C^1 convex extensions of 1-jets from arbitrary subsets of $\mathbb{R}^n$, and applications

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Two related problems

Problem (Analytical version)

Given E a subset of $\mathbb{R}^n$, and a 1-jet $(g, G) : E \to \mathbb{R} \times \mathbb{R}^n$, how can we tell whether there is a *convex* function $F : \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that $(F, \nabla F) = (f, G)$ on E?

Problem (Geometrical version)

Given an arbitrary subset C of X and a collection $\mathcal{H}$ of affine hyperplanes of X such that every $H \in \mathcal{H}$ passes through some point $x_H \in C$, what conditions on $\mathcal{H}$ are necessary and sufficient for the existence of a C^1 *convex* hypersurface in X such that H is tangent to S at x_H for every $H \in \mathcal{H}$?

Previous results on smooth convex extension

M. Ghomi (2002) and M. Yan (2013) considered the problem: if $f : C \subset \mathbb{R}^n \to \mathbb{R}$ is convex and of class C^m on a neighborhood of C (so in fact we are given an m-jet on C), when does there exist a *convex* function $F \in C^m(\mathbb{R}^n)$ such that $F = f$ on C?

One of the main obstacles when trying to apply Whitney extension techniques in the convex case is the fact that partitions of unity destroy convexity.
This difficulty can be overcome, as these authors showed, if C is convex and compact and $D^2 f > 0$ on ∂C.

These are very strong assumptions.

An easier situation: the compact case

Unfortunately the solution for the $C^1_{\text{conv}}(\mathbb{R}^n)$ extension problem for 1-jets is much more complicated than the $C^{1,1}_{\text{conv}}$ case that we studied this morning. Let's start with an easy particular situation: assume that $E \subset \mathbb{R}^n$ is compact. Then there is a couple of simple conditions which are necessary and sufficient for our problem to have a solution:

Theorem (Azagra-Mudarra, 2016)

Let E be a compact subset of $\mathbb{R}^n$. Let $f : E \to \mathbb{R}$ be an arbitrary function, and $G : E \to \mathbb{R}^n$ be a continuous mapping. Then f has a convex, C^1 extension F to all of $\mathbb{R}^n$, with $\nabla F = G$ on E, if and only if (f, G) satisfies:

(C) $f(x) \geq f(y) + \langle G(y), x - y\rangle$

(CW^1) $f(x) = f(y) + \langle G(y), x - y\rangle \implies G(x) = G(y)$

for all $x, y \in E$.

Moreover, the extension $(F, \nabla F)$ of (f, G) can be taken to be Lipschitz on $\mathbb{R}^n$ with

$$\text{Lip}(F) := \sup_{x\in\mathbb{R}^n} |\nabla F(x)| \leq \kappa(n) \sup_{y\in E} |G(y)|,$$

where $\kappa(n)$ only depends on n.

For the proof, the following strategy works:

1. Define $m(x) = m(f, G)(x) = \sup_{y \in E}\{f(y) + \langle G(y), x - y\rangle\}$, the *minimal convex extension* of (f, G). This function is not necessarily differentiable, but it is Lipschitz and convex on $\mathbb{R}^n$, and $m = f$ on E.

2. Show that m is differentiable on E, with $\nabla m(x) = G(x)$ for each $x \in E$.

3. Use Whitney's approximation techniques to construct a (not necessarily convex) differentiable function g such that $g = f$ on C, $g \geq m$ on $\mathbb{R}^n$, and $\lim_{|x|\to\infty} g(x) = \infty$.

4. Take $F = \text{conv}(g)$ and use

Theorem (Kirchheim-Kristensen, 2001)

If $\psi : \mathbb{R}^n \to \mathbb{R}$ is differentiable and $\lim_{|x|\to\infty} \psi(x) = \infty$, then $conv(\psi) \in C^1(\mathbb{R}^n)$.

to ensure that $F \in C^1_{\text{conv}}(\mathbb{R}^n)$.

