Explicit formulas for $C^{1,1}$ and $C^{1,\omega}_{\text{conv}}$ extensions of 1-jets in Hilbert and superreflexive spaces

D. Azagra, E. Le Gruyer, C. Mudarra

ICMAT-Universidad Complutense de Madrid, INSA de Rennes, ICMAT

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Two related problems

Problem ($C^{1,1}$ convex extension of 1-jets)

Given E a subset of a Hilbert space X, and a 1-jet (f, G) on E (meaning a pair of functions $f : E \to \mathbb{R}$ and $G : E \to X$), how can we tell whether there is a $C^{1,1}$ *convex* function $F : X \to \mathbb{R}$ which extends this jet (meaning that $F(x) = f(x)$ and $\nabla F(x) = G(x)$ for all $x \in E$)?

Problem ($C^{1,1}$ extension of 1-jets)

Given E a subset of a Hilbert space X, and a 1-jet (f, G) on E, how can we tell whether there is a $C^{1,1}$ function $F : X \to \mathbb{R}$ which extends (f, G)?

Why are these two problems related?

Recall a well known result: a function $F : X \to \mathbb{R}$ is of class $C^{1,1}$, with $\text{Lip}(\nabla F) \leq M$, if and only if $F + \frac{M}{2}\|\cdot\|^2$ is convex and $F - \frac{M}{2}\|\cdot\|^2$ is concave.

So, if we are given a 1-jet (f, G) defined on $E \subset X$ which can be extended to $(F, \nabla F)$ with $F \in C^{1,1}(X)$ and $\text{Lip}(\nabla F) \leq M$, then the function $H = F + \frac{M}{2}\|\cdot\|^2$ will be convex and of class $C^{1,1}$.

Conversely, if we can find a convex and $C^{1,1}$ function H such that $(H, \nabla H)$ is an extension of the jet
$E \ni y \mapsto \left(f(y) + \frac{M}{2}\|y\|^2,\ G(y) + My\right)$,
then
$X \ni y \mapsto \left(H(y) - \frac{M}{2}\|y\|^2,\ \nabla H(y) - My\right)$
will be a $C^{1,1}$ extension of (f, G).
As we will see, it's easier to solve first the $C^{1,1}_{\text{conv}}$ extension problem for jets, with an explicit formula, and then use this formula and the previous remarks to solve the $C^{1,1}$ extension problem for jets.

Previous solutions to these problems

The $C^{1,1}$ version of the classical Whitney extension theorem theorem tells us that there exists a function $F \in C^{1,1}(\mathbb{R}^n)$ with $F = f$ on C and $\nabla F = G$ on E if and only there exists a constant $M > 0$ such that

$$|f(x) - f(y) - \langle G(y), x-y\rangle| \leq M|x-y|^2, \quad \text{and} \quad |G(x) - G(y)| \leq M|x-y|$$

for all $x, y \in E$.

We can trivially extend (f, G) to the closure $\overline{E}$ of E so that the inequalities hold on $\overline{C}$ with the same constant M. The function F can be explicitly defined by

$$F(x) = \begin{cases} f(x) & \text{if } x \in \overline{C} \\ \sum_{Q \in \mathcal{Q}} \left(f(x_Q) + \langle G(x_Q), x - x_Q \rangle \right) \varphi_Q(x) & \text{if } x \in \mathbb{R}^n \setminus \overline{C}, \end{cases}$$

where $\mathcal{Q}$ is a family of *Whitney cubes* that cover the complement of the closure $\overline{C}$ of C, $\{\varphi_Q\}_{Q \in \mathcal{Q}}$ is the usual Whitney partition of unity associated to $\mathcal{Q}$, and x_Q is a point of $\overline{C}$ which minimizes the distance of $\overline{C}$ to the cube Q.
Recall also that $\mathrm{Lip}(\nabla F) \leq k(n)M$, where $k(n)$ is a constant depending only on n (but with $\lim_{n\to\infty} k(n) = \infty$).

In 1973 J.C. Wells improved this result and extended it to Hilbert spaces.

Theorem (Wells, 1973)

Let E be an arbitrary subset of a Hilbert space X, and $f : E \to \mathbb{R}$, $G : E \to X$. There exists $F \in C^{1,1}(X)$ such that $F_{|_E} = f$ and $(\nabla F)_{|_E} = G$ if and only if there exists $M > 0$ so that

$$f(y) \leq f(x) + \frac{1}{2}\langle G(x) + G(y), y - x\rangle + \frac{M}{4}\|x-y\|^2 - \frac{1}{4M}\|G(x) - G(y)\|^2 \qquad (W^{1,1})$$

for all $x, y \in E$.
Also, F is a maximal extension: if H is another $C^{1,1}$ function with $H_{|_E} = f$, $(\nabla H)_{|_E} = G$, and $Lip(\nabla H) \leq M$, then $H \leq F$.

We will say jet (f, G) on $E \subset X$ satisfies condition $(W^{1,1})$ if it satisfies the inequality of the theorem.

It can be checked that this condition is absolutely equivalent to the condition in the $C^{1,1}$ version of Whitney's extension theorem.

Well's proof was quite complicated, and didn't provide any explicit formula for the extension when E is infinite.

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In 2009 Erwan Le Gruyer showed, by very different means, another version of Well's result.

