KIRSZBRAUN’S THEOREM VIA AN EXPLICIT FORMULA

D. AZAGRA, E. LE GRUYER, AND C. MUDARRA

Abstract. Let $X, Y$ be two Hilbert spaces, $E$ a subset of $X$ and $G : E \to Y$ a Lipschitz mapping.
A famous theorem of Kirszbraun’s states that there exists $\tilde{G} : X \to Y$ with $\tilde{G} = G$ on $E$ and $\text{Lip}(\tilde{G}) = \text{Lip}(G)$. In this note we show that in fact the function

$$
\tilde{G} := \nabla_{Y} (\text{conv}(g))(\cdot, 0),
$$

where

$$
g(x, y) = \inf_{z \in E} \{ \langle G(z), y \rangle + \frac{M}{2} \| (x - z, y) \|^2 \} + \frac{M}{2} \| (x, y) \|^2,
$$

defines such an extension.

In 1934 M.D. Kirszbraun \cite{15} proved that, for every subset $E$ of $\mathbb{R}^n$ and every Lipschitz function $f : E \to \mathbb{R}^m$, there exists a Lipschitz extension $F : \mathbb{R}^n \to \mathbb{R}^m$ of $f$ such that $\text{Lip}(F) = \text{Lip}(f)$. Here $\text{Lip}(\varphi)$ denotes the Lipschitz constant of $\varphi$, that is,

$$
\text{Lip}(\varphi) = \sup_{x \neq y} \frac{\| \varphi(x) - \varphi(y) \|}{\| x - y \|}.
$$

This theorem was generalized for Hilbert spaces $X, Y$ in place of $\mathbb{R}^n$ and $\mathbb{R}^m$ by F.A. Valentine \cite{21} in 1945, and the result is often referred to as the Kirszbraun-Valentine theorem. The proof is rather nonconstructive, in the sense that it requires to use Zorn’s lemma or transfinite induction at least in the nonseparable case. In the separable case the proof can be made by induction, considering a dense sequence $\{ x_k \}$ in $X$ and at each step managing to extend $f$ from $E \cup \{ x_1, \ldots, x_m \}$ to $E \cup \{ x_1, \ldots, x_{m+1} \}$ while preserving the Lipschitz constant of the extension by using Helly’s theorem or intersection properties of families of balls, but still it is not clear what the extension looks like. Several other proofs and generalizations which are not constructive either have appeared in the literature; see \cite{18, 14, 10, 6, 19, 3}; apart from Zorn’s lemma or induction these proofs are based on intersection properties of arbitrary families of balls, or on maximal extensions of non-expansive operators and Fitzpatrick functions. In 2008 H.H. Bauschke and X. Wang \cite{5} gave the first constructive proof of the Kirszbraun-Valentine theorem of which we are aware; they relied on their previous work \cite{4} on extension and representation of monotone operators and the Fitzpatrick function. See also \cite{11}, where some of these techniques are used to construct definable versions of Helly’s and Kirszbraun theorems in arbitrary definably complete expansions of ordered fields. Finally, in 2015 E. Le Gruyer and T-V. Phan provided sup-inf explicit extension formulas for Lipschitz mappings between finite dimensional spaces by relying on Le Gruyer’s solution to the minimal $C^{1,1}$ extension problem for 1-jets; see \cite{17} Theorem 32 and 33] and \cite{16}.

In this note we present a short proof of the Kirszbraun-Valentine theorem in which the extension is given by an explicit formula. This proof is based on our previous work concerning $C^{1,1}$ extensions of 1-jets with optimal Lipschitz constants of the gradients \cite{2}. See \cite{9} for an alternative construction of such $C^{1,1}$ extensions on the Hilbert space, and \cite{7, 11, 12, 13} for the much difficult question of extending functions (as opposed to jets) to $C^{1,1}$ or $C^{m,1}$ functions on $\mathbb{R}^n$.

Date: Oct 01, 2018.

