq k(n)\|G\|_\infty/3$, which we may do) that

$$\mathrm{Lip}(F) \leq 6\,k(n) \sup_{y\in C} |G(y)| \tag{2.19}$$

and also, assuming $0 \in C$, that

$$\|F\|_1 \leq 6\,k(n)\ \inf\{\|\varphi\|_1 : \varphi \in C^1(\mathbb{R}^n), \varphi_{|C} = f, (\nabla\varphi)_{|C} = G\}. \tag{2.20}$$

The proof of Theorem 1.7 is complete.

2.2. **Proof of Theorem 1.10.** (2) $\implies$ (1): Set $C = K \cup \{0\}$, choose a number $\alpha > 0$ sufficiently close to 1 so that

$$0 < 1 - \alpha < \min_{y\in K} \langle N(y), y\rangle$$

(this is possible thanks to condition $(\mathcal{O})$, continuity of N and compactness of K; notice in particular that $0 \notin K$), and define $f : C \to \mathbb{R}$ and $G : C \to \mathbb{R}^n$ by

$$f(y) = \begin{cases} 1 & \text{if } y \in K \\ \alpha & \text{if } y = 0, \end{cases} \quad \text{and } G(y) = \begin{cases} N(y) & \text{if } y \in K \\ 0 & \text{if } y = 0. \end{cases}$$

By using conditions $(\mathcal{K})$ and $(\mathcal{KW}^1)$, it is straightforward to check that f and G satisfy conditions (C) and (CW^1). Therefore, according to Theorem 1.7, there exists a convex function $F \in C^1(\mathbb{R}^n)$ such that $F = f$ and $\nabla F = G$ on C. Moreover, from the proof of Theorem 1.7, it is clear that F can be taken so as to satisfy $\lim_{|x|\to\infty} F(x) = \infty$. If we define $V = \{x \in \mathbb{R}^n : F(x) \leq 1\}$ we then have that V is a compact convex body with $0 \in \mathrm{int}(V)$ (because $F(0) = \alpha < 1$), and $\nabla F(y) = N(y)$ is outwardly normal to $\{x \in \mathbb{R}^n : F(x) = 1\} = \partial V$ at each $y \in K$. Moreover, $F(y) = f(y) = 1$ for each $y \in K$, hence $K \subseteq \partial V$.

(1) $\implies$ (2): Let μ_C be the Minkowski functional of C. By composing μ_C with a C^1 convex function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(t) = |t|$ if and only if $|t| \geq 1/2$, we obtain a function $F(x) := \phi(\mu_C(x))$ which is of class C^1 and convex on $\mathbb{R}^n$ and coincides with μ_C on a neighborhood of ∂C. By Theorem 1.7 we then have that F satisfies conditions (C) and (CW^1) on ∂C. On the other hand $\nabla\mu_C(y)$ is outwardly normal to ∂C at every $y \in \partial C$ and $\nabla F = \nabla\mu_C$ on ∂C, and hence we have $\nabla F(y) = N(y)$ for every $y \in \partial C$. These facts, together with the assumption $K \subseteq \partial V$, are easily checked to imply that N and K satisfy conditions $(\mathcal{K})$ and $(\mathcal{KW}^1)$. Finally, because $0 \in \mathrm{int}(C)$ and $\nabla\mu_C(x)$ is outwardly normal to ∂C for every $x \in \partial C$, we have that $\langle \nabla\mu_C(x), x\rangle > 0$ for every $x \in \partial C$, and in particular condition $(\mathcal{O})$ is satisfied as well. $\square$

Let us conclude this section with a couple of examples. We first observe that $m(f)$ need not be differentiable outside C, even in the case when C is a convex body and f is C^∞ on C.

Example 2.9. Let g be the function $g(x,y) = \max\{x+y-1, -x+y-1, \frac{1}{3}y\}$. Using for instance the smooth maxima introduced in [1], one can smooth away the edges of the graph of g produced by the intersection of the plane $z = \frac{1}{3}y$ with the planes $z = y \pm x - 1$, thus obtaining a smooth convex function f defined on $C := g^{-1}(-\infty, 0] \cap \{(x,y) : y \geq -1\}$. However, $m(f)$ will not be everywhere differentiable, because for $y \geq 2$ we have $m(f)(x,y) = \max\{x+y-1, -x+y-1\}$, and this max function is not smooth on the line $x = 0$. We leave the details to the interested reader.

The following example shows that when C has empty interior there are convex functions $f : C \to \mathbb{R}$ and continuous mappings $G : C \to \mathbb{R}^n$ which satisfy (W^1) but do not satisfy (CW^1).

Example 2.10. Let C be the segment $\{0\} \times [0,1]$ in $\mathbb{R}^2$, and f, G be defined by $f(0,y) = 0$ and $G(0,y) = (y,0)$. If we define $\widetilde{f}(x,y) = xy$ then it is clear that $\widetilde{f}$ is a C^1 extension of f to $\mathbb{R}^2$ which satisfies $\nabla\widetilde{f}(0,y) = G(0,y)$ for $(0,y) \in C$. Therefore the pair f, G satisfies Whitney's extension condition (W^1). However, since f is constant on the segment C and $G(0,1) = (1,0) \neq (0,0) = G(0,0)$, it is clear that the pair f, G does not satisfy (CW^1). In particular f does not have any convex C^1 extension F to $\mathbb{R}^n$ with $\nabla F = G$ on C.