Theorem (Erwan Le Gruyer, 2009)

Given a Hilbert space X, a subset E of X, and functions $f : E \to \mathbb{R}$, $G : E \to X$, a necessary and sufficient condition for the 1-jet (f, G) to have a $C^{1,1}$ extension $(F, \nabla F)$ to the whole space X is that

$$\Gamma(f, G, E) := \sup_{x,y \in E} \left(\sqrt{A_{x,y}^2 + B_{x,y}^2} + |A_{x,y}| \right) < \infty, \tag{2.1}$$

where

$$A_{x,y} = \frac{2(f(x) - f(y)) + \langle G(x) + G(y), y - x \rangle}{\|x - y\|^2} \quad \textit{and}$$

$$B_{x,y} = \frac{\|G(x) - G(y)\|}{\|x - y\|} \quad \textit{for all} \quad x, y \in E, x \neq y.$$

Moreover, $\Gamma(F, \nabla F, X) = \Gamma(f, G, E) = \|(f, G)\|_E$, *where*

$$\|(f, G)\|_E := \inf\{\mathrm{Lip}(\nabla H) \ : \ H \in C^{1,1}(X) \ \textit{and} \ (H, \nabla H) = (f, G) \ \textit{on} \ E\}$$

is the trace seminorm of the jet (f, G) on E.

The number $\Gamma(f, G, E)$ is the smallest $M > 0$ for which (f, G) satisfies Well's condition $(W^{1,1})$ with constant $M > 0$.

In particular Le Gruyer's condition is also absolutely equivalent to the condition in the $C^{1,1}$ version of Whitney's extension theorem.

Le Gruyer's theorem didn't provide any explicit formula for the extension either (it uses Zorn's lemma).

What about the convex case?

Theorem (Azagra-Mudarra, 2016)

Let E be a subset of $\mathbb{R}^n$, and $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$ be functions. There exists a convex *function $F \in C^{1,\omega}(\mathbb{R}^n)$ if and only if there exists $M > 0$ such that, for all $x, y \in E$,*

$$f(x) - f(y) - \langle G(y), x-y \rangle \geq \frac{1}{2} |G(x) - G(y)| \, \omega^{-1} \left(\frac{1}{2M} |G(x) - G(y)| \right).$$

Moreover, $\sup_{x \neq y} \frac{|\nabla F(x) - \nabla F(y)|}{|x-y|} \leq k(n)M.$

Here, as in Whitney's theorem, $k(n)$ only depends on n, but goes to ∞ as $n \to \infty$ (not surprising, as Whitney's extension techniques were used in the proof, which was constructive).
We say that (f, G) satisfies condition $(CW^{1,\omega})$ if it satisfies the inequality of the theorem for some $M > 0$.

When $\omega(t) = t$ we call this condition $(CW^{1,1})$:

$$f(x) - f(y) - \langle G(y), x - y \rangle \geq \frac{1}{2M} |G(x) - G(y)|^2.$$

By using Le Gruyer's technique it can be shown that in the $C^{1,1}_{\text{conv}}$ case the above theorem is true for any Hilbert space, and with $k(n) = 1$ (optimal constant). However, again the proof is not constructive.

A constructive proof and a formula with optimal constants was provided by

Theorem (Daniilidis-Haddou-Le Gruyer-Ley, 2017)

If (f, G) is a 1-jet defined on a subset E of a Hilbert space and (f, G) satisfies $(CW^{1,1})$ for some M then the formula

$$F(x) = \sup_{\varepsilon \in (0, \frac{1}{M})} \inf_{z \in X} \sup_{y \in X} \sup_{u \in E} \{ f(u) + \langle G(u), z - u \rangle - \frac{\|y - z\|^2}{2\varepsilon} + \frac{\|z - x\|^2}{2\varepsilon} \}$$

defines a $C^{1,1}(X)$ convex function such that $(F, \nabla F)$ extends (f, G), and $\text{Lip}\, \nabla F \leq M$.

New results

Unfortunately the previous formula does not work for nonlinear $\omega(t)$.

At the same time we proved:

Theorem (Azagra-Le Gruyer-Mudarra, 2017)

A jet (f, G) defined on a subset E of a Hilbert space X has a $C^{1,\omega}$ convex extension from E to X if and only if it satisfies condition $(CW^{1,\omega})$ on E. The formula

$$F(x) = conv\left(\inf_{y\in E}\{f(y) + \langle G(y), x-y\rangle + M\varphi\left(\|x-y\|\right)\}\right)$$

where $\varphi(t) = \int_0^t \omega(s)ds$, *defines such an extension, with the property that*

$$\sup_{x,y\in E, x\neq y} \frac{\|\nabla F(x) - \nabla F(y)\|}{\omega\left(\|x-y\|\right)} \leq 8M.$$

Recall that

$$\operatorname{conv}(g)(x) = \sup\{h(x) \,:\, h \text{ is convex and continuous}, h \leq g\}.$$

Another expression for $\operatorname{conv}(g)$ is given by

$$\operatorname{conv}(g)(x) = \inf\left\{ \sum_{j=1}^{k} \lambda_j g(x_j) \,:\, \lambda_j \geq 0, \sum_{j=1}^{k} \lambda_j = 1,\, x = \sum_{j=1}^{k} \lambda_j x_j,\, k \in \mathbb{N} \right\}.$$

In the most important case that $\omega(t) = t$ we can find optimal constants:

Theorem (Azagra-Le Guyer-Mudarra, 2017)

Let (f, G) be a 1-jet defined on an arbitrary subset E of a Hilbert space X. There exists $F \in C^{1,1}_{conv}$ such that $(F, \nabla F)$ extends (f, G) if and only if

$$f(x) \geq f(y) + \langle G(y), x - y \rangle + \frac{1}{2M} |G(x) - G(y)|^2 \quad \textit{for all} \quad x, y \in E,$$

where

$$M = M(G, E) := \sup_{x,y \in E,\, x \neq y} \frac{|G(x) - G(y)|}{|x - y|}.$$

The function

$$F(x) = conv \left(\inf_{y \in E} \{ f(y) + \langle G(y), x - y \rangle + \frac{M}{2} |x - y|^2 \} \right)$$

defines such an extension, with the property that $Lip(\nabla F) \leq M$. Moreover, for any other such extension H, we have $H \leq F$.