5. Check that $(F, \nabla F)$ extends (f, G), etc.

Exploring the non-compact case: some examples

Now assume that E is not compact. The previous conditions (C) and (CW^1) are no longer sufficient. The reasons for this insufficiency can be mainly classified into two kinds of difficulties that only arise if the set E is unbounded and G is not uniformly continuous on E:

1. There may be no convex extension of the jet (f, G) to the whole of $\mathbb{R}^n$.
2. Even when there are convex extensions of (f, G) defined on all of $\mathbb{R}^n$, and even when some of these extensions are differentiable in some neighborhood of E, there may be no $C^1(\mathbb{R}^n)$ convex extension of (f, G).

Let us give an example of each situation.

Finding an example of the first situation is easy: you just have to avoid the shower...

... and get in a bathtub instead

Example (Schulz-Schwartz, 1979)

Let $C = \{(x, y) \in \mathbb{R}^2 : x > 0, xy \geq 1\}$*, and define*

$$f(x, y) = -2\sqrt{xy}$$

for every $(x, y) \in C$*. The set* C *is convex and closed, with a nonempty interior, and* f *is convex on a convex open neighborhood of* C*. However,* f *does not have any convex extension to all of* $\mathbb{R}^2$*.*

For jets of the form (f, G) with $f : C \subset \mathbb{R}^n \to \mathbb{R}$ convex and $G(x) \in \partial f(x)$, Schulz and Schwartz (1979) found a necessary and sufficient condition for the existence of a continuous convex function $H : \mathbb{R}^n \to \mathbb{R}$ such that $H(x) = f(x)$ and $G(x) \in \partial H(x)$ for every $x \in C$:

(EX) $\lim_{k\to\infty} |G(x_k)| = +\infty \implies \lim_{k\to\infty} \frac{\langle G(x_k), x_k\rangle - f(x_k)}{|G(x_k)|} = \infty,$

for every sequence $(x_k) \subset C$.

In fact, the extension $H(x)$ can be taken to be $m(x) = \sup_{y\in C}\{f(y) + \langle G(y), x - y\rangle\}$.

By replacing C with an arbitrary set E, it's easy to see that the same condition guarantees that m is everywhere finite and $G(x) \in \partial m(x)$ for every $x \in E$.

The second difficulty is more difficult to explain and overcome: it has to do with:

- the global geometry of convex functions
- the possible presence of *corners at infinity*.

Global geometry of convex functions

Definition (Essential coerciveness)

Let $P_X : \mathbb{R}^n \to X$ be the orthogonal projection onto a subspace $X \subseteq \mathbb{R}^n$. We say that $f : E \subseteq \mathbb{R}^n \to \mathbb{R}$ is *essentially coercive in the direction of X* provided that there is $\ell \in (\mathbb{R}^n)^*$ such that, for every sequence $(x_k)_k \subset E$,

$$\lim_{k\to\infty} |P(x_k)| = \infty \implies \lim_{k\to\infty} (f-\ell)(x_k) = \infty.$$

If $X = \mathbb{R}^n$, we just say that f is *essentially coercive*.

Theorem (Decomposition theorem)

For every convex function $f : \mathbb{R}^n \to \mathbb{R}$ there exist a unique linear subspace $X = X_f$ of $\mathbb{R}^n$, a unique $v \in X_f^{\perp}$, and a unique essentially coercive function $c : X \to \mathbb{R}$ such that

$$f = c \circ P_X + \langle v, \cdot \rangle.$$

Besides, if Y is a linear subspace of $\mathbb{R}^n$ such that f is essentially coercive in the direction of Y, then $Y \subseteq X_f$.