3. Proofs of the $C^{1,\omega}$ results

3.1. **Necessity.** Let us prove the necessity of condition $(CW^{1,\omega})$ in Theorem 1.2. We will use the following.

Lemma 3.1. *Let $\omega : [0,\infty) \to [0,\infty)$ be a modulus of continuity, let a, b, η be real numbers with $a > 0$, $\eta \in (0, \frac{1}{2}]$, and define $h : [0,\infty) \to \mathbb{R}$ by*

$$h(s) = -as + b + \omega(s)s.$$

Assume that $b < \eta a\, \omega^{-1}(\eta a)$. Then we have

$$h\left(\omega^{-1}(\eta a)\right) < 0.$$

Proof. We can write

$$h\left(\omega^{-1}(\eta a)\right) = -a(1-2\eta)\omega^{-1}(\eta a) + b - \eta a\, \omega^{-1}(\eta a),$$

and the result follows at once. □

Proposition 3.2. *Let $f \in C^{1,\omega}(\mathbb{R}^n)$ be convex. Then*

$$f(x) - f(y) - \langle \nabla f(y), x-y \rangle \geq \frac{1}{2}|\nabla f(x) - \nabla f(y)|\, \omega^{-1}\left(\frac{1}{2M}|\nabla f(x) - \nabla f(y)|\right)$$

for all $x, y \in \mathbb{R}^n$, where

$$M = M(\nabla f, \mathbb{R}^n) = \sup_{x,y \in \mathbb{R}^n,\, x \neq y} \frac{|\nabla f(x) - \nabla f(y)|}{\omega\left(|x-y|\right)}.$$

In particular, the pair $(f, \nabla f)$ satisfies $(CW^{1,\omega})$, with $\eta = 1/2$, on every compact subset $C \subset \mathbb{R}^n$.

Proof. Suppose that there exist different points $x, y \in \mathbb{R}^n$ such that

$$f(x)-f(y)-\langle \nabla f(y), x-y\rangle < \frac{1}{2}|\nabla f(x)-\nabla f(y)|\,\omega^{-1}\left(\frac{1}{2M}|\nabla f(x)-\nabla f(y)|\right),$$

and we will get a contradiction.

Case 1. Assume further that $M = 1$, $f(y) = 0$, and $\nabla f(y) = 0$. By convexity this implies $f(x) \geq 0$. Then we have

$$0 \leq f(x) < \frac{1}{2}|\nabla f(x)|\,\omega^{-1}\left(\frac{1}{2}|\nabla f(x)|\right).$$

Call $a = |\nabla f(x)| > 0$, $b = f(x)$, set

$$v = -\frac{1}{|\nabla f(x)|}\nabla f(x),$$

and define

$$\varphi(t) = f(x + tv)$$

for every $t \in \mathbb{R}$. We have $\varphi(0) = b$, $\varphi'(0) = -a$, and ω is a modulus of continuity of the derivative φ'. This implies that

$$|\varphi(t) - b + at| \leq t\omega(t)$$

for every $t \in \mathbb{R}^+$, hence also that

$$\varphi(t) \leq h(t) \text{ for all } t \in \mathbb{R}^+,$$

where $h(t) = -at + b + t\omega(t)$. By assumption,

$$b < \frac{1}{2}a\omega^{-1}\left(\frac{1}{2}a\right),$$

and then Lemma 3.1 implies that

$$f\left(x + \omega^{-1}\left(\frac{1}{2}a\right)v\right) = \varphi\left(\omega^{-1}\left(\frac{1}{2}a\right)\right) \leq h\left(\omega^{-1}\left(\frac{1}{2}a\right)\right) < 0,$$

which is in contradiction with the assumptions that f is convex, $f(y) = 0$, and $\nabla f(y) = 0$. This shows that

$$f(x) \geq \frac{1}{2}|\nabla f(x)|\,\omega^{-1}\left(\frac{1}{2}|\nabla f(x)|\right).$$

Case 2. Assume only that $M = 1$. Define

$$g(z) = f(z) - f(y) - \langle \nabla f(y), z - y\rangle$$

for every $z \in \mathbb{R}^n$. Then $g(y) = 0$ and $\nabla g(y) = 0$. By Case 1, we get

$$g(x) \geq \frac{1}{2}|\nabla g(x)|\,\omega^{-1}\left(\frac{1}{2}|\nabla g(x)|\right),$$

and since $\nabla g(x) = \nabla f(x) - \nabla f(y)$ the Proposition is thus proved in the case when $M = 1$.