Corollary (Wells 1973, Le Gruyer 2009, Azagra-Le Gruyer-Mudarra 2017)

Let E be an arbitrary subset of a Hilbert space X, and $f : E \to \mathbb{R}$, $G : E \to X$. There exists $F \in C^{1,1}(X)$ such that $F_{|_E} = f$ and $(\nabla F)_{|_E} = G$ if and only if there exists $M > 0$ so that

$$f(y) \leq f(x) + \frac{1}{2}\langle G(x)+G(y), y-x\rangle + \frac{M}{4}|x-y|^2 - \frac{1}{4M}|G(x)-G(y)|^2 \quad (W^{1,1})$$

for all $x, y \in E$. Moreover,

$$F = conv(g) - \tfrac{M}{2}|\cdot|^2, \text{ where}$$
$$g(x) = \inf_{y\in E}\{f(y) + \langle G(y), x-y\rangle + \tfrac{M}{2}|x-y|^2\} + \tfrac{M}{2}|x|^2, \quad x \in X,$$

defines such an extension, with the additional property that $Lip(\nabla F) \leq M$. Also, F is a maximal extension: if H is another $C^{1,1}$ function with $H_{|_E} = f$, $(\nabla H)_{|_E} = G$, and $Lip(\nabla H) \leq M$, then $H \leq F$.

Key to the proof of the Corollary: it's well known that $F : X \to \mathbb{R}$ is of class $C^{1,1}$, with $\text{Lip}(\nabla F) \leq M$, if and only if $F + \frac{M}{2}|\cdot|^2$ is convex and $F - \frac{M}{2}|\cdot|^2$ is concave. This result generalizes to jets:

Lemma

Given an arbitrary subset E of a Hilbert space X and a 1-jet (f, G) defined on E, we have:

(f, G) satisfies $(W^{1,1})$ on E, with constant $M > 0$, if and only if the 1-jet $(\tilde{f}, \tilde{G})$ defined by $\tilde{f} = f + \frac{M}{2}|\cdot|^2$, $\tilde{G} = G + MI$, satisfies $(CW^{1,1})$ on E with constant $2M$.

(See "Further details" below.)

Necessity of $(W^{1,1})$

Proposition

(i) *If (f,G) satisfies $(W^{1,1})$ on E with constant M, then G is M-Lipschitz on E.*

(ii) *If F is a function of class $C^{1,1}(X)$ with $\mathrm{Lip}(\nabla F) \leq M$, then $(F, \nabla F)$ satisfies $(W^{1,1})$ on $E = X$ with constant M.*

Shown by Wells (or see “Further details” below).

Proof of the Corollary: (f,G) satisfies $(W^{1,1})$ with constant M if and only if $(\widetilde{f},\widetilde{G}) := (f+\frac{M}{2}|\cdot|^2, g+MI)$ satisfies $(CW^{1,1})$ with constant $2M$. Then, by the $C^{1,1}$ convex extension theorem for jets,

$$\tilde{F} = \mathrm{conv}(\widetilde{g}), \quad \tilde{g}(x) = \inf_{y\in E}\{\tilde{f}(y) + \langle \tilde{G}(y), x-y\rangle + M|x-y|^2\}, \quad x\in X,$$

is convex and of class $C^{1,1}$ with $(\tilde{F},\nabla\tilde{F}) = (\tilde{f},\tilde{G})$ on E, and $\mathrm{Lip}(\nabla\tilde{F}) \leq 2M$. By an easy calculation,

$$\tilde{g}(x) = \inf_{y\in E}\{f(y) + \langle G(y), x-y\rangle + \tfrac{M}{2}|x-y|^2\} + \tfrac{M}{2}|x|^2, \quad x\in X.$$

Now, by the necessity of $(CW^{1,1})$, $(\tilde{F},\nabla\tilde{F})$ satisfies condition $(CW^{1,1})$ with constant $2M$ on X. Thus, if

$$F(x) = \tilde{F}(x) - \frac{M}{2}|x|^2, \quad x\in X$$

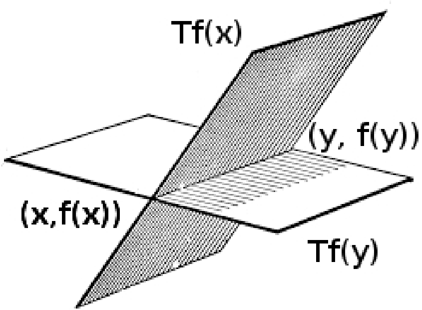
then (again by the preceding lemma) $(F,\nabla F)$ satisfies $(W^{1,1})$ with constant M on X. Hence, by the previous proposition, F is of class $C^{1,1}(X)$, with $\mathrm{Lip}(\nabla F)\leq M$. From the definition of $\tilde{f}, \tilde{G}, \tilde{F}$ and F it is immediate that $F=f$ and $\nabla F = G$ on E.

(For the maximality of F, see “Further details below”.)