2010 Mathematics Subject Classification. 54C20, 52A41, 26B05, 53A99, 53C45, 52A20, 58C25, 35J96.

Key words and phrases. Lipschitz function, Kirszbraun Theorem.
If $X$ is a Hilbert space, $E \subset X$ is an arbitrary subset and $(f,G) : E \to \mathbb{R} \times X$ is a 1-jet on $E$, we will say that $(f,G)$ satisfies condition $(W^{1,1})$ with constant $M > 0$ on $E$ provided that

\[ f(y) \leq f(x) + \frac{1}{2} \langle G(x) + G(y), y - x \rangle + \frac{M}{4} \|x - y\|^2 - \frac{1}{4M} \|G(x) - G(y)\|^2, \quad \text{for all } x, y \in E. \]

In [22] [16] it was proved that condition $(W^{1,1})$ with constant $M > 0$ is a necessary and sufficient condition on $f : E \to \mathbb{R}$, $G : E \to X$ for the existence of a function $F \in C^{1,1}(X)$ with $\text{Lip}(\nabla F) \leq M$ and such that $F = f$ and $\nabla F = G$ on $E$. More recently, as a consequence of a similar extension theorem for $C^{1,1}$ convex functions, we have found an explicit formula for such an extension $F$.

**Theorem 1.** [2] Theorem 3.4] Let $E$ be a subset of a Hilbert space $X$. Given a 1-jet $(f,G)$ satisfying condition $(W^{1,1})$ with constant $M$ on $E$, the formula

\[ F = \text{conv}(g) - \frac{M}{2}\| \cdot \|^2, \]

\[ g(x) = \inf_{y \in E} \{ f(y) + \langle G(y), x - y \rangle + \frac{M}{2}\|x - y\|^2 \} + \frac{M}{2}\|x\|^2, \quad x \in X, \]

defines a $C^{1,1}(X)$ function with $F|_E = f$, $(\nabla F)|_E = G$, and $\text{Lip}(\nabla F) \leq M$.

Here $\text{conv}(g)$ denotes the convex envelope of $g$, defined by

\[ \text{conv}(g)(x) = \sup \{ h(x) : h \text{ is convex, proper and lower semicontinuous, } h \leq g \}. \]

Another expression for $\text{conv}(g)$ is given by

\[ \text{conv}(g)(x) = \inf \left\{ \sum_{j=1}^{k} \lambda_j g(x_j) : \lambda_j \geq 0, \sum_{j=1}^{k} \lambda_j = 1, x = \sum_{j=1}^{k} \lambda_j x_j, k \in \mathbb{N} \right\}, \]

and also by the Fenchel biconjugate of $g$, that is,

\[ \text{conv}(g) = g^{**}, \]

where

\[ h^*(x) := \sup_{v \in X} \{ \langle v, x \rangle - h(v) \}; \]

see [8] Proposition 4.4.3 for instance. In the case that $X$ is finite dimensional, say $X = \mathbb{R}^n$, the expression (3) can be made simpler: by using Carathéodory’s Theorem one can show that it is enough to consider convex combinations of at most $n + 1$ points. That is to say, if $g : \mathbb{R}^n \to \mathbb{R}$ then

\[ \text{conv}(g)(x) = \inf \left\{ \sum_{j=1}^{n+1} \lambda_j g(x_j) : \lambda_j \geq 0, \sum_{j=1}^{n+1} \lambda_j = 1, x = \sum_{j=1}^{n+1} \lambda_j x_j \right\}; \]

see [20] Corollary 17.1.5 for instance.