Case 3. In the general case, we may assume $M > 0$ (the result is trivial for $M = 0$). Consider $\psi = \frac{1}{M} f$, which satisfies the assumption of Case 2. Therefore

$$\psi(x)-\psi(y)-\langle \nabla\psi(y), x-y\rangle \geq \frac{1}{2}|\nabla\psi(x)-\nabla\psi(y)|\,\omega^{-1}\left(\frac{1}{2}|\nabla\psi(x)-\nabla\psi(y)|\right),$$

which is equivalent to the desired inequality. □

3.2. **Sufficiency.** Conversely, let us now show that condition $(CW^{1,\omega})$ is sufficient in Theorem 1.2. In the rest of this section C will be a compact subset of $\mathbb{R}^n$, and for $f : C \to \mathbb{R}$ and $G : C \to \mathbb{R}^n$ we will denote

$$m_C(f, G)(x) = \sup_{y\in C}\{f(y) + \langle G(y), x - y\rangle\}$$

for every $x \in \mathbb{R}^n$.

Lemma 3.3. *Assume that $f : C \to \mathbb{R}$ and $G : C \to \mathbb{R}^n$ satisfies $(CW^{1,\omega})$ on C. We have the following inequalities:*

(i) *For all $x, y \in C$,*

$$0 \leq f(x) - f(y) - \langle G(y), x - y\rangle \leq \langle G(y) - G(x), y - x\rangle.$$

(ii) *If $h \in \mathbb{R}^n$, $x_0, y \in C$ are such that*

$$m_C(f, G)(x_0 + h) = f(y) + \langle G(y), x_0 + h - y\rangle,$$

then

$$f(x_0) - f(y) - \langle G(y), x_0 - y\rangle \leq \langle G(y) - G(x_0), h\rangle.$$

(iii) *For $h \in \mathbb{R}^n$, $x_0, y \in C$ such that*

$$m_C(f, G)(x_0 + h) = f(y) + \langle G(y), x_0 + h - y\rangle,$$

we have that

$$|G(x_0) - G(y)| \leq \frac{2M}{\eta}\omega\left(|h|\right),$$

where we denote

$$M = M(G, C) = \sup_{x,y\in C,\, x\neq y} \frac{|G(x) - G(y)|}{\omega\left(|x - y|\right)}. \tag{3.1}$$

Proof.

(i) It is enough to use that $f(y) \geq f(x) + \langle G(x), y - x\rangle$ (which in turn follows from $(CW^{1,\omega})$).

(ii) By the definition of $m_C(f, G)$, we have that

$$f(x_0) + \langle G(x_0), h\rangle \leq f(y) + \langle G(y), x_0 + h - y\rangle.$$

Then, it is easy to deduce that

$$f(x_0) - f(y) - \langle G(y), x_0 - y\rangle \leq \langle G(y) - G(x_0), h\rangle.$$

(iii) By (ii) we have that

$$f(x_0) - f(y) - \langle G(y), x_0 - y\rangle \leq |G(y) - G(x_0)||h|,$$

and from condition $(CW^{1,\omega})$ we immediately deduce that

$$|G(x_0) - G(y)| \leq \frac{2M}{\eta}\omega\left(|h|\right).$$

□

In the sequel we assume that $f : C \to \mathbb{R}$ is a function, $G : C \to \mathbb{R}^n$ has modulus of continuity ω, and (f, G) satisfies $(CW^{1,\omega})$ on C. Since the operation of adding an affine function to a given function φ preserves both the modulus of continuity of its gradient $\nabla\varphi$ and the convexity properties that φ may have, it is obvious that it suffices to prove Theorem 1.2 in the case that $0 \in C$, $f(0) = 0$ and $G(0) = 0$. Thus we assume in what follows that $0 \in C$, $f(0) = 0$ and $G(0) = 0$.

Note that, by Remark 1.1, (f, G) also satisfies $(W^{1,\omega})$ on C. Thus, by (the proof of) Glaeser's $C^{1,\omega}$ version of Whitney's Extension Theorem (see [12, 19]), bearing in mind Lemma 3.3(i) and recalling that $f(0) = 0$ and $G(0) = 0$, we may further assume that f is extended to a function of class $C^{1,\omega}(\mathbb{R}^n)$, which we will keep denoting f, such that $\nabla f = G$ on C, and

$$M(\nabla f, \mathbb{R}^n) = \sup_{x,y\in\mathbb{R}^n,\, x\neq y} \frac{|\nabla f(x) - \nabla f(y)|}{\omega\left(|x - y|\right)} \leq c(n)M, \tag{3.2}$$

where M is defined in (3.1) and $c(n) > 0$ is a constant depending only on the dimension n. Using again that $f(0) = 0$ and $G(0) = 0$, it is not difficult to see that f can even be assumed to have compact support (keeping an estimation of the type (3.2), with $c(n)$ independent of f and C, of course). This can be checked either by multiplying f by a suitable cut-off function and make some tedious but standard calculations, or directly by defining f as in equation (2.2), but summing only on those cubes Q_k for which $\operatorname{diam}(Q_k) \leq R$, where $R > 0$ is such that $C \subset B(0, R)$, and then making some estimations similar to those in the proof of the Whitney extension theorem for $C^{1,\omega}$ functions.

Lastly, since the case $M = 0$ is trivial, we also assume $M > 0$ in what follows. We will also denote

$$m_C(f)(x) = m_C(f, \nabla f)(x) \tag{3.3}$$

for brevity. As we already noted in Section 2, the function $m_C(f) : \mathbb{R}^n \to \mathbb{R}$ is convex and Lipschitz.