Sketch of the proof of the $C^{1,1}_{\text{conv}}(X)$ extension result for 1-jets

Necessity:

$$f(x) = f(y) + \langle \nabla f(y), x - y\rangle \implies \nabla f(x) = \nabla f(y)$$



A quantitative refinement of this geometrical idea leads to

$$f(x) \geq f(y) + \langle G(y), x - y\rangle + \frac{1}{2M}|G(x) - G(y)|^2 \quad \text{for all} \quad x, y.$$

Sketch of the proof of the $C^{1,1}_{\mathrm{conv}}(X)$ extension result for 1-jets

Sufficiency: let's first recall some known facts.

Proposition

For a continuous convex function $f : X \to \mathbb{R}$, the following statements are equivalent.

(i) *There exists $M > 0$ such that*

$$f(x+h) + f(x-h) - 2f(x) \leq M|h|^2 \quad \text{for all} \quad x, h \in X.$$

(ii) *f is differentiable on X with $\mathrm{Lip}(\nabla f) \leq M$.*

Sketch of the proof of the $C^{1,1}_{\text{conv}}(X)$ extension result for 1-jets

Sufficiency: some useful results.

Theorem ($C^{1,1}$ convex envelopes)

Let X be a Banach space. Suppose that a function $g : X \to \mathbb{R}$ has a convex continuous minorant, and satisfies

$$g(x+h) + g(x-h) - 2g(x) \leq M|h|^2 \quad \textit{for all} \quad x, h \in X.$$

Then $H := conv(g)$ is a continuous convex function satisfying the same property. Hence H is of class $C^{1,1}(X)$, with $\text{Lip}(\nabla \psi) \leq M$.

In particular, for a function $\varphi \in C^{1,1}(X)$, we have that $conv(\varphi) \in C^{1,1}(X)$, with $\text{Lip}(\nabla conv(\varphi)) \leq \text{Lip}(\nabla \varphi)$.

Proof of the $C^{1,1}$ smoothness of this convex envelope:
Given $x, h \in X$ and $\varepsilon > 0$, we can find $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n > 0$ such that

$$H(x) \geq \sum_{i=1}^{n} \lambda_i g(x_i) - \varepsilon, \quad \sum_{i=1}^{n} \lambda_i = 1 \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i x_i = x.$$

Since $x \pm h = \sum_{i=1}^{n} \lambda_i (x_i \pm h)$, we have $H(x \pm h) \leq \sum_{i=1}^{n} \lambda_i g(x_i \pm h)$.
Therefore

$$H(x+h)+H(x-h)-2H(x) \leq \sum_{i=1}^{n} \lambda_i \left(g(x_i+h)+g(x_i-h)-2g(x_i)\right)+2\varepsilon,$$

and by the assumption on g we have

$$g(x_i+h)+g(x_i-h)-2g(x_i) \leq M|h|^2 \quad i=1,\ldots,n.$$

Thus

$$H(x+h)+H(x-h)-2H(x) \leq M|h|^2 + 2\varepsilon, \tag{3.1}$$

and since ε is arbitrary we get the inequality of the statement.

Sketch of the proof of the $C^{1,1}_{\text{conv}}(X)$ extension result for 1-jets

Sufficiency:

1. Define $m(x) = \sup_{y\in E}\{f(y) + \langle G(y), x-y\rangle\}$, the *minimal convex extension* of (f, G). This function is not necessarily differentiable.

2. Use condition $(CW^{1,1})$ to check that, for all $y, z \in E$, $x \in X$,

$$f(z) + \langle G(z), x-z\rangle \leq f(y) + \langle G(y), x-y\rangle + \frac{M}{2}|x-y|^2.$$

Hence (taking $\inf_{z\in E}$ on the left and then $\inf_{y\in E}$ on the right)

$$m(x) \leq \inf_{y\in E}\{f(y) + \langle G(y), x-y\rangle + \frac{M}{2}|x-y|^2\} =: g(x)$$

for all $x \in X$. Besides $f \leq m \leq g \leq f$ on E, and in particular $m = f = g$ on E. Since m is convex, $m \leq F := \text{conv}(g) \leq g$. Therefore $F = f$ on E.

3. Check that $g(x+h) + g(x-h) - 2g(x) \leq M|h|^2$. This inequality is preserved when we take $F = \text{conv}(g)$. Since F is convex, this implies $F \in C^{1,1}(X)$. Also check that $\nabla F(x) = G(x)$ for every $x \in E$.

Details

Details: **Step 2.**

Lemma

We have that $f(z) + \langle G(z), x - z\rangle \leq f(y) + \langle G(y), x - y\rangle + \frac{M}{2}|x - y|^2$ *for every* $y, z \in E,\ x \in X$.

Proof: Given $y, z \in E,\ x \in X$, condition $(CW^{1,1})$ implies

$$f(y) + \langle G(y), x - y\rangle + \frac{M}{2}|x - y|^2$$
$$\geq f(z) + \langle G(z), y - z\rangle + \frac{1}{2M}|G(y) - G(z)|^2 + \langle G(y), x - y\rangle + \frac{M}{2}|x - y|^2 =$$
$$f(z) + \langle G(z), x - z\rangle + \frac{1}{2M}|G(y) - G(z)|^2 + \langle G(z) - G(y), y - x\rangle + \frac{M}{2}|x - y|^2$$
$$= f(z) + \langle G(z), x - z\rangle + \frac{1}{2M}|G(y) - G(z) + 2M(y - x)|^2$$
$$\geq f(z) + \langle G(z), x - z\rangle.$$

Details: **Step 3.**

Lemma

We have $g(x+h)+g(x-h)-2g(x)\leq M|h|^2$ *for all* $x,h\in X$.