Remark

For a differentiable convex function $f : \mathbb{R}^n \to \mathbb{R}$, the following conditions are equivalent:

1. *f is essentially coercive in the direction of a subspace $X \subseteq \mathbb{R}^n$*
2. *$span\{\nabla f(x) - \nabla f(y) : x, y \in \mathbb{R}^n\} = X$.*

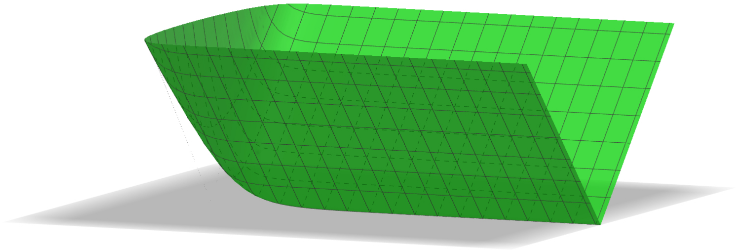
In particular, a convex function is essentially coercive if and only if

$$\mathrm{span}\{\nabla f(x) - \nabla f(y) : x, y \in \mathbb{R}^n\} = \mathbb{R}^n.$$

Corners at infinity

What is a corner at infinity?

What is a corner at infinity? To find one it's not enough to avoid the shower. You have to go to Egypt and do a handstand in the desert.



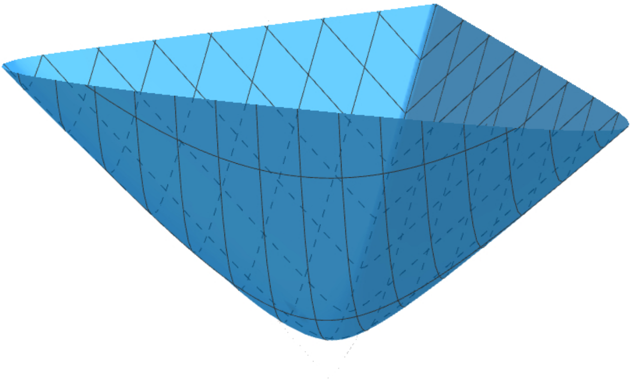
The function $f(x,y)=\sqrt{x^2+e^{-2y}}$ is essentially coercive, and has a corner at infinity directed by the line $x=0$: for the sequences
$z_k=\left(\frac{1}{k},k\right)$, $x_k=(0,k)$, one has that

$$\lim_{k\to\infty} f(x_k)-f(z_k)-\langle\nabla f(z_k),x_k-z_k\rangle=0,$$

and yet

$$\lim_{k\to\infty}|\nabla f(x_k)-\nabla f(z_k)|=1>0$$

A suitable linear perturbation of the same function is coercive (and of course still has a corner at infinity).



Definition (Corners at infinity of 1-jets)

Let X be a proper linear subspace of $\mathbb{R}^n$, $X^\perp$ be its orthogonal complement, and $P : \mathbb{R}^n \to X$ be the orthogonal projection. We say that a jet $(f, G) : E \subset \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n$ has a corner at infinity in a direction of $X^\perp$ provided that there exist two sequences $(x_k)_k, (z_k)_k$ in E such that $(P(x_k))_k$ and $(G(z_k))_k$ are bounded, $\lim_{k\to\infty} |x_k| = \infty$,

$$\lim_{k\to\infty} \left(f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k\rangle\right) = 0,$$

and yet

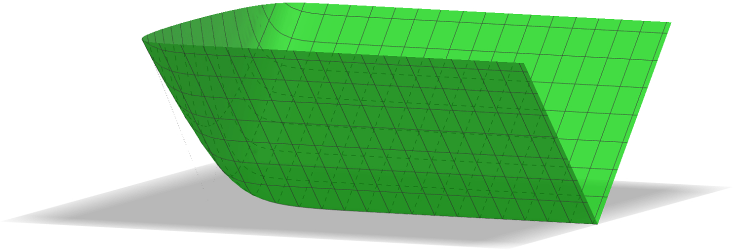
$$\limsup_{k\to\infty} |G(x_k) - G(z_k)| > 0.$$

In particular we will say that the jet (f, G) has a corner at infinity in the direction of the line $\{tv : t \in \mathbb{R}\}$ provided that there exist sequences $(x_k)_k, (z_k)_k$ satisfying the above properties with $X = v^\perp$.