Lemma 3.4. *Under the above assumptions, we have:*

$$|f(x) - m_C(f)(x)| \leq \left(c(n) + \frac{2}{\eta}\right) M\omega(d(x, C))d(x, C), \quad \textit{for all} \quad x \in \mathbb{R}^n.$$

Proof. Let $x \in \mathbb{R}^n$, $x_0 \in C$ be such that $|x - x_0| = d(x, C)$, and take a point $y \in C$ such that $m_C(f)(x) = f(y) + \langle \nabla f(y), x - y\rangle$. We have

$$|f(x) - m_C(f)(x)| \leq$$
$$|f(x) - f(x_0) - \langle \nabla f(x_0), x - x_0\rangle| + |m_C(f)(x) - f(x_0) - \langle \nabla f(x_0), x - x_0\rangle|.$$

Using the mean value theorem and (3.2), the first term satisfies

$$\begin{aligned}&|f(x) - f(x_0) - \langle \nabla f(x_0), x - x_0\rangle| \leq M(\nabla f, \mathbb{R}^n)\omega(|x - x_0|)|x - x_0| \\ &\leq c(n)M\omega(d(x, C))\, d(x, C).\end{aligned}$$

The second term can be estimated as follows:

$$\begin{aligned}0 &\leq m_C(f)(x) - f(x_0) - \langle \nabla f(x_0), x - x_0\rangle \\ &= f(y) + \langle \nabla f(y), x - y\rangle - f(x_0) - \langle \nabla f(x_0), x - x_0\rangle \\ &\leq f(y) + \langle \nabla f(y), x - y\rangle - f(y) - \langle \nabla f(y), x_0 - y\rangle - \langle \nabla f(x_0), x - x_0\rangle \\ &= \langle \nabla f(y) - \nabla f(x_0), x - x_0\rangle \leq |\nabla f(y) - \nabla f(x_0)||x - x_0|\end{aligned}$$

By Lemma 3.3 (iii) we have

$$|\nabla f(y) - \nabla f(x_0)| \leq \frac{2M}{\eta}\omega(|x - x_0|) = \frac{2M}{\eta}\omega(d(x, C)).$$

This implies that

$$0 \leq m_C(f)(x) - f(x_0) - \langle \nabla f(x_0), x - x_0\rangle \leq \frac{2M}{\eta}\omega(d(x, C))d(x, C).$$

In particular, $m_C(f)$ is ω-differentiable at all points $x_0 \in C$, with $\nabla m_C(f)(x_0) = \nabla f(x_0)$. [1] Therefore

$$|f(x) - m_C(f)(x)| \leq \left(c(n) + \frac{2}{\eta}\right) M\omega(d(x, C))d(x, C).$$

□

Our next step is constructing a function φ of class $C^{1,\omega}(\mathbb{R}^n)$ which vanishes on C and is greater than or equal to the function $|f - m_C(f)|$ on $\mathbb{R}^n \setminus C$. To this purpose we need to use a Whitney decomposition of an open set into cubes and the corresponding partition of unity, see [19, Chapter VI] for an exposition of this technique. So let $\{Q_k\}_k$ be a decomposition of $\mathbb{R}^n \setminus C$ into Whitney cubes, and for a fixed number $0 < \varepsilon_0 < 1/4$, consider the corresponding cubes $\{Q_k^*\}_k$ with the same center as Q_k and dilated by the factor $1 + \varepsilon_0$. We next sum up some of the most important properties of this cubes.

Proposition 3.5. *The families $\{Q_k\}_k$ and $\{Q_k^*\}_k$ are sequences of compact cubes for which:*

(i) $\bigcup_k Q_k = \mathbb{R}^n \setminus C$.

[1] Recall that a function ψ is said to be ω-differentiable on C provided there exists a constant $K > 0$ such that $|\psi(x) - \psi(y) - \langle \nabla\psi(y), x - y\rangle| \leq K|x - y|\,\omega(|x - y|)$ for every $y \in C$ and every x in a neighbourhood of C.

(ii) *The interiors of* Q_k *are mutually disjoint.*
(iii) $\operatorname{diam}(Q_k) \leq d(Q_k, F) \leq 4 \operatorname{diam}(Q_k)$ *for all* k.
(iv) *If two cubes* Q_l *and* Q_k *touch each other, that is* $\partial Q_l \cap \partial Q_k \neq \emptyset$, *then* $\operatorname{diam}(Q_l) \approx \operatorname{diam}(Q_k)$.
(v) *Every point of* $\mathbb{R}^n \setminus C$ *is contained in an open neighbourhood which intersects at most* $N = (12)^n$ *cubes of the family* $\{Q_k^*\}_k$.
(vi) *If* $x \in Q_k$, *then* $d(x, C) \leq 5 \operatorname{diam}(Q_k)$.
(vii) *If* $x \in Q_k^*$, *then* $\frac{3}{4} \operatorname{diam}(Q_k) \leq d(x, C) \leq (6 + \varepsilon_0) \operatorname{diam}(Q_k)$ *and in particular* $Q_k^* \subset \mathbb{R}^n \setminus C$.
(viii) *If two cubes* Q_l^* *and* Q_k^* *are not disjoint, then* $\operatorname{diam}(Q_l) \approx \operatorname{diam}(Q_k)$.