Proof: Given $x,h\in X$ and $\varepsilon>0$, by definition of g, we can pick $y\in E$ with

$$g(x)\geq f(y)+\langle G(y),x-y\rangle+\frac{M}{2}|x-y|^2-\varepsilon.$$

We then have

$$\begin{aligned} g(x+h)+g(x-h)-2g(x) &\leq f(y)+\langle G(y),x+h-y\rangle+\frac{M}{2}|x+h-y|^2\\ &+f(y)+\langle G(y),x-h-y\rangle+\frac{M}{2}|x-h-y|^2\\ &-2\left(f(y)+\langle G(y),x-y\rangle+\frac{M}{2}|x-y|^2\right)+2\varepsilon\\ &=\frac{M}{2}\left(|x+h-y|^2+|x-h-y|^2-2|x-y|^2\right)+2\varepsilon\\ &=M|h|^2+2\varepsilon. \end{aligned}$$

Details: **Step 3.**

Also note that $m \leq F$ on X and $F = m$ on E, where m is convex and F is differentiable on X. This implies that m is differentiable on E with $\nabla m(x) = \nabla F(x)$ for all $x \in E$.

It is clear, by definition of m, that $G(x) \in \partial m(x)$ (the subdifferential of m at x) for every $x \in E$, and these observations show that $\nabla F = G$ on E.

Similar $C^{1,\alpha}$ convex extension results in superreflexive spaces

Theorem (Azagra-Le Gruyer-Mudarra, 2017)

Let E be a subset of a superreflexive Banach space (with a power-type uniformly differentiable equivalent norm, i.e., for some $\alpha \in (0,1]$*,* $\|x+h\|^{1+\alpha} + \|x-h\|^{1+\alpha} - 2\|x\|^{1+\alpha} \leq C\|h\|^{1+\alpha}$ *for all* x, h*).*
Then (f, G) *has an extension* $(F, \nabla F)$*, with F convex and of class* $C^{1,\alpha}(X)$*, if and only if*

$$f(x) \geq f(y) + G(y)(x-y) + \frac{\alpha}{(1+\alpha)M^{1/\alpha}}\|G(x)-G(y)\|_*^{1+\frac{1}{\alpha}} \quad x, y \in E,$$

where $M = M_\alpha(G) := \sup_{x \neq y,\, x,y \in E} \frac{\|G(x)-G(y)\|_*}{\|x-y\|^\alpha} < \infty$*.*
Moreover, the formula

$$F(x) = conv\left(\inf_{y\in E}\{f(y) + G(y)(x-y) + \tfrac{M}{1+\alpha}\|x-y\|^{1+\alpha}\}\right),$$

defines such an extension, with $M_\alpha(DF) \leq \frac{2^{1+\alpha}C}{1+\alpha}M$*.*

Unfortunately, one cannot use the same kind of method as in the $C^{1,1}$ case to solve $C^{1,\alpha}$ extension problems for general (not necessarily convex) 1-jets in Hilbert spaces.
The exponent $\alpha = 1$ is miraculous in this respect: it is not true in general that, given a function $f \in C^{1,\alpha}(\mathbb{R})$, there exists a constant C such that $f + C|\cdot|^{1+\alpha}$ is convex.

Example

Let $0 < \alpha < 1$ *and define* $f : \mathbb{R} \to \mathbb{R}$ *by*

$$f(t) = \begin{cases} 0 & \text{if} \quad t \leq 1 \\ -(t-1)^{1+\alpha} & \text{if} \quad t \geq 1. \end{cases}$$

Then $f \in C^{1,\alpha}(\mathbb{R})$*, but there is no constant* $C > 0$ *for which* $f + C|\cdot|^{1+\alpha}$ *is convex.*

Two applications of the $C^{1,1}$ results

Corollary (Kirszbraun's Theorem with an explicit formula)

Let X, Y be two Hilbert spaces, E a subset of X and $G : E \to Y$ a Lipschitz mapping. There exists $\widetilde{G} : X \to Y$ with $\widetilde{G} = G$ on E and $\mathrm{Lip}(\widetilde{G}) = \mathrm{Lip}(G)$. In fact, if $P_Y : X \times Y \to Y$ denotes the natural projection $P_Y(x, y) = y$, then the function

$$\widetilde{G} := P_Y(\nabla(conv(g) - \tfrac{M}{2}\| \cdot \|^2)),$$

where

$$g(x, y) = \inf_{z \in E}\{\langle G(z), y\rangle + \tfrac{M}{2}\|(x - z, y)\|^2\} + \tfrac{M}{2}\|(x, y)\|^2,$$

extends G from E to X.