Three significant examples

Examples:

1. The 1-jet (f_1, G_1) defined on $E_1 = \mathbb{R}^2$, where $f_1(x,y) = \sqrt{x^2 + e^{-2y}}$ and $G_1 = \nabla f_1$, has a corner at infinity in the direction of $\{x = 0\}$. Notice that f_1 is convex, essentially coercive, and C^∞.



Examples:

2. Consider $E_2 = \{(x,y) \in \mathbb{R}^2 : y \leq \log|x|\} \cup \{(x,y) \in \mathbb{R}^2 : |x| \geq 1\}$, $f(x,y) = |x|$, $G(x,y) = (-1,0)$ if $x < 0$, $G(x,y) = (1,0)$ if $x > 0$. Since the convex function $(x,y) \mapsto |x|$ and its derivative are smooth in $\mathbb{R}^2 \setminus \{x = 0\}$ and extend the jet (f,G) from E_2 to this region, it is clear that (f,G) satisfies (EX), (W^1), (C), and (CW^1) on E. This jet also has a corner at infinity directed by $\{x = 0\}$. Note that

$$\operatorname{span}\{G(x,y) - G(u,v) : (x,y),(u,v) \in E_3\} = \{y = 0\} \neq \mathbb{R}^2.$$

We claim that there is no C^1 convex extension of (f,G) to $\mathbb{R}^2$. An intuitive geometrical argument that can be made rigorous is that in order to allow the *corner at infinity* that this jet has along the line $\{x = 0\}$, any $C^1_{\text{conv}}(\mathbb{R}^2)$ extension of this datum would have to be essentially coercive in the direction of this line, but the requirement that $f(x,y) = |x|$ for $|x| \geq 1$ precludes the existence of any such extension.

Another way to see that there is no C^1 convex extension of f to $\mathbb{R}^2$ is solving the following exercise. Suppose a function $g : \mathbb{R}^2 \to \mathbb{R}$ is convex and has the following two properties:

1. $g(x,y) = |x|$ for $|x| \geq 1$;
2. there exists a sequence $((x_k, y_k))_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} x_k = 0$ and $g(x_k, y_k) = |x_k|$ for all k.

Then $g(x,y) = |x|$ for all $(x,y) \in \mathbb{R}^2$.

Examples:

3. However, if in the previous example we replace E_2 with $E_3 = \{(x, y) \in \mathbb{R}^2 : y \leq \log |x|\}$, then the situation changes completely: the jet (f, G) defined on E_3 by $f(x, y) = |x|$, $G(x, y) = (-1, 0)$ if $x < 0$, $G(x, y) = (1, 0)$ if $x > 0$, has a C^1 convex extension to $\mathbb{R}^2$. It also has a corner at infinity directed by $\{x = 0\}$, and we have that

$$\text{span}\{G(x, y) - G(u, v) : (x, y), (u, v) \in E_3\} = \{y = 0\} \neq \mathbb{R}^2.$$

Q.: What's the difference between Example 2 and Example 3?

Examples:

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$$\text{span}\{G(x,y) - G(u,v) : (x,y),(u,v) \in E_3\} = \{y = 0\} \neq \mathbb{R}^2.$$

Q.: What's the difference between Example 2 and Example 3?
A.: In Example 3, there is enough room in E_3 to add new jets to the problem (one more point p is just enough in this case), so that we obtain a new extension problem for which we have

$$\text{span}\{G(x,y) - G(u,v) : (x,y),(u,v) \in E_3 \cup \{p\}\} = \mathbb{R}^2.$$

What can we learn from these three examples?