Here the notation $A_k \approx B_l$ means that there exist positive constants γ, Γ, depending only on the dimension n, such that $\gamma A_k \leq B_l \leq \Gamma A_k$ for all k, l satisfying the properties specified in each case.

The following Proposition summarizes the basic properties of the Whitney partition of unity associated to these cubes.

Proposition 3.6. *There exists a sequence of functions* $\{\varphi_k\}_k$ *defined on* $\mathbb{R}^n \setminus C$ *such that*

(i) $\varphi_k \in C^\infty(\mathbb{R}^n \setminus C)$.
(ii) $0 \leq \varphi_k \leq 1$ *on* $\mathbb{R}^n \setminus C$ *and* $\operatorname{supp}(\varphi_k) \subseteq Q_k^*$.
(iii) $\sum_k \varphi_k = 1$ *on* $\mathbb{R}^n \setminus C$.
(vi) *For all multiindex* α *there exists a constant* $A_\alpha > 0$, *depending only on* α *and on the dimension* n, *such that*

$$|D^\alpha \varphi_k(x)| \leq A_\alpha \operatorname{diam}(Q_k)^{-|\alpha|},$$

for all $x \in \mathbb{R}^n \setminus C$ *and for all* k.

The statements contained in the following Proposition must look fairly obvious to those readers well acquainted with Whitney's techniques, but we have not been able to find an explicit reference, so we include a proof for the general reader's convenience.

Proposition 3.7. *Consider the family of cubes* $\{Q_k\}_k$ *asociated to* $\mathbb{R}^n \setminus C$ *and its partition of unity* $\{\varphi_k\}_k$ *as in the preceding Propositions.*

(i) *Suppose that there is a sequence of nonnegative numbers* $\{p_k\}_k$ *and a positive constant* $\lambda > 0$ *such that* $p_k \leq \lambda\, \omega(\operatorname{diam}(Q_k)) \operatorname{diam}(Q_k)$, *for every* k. *Then, the function defined by*

$$\varphi(x) = \begin{cases} \sum_k p_k\, \varphi_k(x) & \text{if} \quad x \in \mathbb{R}^n \setminus C, \\ 0 & \text{if} \quad x \in C \end{cases}$$

is of class $C^{1,\omega}(\mathbb{R}^n)$ *and there is a constant* $\gamma(n) > 0$, *depending only on the dimension* n, *such that*

$$M(\nabla\varphi, \mathbb{R}^n) := \sup_{x,y \in \mathbb{R}^n,\, x \neq y} \frac{|\nabla\varphi(x) - \nabla\varphi(y)|}{\omega\left(|x-y|\right)} \leq \gamma(n)\lambda. \tag{3.4}$$

(ii) *There exists a function* $\Phi \in C^{1,\omega}(\mathbb{R}^n)$ *such that* $\Phi = 0$ *on* C *and*

$$\frac{\omega(d(x,C))d(x,C)}{6+\varepsilon_0} \leq \Phi(x) \leq \frac{(6+\varepsilon_0)16}{9}\omega(d(x,C))d(x,C), \quad \textit{for all} \quad x \in \mathbb{R}^n.$$

In addition, we have:

$$M(\nabla\Phi, \mathbb{R}^n) = \sup_{x,y\in\mathbb{R}^n,\, x\neq y} \frac{|\nabla\Phi(x) - \nabla\Phi(y)|}{\omega\left(|x-y|\right)} \leq (6+\varepsilon_0)\gamma(n).$$

Proof. Let us start with the proof of (i). Since every point in $\mathbb{R}^n \setminus C$ has an open neighbourhood which intersects at most $N = (12)^n$ cubes, and all the functions φ_k are of class C^∞, it is clear that $\varphi \in C^\infty(\mathbb{R}^n \setminus C)$. Given a point $x \in \mathbb{R}^n$, by our assumptions on the sequence $\{p_k\}_k$, we easily have

$$\varphi(x) = \sum_{Q_k^* \ni x} p_k \varphi_k(x) \leq \lambda \sum_{Q_k^* \ni x} \omega(\operatorname{diam}(Q_k)) \operatorname{diam}(Q_k)\varphi_k(x).$$

Using Proposition 3.5 we have that $\operatorname{diam}(Q_k) \leq \frac{4}{3}d(x,C)$ for those k such that $x \in Q_k^*$. Then, we can write

$$\varphi(x) \leq \lambda \sum_{Q_k^* \ni x} \omega\left(\frac{4}{3}d(x,C)\right)\frac{4}{3}d(x,C)\varphi_k(x) \leq$$
$$\lambda\left(\frac{4}{3}\right)^2 \omega(d(x,C))d(x,C) \sum_{Q_k^* \ni x} \varphi_k(x) = \left(\frac{4}{3}\right)^2 \lambda\omega(d(x,C))d(x,C).$$