Proof of this version of Kirszbraun's theorem:

Consider the 1-jet (f^*, G^*) defined on $E \times \{0\} \subset X \times Y$ by $f^*(x,0) = 0$ and $G^*(x,0) = (0, G(x))$. If $M := \mathrm{Lip}(G)$, one easily checks that (f^*, G^*) satisfies condition $(W^{1,1})$ on $E \times \{0\}$ with constant M. Therefore the function

$$F = \mathrm{conv}(g) - \frac{M}{2}\|\cdot\|^2, \quad \text{where} \quad g(x,y) = \inf_{z \in E}\{f^*(z,0) + \langle G^*(z,0), (x-z,y)\rangle + \frac{M}{2}\|(x-z,y)\|^2\} + \frac{M}{2}\|(x,y)\|^2,$$

is of class $C^{1,1}(X \times Y)$ with $(F, \nabla F) = (f^*, G^*)$ on $E \times \{0\}$ and $\mathrm{Lip}(\nabla F) \leq M$. The expression defining g can be simplified as below and therefore

$$\widetilde{G} := P_Y(\nabla(\mathrm{conv}(g) - \frac{M}{2}\|\cdot\|^2)) \text{ extends } G \text{ to } X.$$

$$\text{Here, } g(x,y) = \inf_{z \in E}\{\langle G(z), y\rangle + \frac{M}{2}\|(x-z,y)\|^2\} + \frac{M}{2}\|(x,y)\|^2.$$

Theorem (Finding $C^{1,1}$ convex hypersurfaces with prescribed tangent hyperplanes at a given subset of a Hilbert space)

Let E be an arbitrary subset of a Hilbert space X, and let $N : E \to S_X$ be a mapping. Then the following statements are equivalent.

1. *There exists a $C^{1,1}$ convex hypersurface S such that $E \subseteq S$ and $N(x)$ is outwardly normal to S at x for every $x \in E$.*
2. *There exists some $\delta > 0$ such that*

$$\langle N(y), y-x \rangle \geq \delta \|N(y)-N(x)\|^2 \quad \text{for all} \quad x, y \in E.$$

Moreover, if we further assume that E is bounded then S can be taken to be closed.

An equivalent reformulation of this result which was suggested to us by Arie Israel is the following.

Theorem

Let E be a subset of a Hilbert space X, and let $\mathcal{H}$ be a collection of affine hyperplanes of X such that every $H \in \mathcal{H}$ passes through some point $x_H \in E$. The following statements are equivalent:

1. *There exists a convex hypersurface S of class $C^{1,1}$ in X such that X has bounded principal curvatures and H is tangent to S at x_H for every $H \in \mathcal{H}$.*
2. *There exists some $M > 0$ such that, for every couple H_1, H_2 of hyperplanes in $\mathcal{H}$, there exists a convex hypersurface $S(H_1, H_2)$ of class $C^{1,1}$ such that the principal curvatures of $S(H_1, H_2)$ are bounded by M and $S(H_1, H_2)$ is tangent to H_1 and H_2 at x_{H_1} and x_{H_2}, respectively.*

Thank you for your attention!

Further details

Details: **Maximality of F in the $C^{1,1}$ extension result for jets.**
Suppose that H is another $C^{1,1}(X)$ function with $H = f$ and $\nabla H = G$ on E and $\mathrm{Lip}(\nabla H) \leq M$. Using these assumptions and Taylor's Theorem we get

$$H(x) + \frac{M}{2}|x|^2 \leq f(y) + \langle G(y), x - y\rangle + \frac{M}{2}|x-y|^2 + \frac{M}{2}|x|^2,$$

for all $x \in X, y \in E$. Taking the infimum over $y \in E$ we get that

$$H(x) + \frac{M}{2}|x|^2 \leq g(x), \quad x \in X.$$

Since H is $C^{1,1}(X)$ with $\mathrm{Lip}(\nabla H) \leq M$, the function $X \ni x \mapsto \tilde{H}(x) = H(x) + \frac{M}{2}|x|^2$ is convex, which implies that

$$\tilde{H} = \mathrm{conv}(\tilde{H}) \leq g.$$

Therefore, $\tilde{H} \leq \tilde{F} = \mathrm{conv}(g)$ on X, from which we obtain that $H \leq F$ on X.

Details: **Necessity of** $(W^{1,1})$**.**
Proof: (i) Given $x, y \in E$, we have

$$f(y) \leq f(x) + \frac{1}{2}\langle G(x) + G(y), y - x\rangle + \frac{M}{4}\|x - y\|^2 - \frac{1}{4M}\|G(x) - G(y)\|^2$$
$$f(x) \leq f(y) + \frac{1}{2}\langle G(y) + G(x), x - y\rangle + \frac{M}{4}\|x - y\|^2 - \frac{1}{4M}\|G(x) - G(y)\|^2.$$

By combining both inequalities we easily get $\|G(x) - G(y)\| \leq M\|x - y\|$.

(ii) Fix $x, y \in X$ and $z = \frac{1}{2}(x + y) + \frac{1}{2M}(\nabla F(y) - \nabla F(x))$.

Details: **Necessity of** $(W^{1,1})$**.**
Using Taylor's theorem we obtain

$$F(z) \leq F(x) + \langle \nabla F(x), \tfrac{1}{2}(y-x) \rangle + \tfrac{1}{2M}(\nabla F(y) - \nabla F(x)) \rangle$$

$$+ \tfrac{M}{2} \left\| \tfrac{1}{2}(y-x) + \tfrac{1}{2M}(\nabla F(y) - \nabla F(x)) \right\|^2$$

and

$$F(z) \geq F(y) + \langle \nabla F(y), \tfrac{1}{2}(x-y) \rangle + \tfrac{1}{2M}(\nabla F(y) - \nabla F(x)) \rangle$$

$$- \tfrac{M}{2} \left\| \tfrac{1}{2}(x-y) + \tfrac{1}{2M}(\nabla F(y) - \nabla F(x)) \right\|^2.$$