**Theorem 2** (Kirszhbraun’s theorem via an explicit formula). Let $X,Y$ be two Hilbert spaces, $E$ a subset of $X$ and $G : E \to Y$ a Lipschitz mapping. There exists $\tilde{G} : X \to Y$ with $\tilde{G} = G$ on $E$ and $\text{Lip}(\tilde{G}) = \text{Lip}(G)$. In fact, if $M = \text{Lip}(G)$, then the function

\[ \tilde{G}(x) := \nabla_Y \left( \text{conv}(g) \right)(x,0), \quad x \in X, \]

where

\[ g(x,y) = \inf_{z \in E} \{ \langle G(z), y \rangle_Y + \frac{M}{2}\|x - z\|^2_X + \frac{M}{2}\|x\|^2_X + M\|y\|^2_Y \}, \quad (x,y) \in X \times Y, \]

defines such an extension.
Here $\| \cdot \|_X$ and $\| \cdot \|_Y$ denote the norm on $X$ and $Y$ respectively. Also, the inner products in $X$ and $Y$ are denoted by $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Y$ respectively. For any function $F$, $\nabla_Y F$ will stand for the $Y$-partial derivatives of $F$, that is the canonical projection from $X \times Y$ onto $Y$ composed with $\nabla F$.

Proof. We consider on $X \times Y$ the norm given by $\|(x,y)\| = \sqrt{\|(x)\|_X^2 + \|(y)\|_Y^2}$ for every $(x,y) \in X \times Y$. Then $X \times Y$ is a Hilbert space whose inner product is $\langle (x,y), (x',y') \rangle = \langle x, x' \rangle_X + \langle y, y' \rangle_Y$ for every $(x,y), (x',y') \in X \times Y$. We define the 1-jet $(f^*, G^*)$ on $E \times \{0\} \subset X \times Y$ by $f^*(x,0) = 0$ and $G^*(x,0) = (0,G(x))$. Using that $G$ is $M$-Lipschitz, it is easy to see that $(f^*, G^*)$ satisfies condition $(W^{1,1})$ on $E \times \{0\}$ with constant $M$, see inequality [9]. Therefore, Theorem 1 asserts that the function $F$ defined by

$$F = \text{conv}(g) - \frac{M}{2} \| \cdot \|_2^2,$$

where

$$g(x,y) = \inf_{z \in E} \{ f^*(z,0) + \langle G^*(z,0), (x-z,y) \rangle + \frac{M}{2} \|(x-z,y)\|_2^2 \} + \frac{M}{2} \|(x,y)\|_2^2,$$

is of class $C^{1,1}(X \times Y)$ with $(F, \nabla F) = (f^*, G^*)$ on $E \times \{0\}$ and $\text{Lip}(\nabla F) \leq M$. In particular, the mapping $X \ni x \mapsto \tilde{G}(x) := \nabla_Y F(x,0) \in Y$ is $M$-Lipschitz and extends $G$ from $E$ to $X$. Finally, the expressions defining $G$ and $g$ can be simplified as

$$\tilde{G}(x) = \nabla_Y \left( \text{conv}(g) - \frac{M}{2} \| \cdot \|_2^2 \right) (x,0) = \nabla_Y \text{conv}(g)(x,0) - \nabla_Y \left( \frac{M}{2} \| \cdot \|_2^2 \right) (x,0) = \nabla_Y (\text{conv}(g))(x,0)$$

and

$$g(x,y) = \inf_{z \in E} \left\{ \langle G(z), y \rangle_Y + \frac{M}{2} \| x - z \|_X^2 \} + \frac{M}{2} \| x \|_X^2 + M \| y \|_Y^2 \right\}.$$

References


ICMAT (CSIC-UAM-UC3-UCM), Departamento de Análisis Matemático y Matemática Aplicada, Facultad de Ciencias Matemáticas, Universidad Complutense, 28040, Madrid, Spain. DISCLAIMER: The first-named author is affiliated with Universidad Complutense de Madrid, but this does not mean this institution has offered him all the support he expected. On the contrary, the Biblioteca Complutense has hampered his research by restricting his access to books.
E-mail address: azagra@mat.ucm.es

INSA de Rennes & IRMAR, 20, Avenue des Buttes de Coësmes, CS 70839 F-35708, Rennes Cedex 7, France
E-mail address: Erwan.Le-Gruyer@insa-rennes.fr

ICMAT (CSIC-UAM-UC3-UCM), Calle Nicolás Cabrera 13-15. 28049 Madrid, Spain
E-mail address: carlos.mudarra@icmat.es