In particular, due to the fact that $\omega(0^+) = 0$, the above estimation shows that φ is differentiable on C with $\nabla\varphi = 0$ on C. We next give an estimation for $\nabla\varphi$. Bearing in mind Proposition 3.6, we set

$$A = A(n) := \max\{\sqrt{n}A_\alpha,\ |\alpha| \leq 2\}. \tag{3.5}$$

Given a point $x \in \mathbb{R}^n$, we have that

$$|\nabla\varphi(x)| \leq \sum_{Q_k^* \ni x} p_k |\nabla\varphi_k(x)| \leq A \sum_{Q_k^* \ni x} p_k \operatorname{diam}(Q_k)^{-1} \leq A\lambda \sum_{Q_k^* \ni x} \omega(\operatorname{diam}(Q_k))$$
$$\leq A\,\lambda \sum_{Q_k^* \ni x} \omega\left(\frac{4}{3}d(x,C)\right) \leq \frac{4AN}{3}\lambda\,\omega(d(x,C)).$$

Summing up,

$$|\nabla\varphi(x)| \leq \frac{4AN}{3}\lambda\,\omega(d(x,C)) \quad \text{for every} \quad x \in \mathbb{R}^n. \tag{3.6}$$

Note that the above shows inequality (3.4) when $x \in \mathbb{R}^n$ and $y \in C$. Hence, in the rest of the proof we only have consider the situation where x, y are such that $x \neq y$ and $x, y \in \mathbb{R}^n \setminus C$. However, we will still have to separately consider two cases. Let us denote $L := [x, y]$, the straight line connecting x to y.

Case 1: $d(L,C) \geq |x-y|$. Take a multiindex α with $|\alpha| = 1$. Because in this case the segment L is necessarily contained in $\mathbb{R}^n \setminus C$, and the function φ is of class C^2 on $\mathbb{R}^n \setminus C$, we can write

$$|D^\alpha \varphi(x) - D^\alpha \varphi(y)| \leq \left(\sup_{z \in L} |\nabla (D^\alpha \varphi)(z)| \right) |x-y|. \tag{3.7}$$

Using Proposition 3.6 and the definition of A in (3.5), we may write

$$|\nabla (D^\alpha \varphi)(z)| \leq \sum_{Q_k^* \ni z} p_k |\nabla (D^\alpha \varphi_k)(z)| \leq A \sum_{Q_k^* \ni z} p_k \operatorname{diam}(Q_k)^{-2} \leq A\lambda \sum_{Q_k^* \ni z} \frac{\omega(\operatorname{diam}(Q_k))}{\operatorname{diam}(Q_k)}.$$

By Proposition 3.5, we have that $(6+\varepsilon_0)\operatorname{diam}(Q_k) \geq d(z,C) \geq d(L,C) \geq |x-y|$ for those k with $Q_k^* \ni z$ and by the properties of ω, we have that

$$A\lambda \sum_{Q_k^* \ni z} \frac{\omega(\operatorname{diam}(Q_k))}{\operatorname{diam}(Q_k)} \leq A\lambda \sum_{Q_k^* \ni z} \frac{\omega\left(\frac{|x-y|}{6+\varepsilon_0}\right)}{\frac{|x-y|}{6+\varepsilon_0}} \leq (6+\varepsilon_0) AN\lambda \frac{\omega(|x-y|)}{|x-y|}.$$

Therefore, by substituting in (3.7) we find that

$$|\nabla \varphi(x) - \nabla \varphi(y)| \leq (6+\varepsilon_0)\sqrt{n} AN\lambda \omega(|x-y|).$$

Case 2: $d(L,C) \leq |x-y|$. Take points $x' \in L$ and $y' \in C$ such that $d(L,C) = |x'-y'| \leq |x-y|$. We have that

$$|x-y'| \leq |x-x'| + |y'-x'| \leq |x-y| + |x'-y'| \leq 2|x-y|,$$

and similarly we obtain $|y-y'| \leq 2|x-y|$. Hence, if we use (3.6) we obtain

$$|\nabla \varphi(x) - \nabla \varphi(y)| \leq |\nabla \varphi(x)| + |\nabla \varphi(y)| \leq \frac{4AN}{3}\lambda \left(\omega(d(x,C)) + \omega(d(y,C))\right)$$
$$\leq \frac{4AN}{3}\lambda \left(\omega(|x-y'|) + \omega(|y-y'|)\right) \leq \frac{8AN}{3}\lambda\omega(2|x-y|) \leq \frac{16AN}{3}\lambda\omega(|x-y|).$$

If we call

$$\gamma(n) = \max\left\{ \frac{4AN}{3}, (6+\varepsilon_0)\sqrt{n}AN, \frac{16AN}{3} \right\} = (6+\varepsilon_0)\sqrt{n}AN,$$

we get (3.4).