Details: **Necessity of** $(W^{1,1})$**.**
Then we get

$$
\begin{aligned}
F(y) \le F(x) &+ \langle \nabla F(x), \tfrac{1}{2}(y-x)\rangle + \tfrac{1}{2M}(\nabla F(y) - \nabla F(x))\rangle \\
&+ \tfrac{M}{2}\left\| \tfrac{1}{2}(y-x)\rangle + \tfrac{1}{2M}(\nabla F(y) - \nabla F(x))\right\|^2 \\
&- \langle \nabla F(y), \tfrac{1}{2}(x-y)\rangle - \tfrac{1}{2M}(\nabla F(y) - \nabla F(x))\rangle \\
&+ \tfrac{M}{2}\left\| \tfrac{1}{2}(x-y) + \tfrac{1}{2M}(\nabla F(y) - \nabla F(x))\right\|^2 \\
= F(x) &+ \frac{1}{2}\langle \nabla F(x) + \nabla F(y), y-x\rangle + \\
&+ \tfrac{M}{4}\|x-y\|^2 - \tfrac{1}{4M}\|\nabla F(x) - \nabla F(y)\|^2
\end{aligned}
$$

Details: **Relationship between** $(CW^{1,1})$ **and** $(W^{1,1})$**.**

Lemma

(f, G) satisfies $(W^{1,1})$ on E, with constant $M > 0$, if and only if $(\tilde{f}, \tilde{G})$, defined by $\tilde{f}(x) = f(x) + \frac{M}{2}\|x\|^2, \ \tilde{G}(x) = G(x) + Mx, \ x \in E$, satisfies property $(CW^{1,1})$ on E, with constant $2M$.

Details: **Relationship between** $(CW^{1,1})$ **and** $(W^{1,1})$**.**
Proof: Suppose first that (f, G) satisfies $(W^{1,1})$ on E with constant $M > 0$. We have, for all $x, y \in E$,

$$\tilde{f}(x) - \tilde{f}(y) - \langle \tilde{G}(y), x - y \rangle - \frac{1}{4M} \|\tilde{G}(x) - \tilde{G}(y)\|^2$$
$$= f(x) - f(y) + \frac{M}{2}\|x\|^2 - \frac{M}{2}\|y\|^2 - \langle G(y) + My, x - y \rangle$$
$$- \frac{1}{4M}\|G(x) - G(y) + M(x-y)\|^2$$
$$\geq \frac{1}{2}\langle G(x) + G(y), x - y \rangle - \frac{M}{4}\|x-y\|^2 + \frac{1}{4M}\|G(x) - G(y)\|^2$$
$$+ f(x) - f(y) + \frac{M}{2}\|x\|^2 - \frac{M}{2}\|y\|^2 - \langle G(y) + My, x - y \rangle$$
$$- \frac{1}{4M}\|G(x) - G(y) + M(x-y)\|^2$$
$$= \frac{M}{2}\|x\|^2 + \frac{M}{2}\|y\|^2 - M\langle x, y \rangle - \frac{M}{2}\|x-y\|^2 = 0.$$

Details: **Relationship between** $(CW^{1,1})$ **and** $(W^{1,1})$.

Conversely, if $(\tilde{f}, \tilde{G})$ satisfies $(CW^{1,1})$ on E with constant $2M$, we have

$$
\begin{aligned}
& f(x) + \frac{1}{2}\langle G(x)+G(y), y-x\rangle + \frac{M}{4}\|x-y\|^2 - \frac{1}{4M}\|G(x)-G(y)\|^2 - f(y) \\
&= \tilde{f}(x) - \frac{M}{2}\|x\|^2 + \frac{1}{2}\langle \tilde{G}(x)+\tilde{G}(y) - M(x+y), y-x\rangle + \frac{M}{4}\|x-y\|^2 \\
&\quad - \frac{1}{4M}\|\tilde{G}(x)-\tilde{G}(y)-M(x-y)\|^2 - \tilde{f}(y) + \frac{M}{2}\|y\|^2 \\
&= \tilde{f}(x) - \tilde{f}(y) + \frac{1}{2}\langle \tilde{G}(x)+\tilde{G}(y), y-x\rangle + \frac{M}{4}\|x-y\|^2 \\
&\quad - \frac{1}{4M}\|\tilde{G}(x)-\tilde{G}(y)-M(x-y)\|^2 \\
&\geq \langle \tilde{G}(y), x-y\rangle + \frac{1}{4M}\|\tilde{G}(x)-\tilde{G}(y)\|^2 + \frac{1}{2}\langle \tilde{G}(x)+\tilde{G(y)}, y-x\rangle \\
&\quad + \frac{M}{4}\|x-y\|^2 - \frac{1}{4M}\|\tilde{G}(x)-\tilde{G}(y)-M(x-y)\|^2 = 0.
\end{aligned}
$$

Details: **Absolute equivalence of $(W^{1,1})$ and the classical Whitney $C^{1,1}$ extension condition.**

Let E be a subset of a Hilbert space X and $(f,G) : E \to \mathbb{R} \times X$ be a 1-jet. Given $M > 0$, we will say that (f,G) satisfies the condition $(W^{1,1}_M)$ on E if the inequality

$$f(y) \leq f(x) + \frac{1}{2}\langle G(x) + G(y), y - x\rangle + \frac{M}{4}\|x-y\|^2 - \frac{1}{4M}\|G(x) - G(y)\|^2,$$

holds for every $x, y \in E$. Also, given $M_1, M_2 > 0$, we will say that (f,G) satisfies the condition $(W^{1,1}_{M_1,M_2})$ on E provided that the inequalities

$$|f(y) - f(x) - \langle G(x), y-x\rangle| \leq M_1\|x-y\|^2, \quad \|G(x) - G(y)\| \leq M_2\|x-y\|,$$

are satisfied for every $x, y \in E$.