- If a given 1-jet doesn't have corners at infinity then the problem of finding C^1 convex extensions looks easier.
- If a given 1-jet has corners at infinity this will make finding C^1 convex extensions more difficult, but still possible in some cases.
- If the span of the derivatives of a given 1-jet is all of $\mathbb{R}^n$ the problem looks also easier.
- If there are corners at infinity and the span of the putative derivatives is not all of $\mathbb{R}^n$ the problem looks more difficult, and a possible strategy to solve it might be to add new data to the problem so as to reduce it to the previous situation. (Unfortunately this strategy will not work in some cases, but it does go in the correct direction, as we will see.)

The fully geometrically determined case

So let's start with a promising situation: assume that

$$\mathrm{span}\{G(x) - G(y) : x, y \in E\} = \mathbb{R}^n.$$

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How can we solve our problem in this case?

If E is not compact, it's natural to try to replace CW^1 with a similar condition in which sequences take the place of points, that is, if $(x_k)_k, (z_k)_k$ are sequences in E then

$$\lim_{k\to\infty} \left(f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k\rangle\right) = 0 \implies \lim_{k\to\infty} |G(x_k) - G(z_k)| = 0.$$

This means that (f, G) does not have corners at infinity.

But Example 1 tells us that if E is unbounded then this condition is not necessary for the existence of a convex function $F \in C^1(\mathbb{R}^n)$ such that $(F, \nabla F) = (f, G)$: even real analytic convex functions may have corners at infinity!

So, let's try to restrict this condition to *bounded* sequences (x_k). If $(G(z_k))$ is not bounded too, then by using condition (EX), up to extracting a subsequence, we would have

$$\lim_{k\to\infty} \frac{\langle G(z_k), z_k\rangle - f(z_k)}{|G(z_k)|} = \infty,$$

hence

$$\langle G(z_k), z_k\rangle - f(z_k) = M_k |G(z_k)|, \quad \text{with } \lim_{k\to\infty} M_k = \infty,$$

and it follows that

$$\begin{aligned} & f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k\rangle = \\ & f(x_k) - f(z_k) + \langle G(z_k), z_k\rangle - \langle G(z_k), x_k\rangle \geq \\ & f(x_k) + (M_k - |x_k|)\, |G(z_k)| \to \infty \end{aligned}$$

(because $(f(x_k))_k$ and $(x_k)_k$ are bounded and $M_k \to \infty$). Thus we have learned that we cannot have $\lim_{k\to\infty} (f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k\rangle) = 0$ unless $(G(z_k))_k$ is bounded.

Corollary (of a more complicated theorem)

Suppose that span $\{G(x)-G(y) : x, y \in E\} = \mathbb{R}^n$. *Then there exists* $F \in C^1_{conv}(\mathbb{R}^n)$ *with* $F_{|_E} = f$ *and* $(\nabla F)_{|_E} = G$ *if and only if:*

(i) G *is continuous and* $f(x) \geq f(y) + \langle G(y), x-y \rangle$ *for all* $x, y \in E$.

(ii) *If* $(x_k)_k \subset E$ *is a sequence for which* $\lim_{k\to\infty} |G(x_k)| = +\infty$, *then*

$$\lim_{k\to\infty} \frac{\langle G(x_k), x_k \rangle - f(x_k)}{|G(x_k)|} = +\infty.$$

(iii) *If* $(x_k)_k, (z_k)_k$ *are sequences in* E *such that* $(x_k)_k$ *and* $(G(z_k))_k$ *are bounded, and*

$$\lim_{k\to\infty} \left(f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k \rangle \right) = 0,$$

then $\lim_{k\to\infty} |G(x_k) - G(z_k)| = 0$.

Moreover, whenever these conditions are satisfied, the extension F *can be taken to be* essentially coercive.

Let's analyze these conditions:

- (i) G is continuous,and $f(x) \geq f(y) + \langle G(y), x-y \rangle$ for all $x, y \in E$.
- (ii) $(x_k)_k \subset E$, $\lim_k |G(x_k)| = \infty \implies \lim_k \frac{\langle G(x_k), x_k \rangle - f(x_k)}{|G(x_k)|} = \infty$.
- (iii) If $(x_k)_k, (z_k)_k \subset E$, with $(P(x_k))_k$ and $(G(z_k))_k$ bounded, then $f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k \rangle \to 0 \implies |G(x_k) - G(z_k)| \to 0$.