Now let us prove the part (ii). Define the sequence

$$p_k = \omega((6+\varepsilon_0)\operatorname{diam}(Q_k))\operatorname{diam}(Q_k), \quad \text{for all} \quad k.$$

We have that $p_k \leq (6+\varepsilon_0)\omega(\operatorname{diam}(Q_k))\operatorname{diam}(Q_k)$ and, by (i), the function defined by

$$\Phi(x) = \begin{cases} \sum_k p_k \varphi_k(x) & \text{if} \quad x \in \mathbb{R}^n \setminus C, \\ 0 & \text{if} \quad x \in C \end{cases}$$

is of class $C^{1,\omega}(\mathbb{R}^n)$, vanishes on C, and satisfies $M(\nabla\Phi,\mathbb{R}^n)\leq(6+\varepsilon_0)\gamma(n)$ (that is, the second part of (ii)). Given a point $x\in\mathbb{R}^n\setminus C$, we know that $(6+\varepsilon_0)\operatorname{diam}(Q_k)\geq d(x,C)$ if $x\in Q_k^*$. Then

$$\Phi(x)=\sum_{Q_k^*\ni x}p_k\,\varphi_k(x)\geq\frac{\omega(d(x,C))d(x,C)}{6+\varepsilon_0}\sum_{Q_k^*\ni x}\varphi_k(x)=\frac{\omega(d(x,C))d(x,C)}{6+\varepsilon_0}.$$

On the other hand, from Proposition 3.5, we have that $\operatorname{diam}(Q_k)\leq\frac{4}{3}d(x,C)$ if $x\in Q_k^*$. This yields

$$\begin{aligned}\Phi(x)=\sum_{Q_k^*\ni x}p_k\,\varphi_k(x)&\leq(6+\varepsilon_0)\omega\left(\frac{4}{3}d(x,C)\right)\left(\frac{4}{3}d(x,C)\right)\\&\leq\frac{(6+\varepsilon_0)16}{9}\omega(d(x,C))d(x,C),\end{aligned}$$

hence the first inequality in (ii) is proved. □

Let us continue with our proof. Consider the numbers

$$p_k:=\sup_{x\in Q_k^*}\{|f(x)-m_C(f)(x)|\},\quad\text{for every}\quad k.$$

By Lemma 3.4,

$$|f(x)-m_C(f)(x)|\leq M^*\omega(d(x,C))d(x,C),\tag{3.8}$$

where

$$M^*=(c(n)+2\eta^{-1})M,\tag{3.9}$$

and if k is such that $x\in Q_k^*$ then the right-hand side of (3.8) is smaller than or equal to

$$M^*\omega((6+\varepsilon_0)\operatorname{diam}(Q_k))((6+\varepsilon_0)\operatorname{diam}(Q_k))\leq(6+\varepsilon_0)^2M^*\omega(\operatorname{diam}(Q_k))\operatorname{diam}(Q_k).$$

This shows that

$$p_k\leq(6+\varepsilon_0)^2M^*\omega(\operatorname{diam}(Q_k))\operatorname{diam}(Q_k)\quad\text{for every}\quad k,$$

and, by Theorem 3.7, the function defined by

$$\varphi(x)=\begin{cases}\sum_k p_k\,\varphi_k(x) & \text{if}\quad x\in\mathbb{R}^n\setminus C,\\ 0 & \text{if}\quad x\in C\end{cases}$$

is of class $C^{1,\omega}(\mathbb{R}^n)$ and

$$M(\nabla\varphi,\mathbb{R}^n)\leq\gamma(n)(6+\varepsilon_0)^2M^*=\gamma(n)(6+\varepsilon_0)^2(c(n)+2\eta^{-1})M.$$

We claim that $\varphi\geq|f-m_C(f)|$. Indeed, if $x\in C$, there is nothing to say. If $x\in\mathbb{R}^n\setminus C$, by the definition of the p_k's we easily have

$$\varphi(x)=\sum_{Q_k^*\ni x}p_k\,\varphi_k(x)\geq\sum_{Q_k^*\ni x}|f(x)-m_C(f)(x)|\varphi_k(x)=|f(x)-m_C(f)(x)|.$$

Now, making use of Proposition 3.7(ii), we obtain a function $\Phi \in C^{1,\omega}(\mathbb{R}^n)$ such that Φ vanishes on C,

$$\Phi(x) \geq \frac{\omega(d(x,C))d(x,C)}{6+\varepsilon_0} \quad \text{for all} \quad x \in \mathbb{R}^n,$$

and $M(\nabla\Phi, \mathbb{R}^n) \leq (6+\varepsilon_0)\gamma(n)$. Let us consider the function $g = f+\varphi+M\Phi$. Our function g is clearly of class $C^{1,\omega}(\mathbb{R}^n)$ and satisfies $g = f$, $\nabla g = \nabla f$ on C. Using that $\varphi \geq |f - m_C(f)|$, we easily obtain that

$$g(x) \geq f(x) + |m_C(f)(x) - f(x)| + M\Phi(x) \geq m_C(f)(x),$$

that is, $g \geq m_C(f)$ on $\mathbb{R}^n$. Since f has compact support and $\varphi \geq 0$, we also have that

$$g(x) \geq -\|f\|_\infty + M\Phi(x) \geq -\|f\|_\infty + M\frac{\omega(d(x,C))d(x,C)}{6+\varepsilon_0},$$

which tends to $+\infty$ as $|x|$ goes to $+\infty$. Hence $\lim_{|x|\to+\infty} g(x) = +\infty$. We can also estimate $M(\nabla g, \mathbb{R}^n)$ as follows,

$$\begin{aligned} M(\nabla g, \mathbb{R}^n) &\leq M(\nabla f, \mathbb{R}^n) + M(\nabla\varphi, \mathbb{R}^n) + M \cdot M(\nabla\Phi, \mathbb{R}^n) \\ &\leq c(n)M + \gamma(n)(6+\varepsilon_0)^2(c(n)+2\eta^{-1})M + (6+\varepsilon_0)\gamma(n)M \\ &= \mu(n,\eta)M, \end{aligned}$$

where

$$\mu(n,\eta) := c(n) + \gamma(n)(6+\varepsilon_0)^2(c(n)+2\eta^{-1}) + (6+\varepsilon_0)\gamma(n). \tag{3.10}$$