Details: **Absolute equivalence of $(W^{1,1})$ and the classical Whitney $C^{1,1}$ extension condition.**

Claim

$$(W^{1,1}_M) \implies (\widetilde{W^{1,1}_{\frac{M}{2},M}})$$

First, we have that, for all $x, y \in E$,

$$f(y) \leq f(x) + \frac{1}{2}\langle G(x) + G(y), y - x\rangle + \frac{M}{4}\|x - y\|^2 - \frac{1}{4M}\|G(x) - G(y)\|^2$$

and

$$f(x) \leq f(y) + \frac{1}{2}\langle G(y) + G(x), x - y\rangle + \frac{M}{4}\|y - x\|^2 - \frac{1}{4M}\|G(y) - G(x)\|^2.$$

Details: **Absolute equivalence of $(W^{1,1})$ and the classical Whitney $C^{1,1}$ extension condition.**
By summing both inequalities we get $\|G(x)-G(y)\| \le M\|x-y\|$. On the other hand, by using $(W^{1,1}_M)$, we can write

$$f(y)-f(x)-\langle G(x),y-x\rangle \le \frac{1}{2}\langle G(x)+G(y),y-x\rangle - \langle G(x),y-x\rangle$$
$$+\frac{M}{4}\|x-y\|^2 - \frac{1}{4M}\|G(x)-G(y)\|^2$$
$$=\frac{1}{2}\langle G(y)-G(x),y-x\rangle + \frac{M}{4}\|x-y\|^2 - \frac{1}{4M}\|G(x)-G(y)\|^2$$
$$=\frac{M}{2}\|x-y\|^2 - \frac{M}{4}\left(\|x-y\|^2 + \frac{1}{M^2}\|G(x)-G(y)\|^2 - 2\langle\frac{1}{M}(G(y)-G(x)),y$$
$$=\frac{M}{2}\|x-y\|^2 - \frac{M}{4}\left\|\frac{1}{M}(G(x)-G(y))-(y-x)\right\|^2 \le \frac{M}{2}\|x-y\|^2.$$

Details: **Absolute equivalence of $(W^{1,1})$ and the classical Whitney $C^{1,1}$ extension condition.**

Also, we have

$$f(x)-f(y)-\langle G(x),x-y\rangle \leq \frac{1}{2}\langle G(x)+G(y),x-y\rangle - \langle G(x),x-y\rangle$$
$$+\frac{M}{4}\|x-y\|^2-\frac{1}{4M}\|G(x)-G(y)\|^2$$
$$=\frac{1}{2}\langle G(y)-G(x),x-y\rangle+\frac{M}{4}\|x-y\|^2-\frac{1}{4M}\|G(x)-G(y)\|^2$$
$$=\frac{M}{2}\|x-y\|^2-\frac{M}{4}\left(\|x-y\|^2+\frac{1}{M^2}\|G(x)-G(y)\|^2-2\langle\frac{1}{M}(G(y)-G(x)),x$$
$$=\frac{M}{2}\|x-y\|^2-\frac{M}{4}\left\|\frac{1}{M}(G(x)-G(y))-(x-y)\right\|^2\leq\frac{M}{2}\|x-y\|^2.$$

This leads us to

$$|f(y)-f(x)-\langle G(x),y-x\rangle|\leq\frac{M}{2}\|x-y\|^2,$$

which proves the first laim.

Details: **Absolute equivalence of $(W^{1,1})$ and the classical Whitney $C^{1,1}$ extension condition.**

Claim

$(\widetilde{W^{1,1}_{M_1,M_2}}) \implies (W^{1,1}_M)$, *where* $M = (3+\sqrt{10})\max\{M_1, M_2\}$.

Using that $f(y) - f(x) - \langle G(x), y-x\rangle \leq M_1\|x-y\|^2$, we can write

$$f(y) - f(x) - \frac{1}{2}\langle G(x)+G(y), y-x\rangle - \frac{M}{4}\|x-y\|^2 + \frac{1}{4M}\|G(x)-G(y)\|^2$$
$$\leq \langle G(x), y-x\rangle + M_1\|x-y\|^2 - \frac{1}{2}\langle G(x)+G(y), y-x\rangle - \frac{M}{4}\|x-y\|^2 + \frac{1}{4M}$$
$$= \frac{1}{2}\langle G(x)-G(y), y-x\rangle + \left(M_1 - \frac{M}{4}\right)\|x-y\|^2 + \frac{1}{4M}\|G(x)-G(y)\|^2$$
$$\leq \frac{1}{2}ab + \left(M_1 - \frac{M}{4}\right)a^2 + \frac{1}{4M}b^2,$$

where $a = \|x-y\|$ and $b = \|G(x)-G(y)\|$. Since G is M_2-Lipschitz, we have the inequality $b \leq M_2 a$.

Details: **Absolute equivalence of $(W^{1,1})$ and the classical Whitney $C^{1,1}$ extension condition.**

So, the last term in the above chain of inequalities is smaller than or equal to

$$(\tfrac{1}{2}M_2 + (M_1 - \tfrac{M}{4}) + \tfrac{1}{4M}M_2^2)a^2 \leq (\tfrac{1}{2}K + (K - \tfrac{M}{4}) + \tfrac{1}{4M}K^2)a^2,$$

where $K = \max\{M_1, M_2\}$. Now, the last term is smaller than or equal to 0 if and only if $-M^2 + 6MK + K^2 \leq 0$. But, in fact, for $M = (3 + \sqrt{10})K$ the term $-M^2 + 6MK + K^2$ is equal to 0. This proves the second Claim.