Finally, let us define $F := \mathrm{conv}(g)$, the convex envelope of the function g, that is

$$\mathrm{conv}(g)(x) = \sup\{h(x) \,:\, h \text{ is convex }, h \leq g\}, \quad x \in \mathbb{R}^n.$$

In order to finish our proof we need to use the following result of Kirchheim and Kristensen on $C^{1,\omega}$ smoothness of convex envelopes.

Theorem 3.8 (Kirchheim-Kristensen)**.** *If $\psi : \mathbb{R}^n \to \mathbb{R}$ is an ω-differentiable function on $\mathbb{R}^n$ such that $\lim_{|x|\to+\infty} \psi(x) = +\infty$, then conv$(\psi)$ is a convex function of class $C^{1,\omega}(\mathbb{R}^n)$, and*

$$M(\nabla\, conv(\psi), \mathbb{R}^n) \leq 4(n+1)M(\nabla\psi, \mathbb{R}^n).$$

As a matter of fact, Theorem 3.8 is a restatement of a particular case of Kirchheim and Kristensen's result in [15], and the estimation on the modulus of continuity of the gradient of the convex envelope does not appear in their original statement, but it does follow, with some extra work, from their proof.

According to Theorem 3.8, our function F is a convex function of class $C^{1,\omega}(\mathbb{R}^n)$. It only remains to see that $F = f$ on C and $\nabla F = \nabla f$ on C. Using the fact that $g \geq m_C(f)$ and $m_C(f)$ is convex, we must have $F \geq m_C(f)$. On the other hand, because $g = f$ on C, for every $x \in C$ we have $F(x) \leq g(x) = f(x) = m_C(f)(x)$. Hence $F = m_C(f) = f$ on C. On the

other hand, since we know that $m_C(f) \leq F$, $m_C(f)(y) = f(y) = F(y)$ for all $y \in C$, and $F \in C^1(\mathbb{R}^n)$, it follows from Lemma 2.8 that

$$\nabla f(y) = \nabla m_C(f)(y) = \nabla F(y) \text{ for all } y \in C.$$

As for the control of the modulus of continuity of the extension, since $M(\nabla g, \mathbb{R}^n) \leq \mu(n,\eta)M$ (see the inequalities before (3.10)), Theorem 3.8 gives us

$$M(\nabla F, \mathbb{R}^n) \leq 4(n+1)\mu(n,\eta)M.$$

Assuming $\eta = 1/2$, let us denote $k(n) = 4(n+1)\mu(n, 1/2)$. We have thus proved that

$$M(\nabla F, \mathbb{R}^n) \leq k(n)M(G, C),$$

where $k(n) > 0$ only depends on the dimension n. Finally, in terms of the norm defined in (1.3), assuming $0 \in C$ and $\eta = 1/2$, we have that $F(0) = f(0)$ and $\nabla F(0) = G(0)$, and therefore

$$\|F\|_{1,\omega} = |F(0)| + |\nabla F(0)| + M(\nabla F, \mathbb{R}^n) \leq k(n)\left(|f(0)| + |G(0)| + M(G, C)\right).$$

3.3. **Proof of Theorem 1.9.** (2) $\implies$ (1): Set $C = K \cup \{0\}$. We may assume $\eta > 0$ small enough, and choose $\alpha > 0$ sufficiently close to 1, so that

$$0 < 1 - \alpha + \frac{\eta}{2M} < \min_{y \in K} \langle N(y), y \rangle$$

(this is possible thanks to condition $(\mathcal{O})$, M-Lipschitzness of N and compactness of K; notice in particular that $0 \notin K$). Now define $f : C \to \mathbb{R}$ and $G : C \to \mathbb{R}^n$ by

$$f(y) = \begin{cases} 1 & \text{if } y \in K \\ \alpha & \text{if } y = 0, \end{cases} \quad \text{and } G(y) = \begin{cases} N(y) & \text{if } y \in K \\ 0 & \text{if } y = 0. \end{cases}$$

By using condition $(\mathcal{KW}^{1,1})$, it is straightforward to check that (f, G) satisfies condition $(CW^{1,1})$. Therefore, according to Theorem 1.3, there exists a convex function $F \in C^{1,1}(\mathbb{R}^n)$ such that $F = f$ and $\nabla F = G$ on C, and the proof of Theorem 1.3 indicates that F can be taken so as to satisfy $\lim_{|x| \to \infty} F(x) = \infty$. We now define $V = \{x \in \mathbb{R}^n : F(x) \leq 1\}$. The rest of the proof is similar to that of Theorem 1.10, and we leave it to the reader's care. □

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