$(i) + (ii) \implies$ finiteness of $m(x)$ for all x, that is, existence of a convex extension (not differentiable in general).

(iii) is a natural generalization of (CW^1): a
$f(x) - f(y) - \langle G(y), x-y \rangle = 0 \implies G(x) = G(y)$ for all $x, y \in E$.

A direct proof of this corollary can be obtained using the ideas of the result for E compact, just replacing points with sequences whenever it's appropriate.

Of course, a C^1 convex extension problem for a given 1-jet (f, G) may have solutions which are not essentially coercive; in fact it may happen that none of its solutions are essentially coercive. A more general, but still partial solution to our main problem, is the following.

Corollary (Azagra-Mudarra, 2017)

Assume that:

(i) G *is continuous and* $f(x) \geq f(y) + \langle G(y), x-y \rangle$ *for all* $x, y \in E$.

(ii) $(x_k)_k \subset E$, $\lim_k |G(x_k)| = \infty \implies \lim_k \frac{\langle G(x_k), x_k \rangle - f(x_k)}{|G(x_k)|} = \infty$.

(iii) *Let* $Y := span\{G(x) - G(y) \, : \, x, y \in E\}$, *and* $P = P_Y$ *the orthogonal projection onto* Y. *If* $(x_k)_k, (z_k)_k \subset E$, *with* $(P(x_k))_k$ *and* $(G(z_k))_k$ *bounded, then*

$$f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k \rangle \to 0 \implies |G(x_k) - G(z_k)| \to 0.$$

Then there exists a convex function $F : \mathbb{R}^n \to \mathbb{R}$ *of class* C^1 *such that* $F_{|_E} = f$ *and* $(\nabla F)_{|_E} = G$.

Condition (iii) of the above corollary can be intuitively rephrased by saying that: 1) our jet satisfies a natural generalization of condition (CW^1); and 2) (f, G) cannot have corners at infinity in any direction contained in the orthogonal complement of $Y = \text{span}\{G(x) - G(y) \, : \, x, y \in E\}$.

A direct proof of this result can be given by following these steps:

1. *Project the given data onto Y.* This process will keep the *corners at infinity* at infinity: we will not encounter any corner at any finite point.
2. Use the previous corollary (i.e., the fully geometrically determined case) to solve the new extension problem in Y, that is: find an $F_Y \in C^1_{\text{conv}}(Y)$ which extends the *projected jet* from $\overline{P(E)}$ to Y
3. Define $F = F_Y \circ P : \mathbb{R}^n \to \mathbb{R}$, so that $(F, \nabla F)$ will extend (f, G) to $\mathbb{R}^n$.

The geometrically underdetermined case

Unfortunately the solution to Problem 2 is necessarily more complicated than the above corollaries. Let's recall Example 3:

Example

Let $E = \{(x,y) \in \mathbb{R}^2 : y \leq \log|x|\}$,
$f(x,y) = |x|$,
$G(x,y) = (-1,0)$ *if* $x < 0$, $G(x,y) = (1,0)$ *if* $x > 0$.
In this case we have
$Y := span\{G(x,y) - G(u,v) \ : \ (x,y),(u,v) \in E\} = \mathbb{R} \times \{0\}$, *and it is easily seen that condition (iii) is not satisfied.*

However, it can be checked that, for $\varepsilon > 0$ *small enough, if we set* $E^* = E \cup \{(0,1)\}$, $f^* = f$ *on* E, $f^*(0,1) = \varepsilon$, $G^* = G$ *on* E, *and* $G^*(0,1) = (0,\varepsilon)$, *then we obtain a new extension problem for which the first corollary gives us a positive solution.*

That is, the problem of finding a C^1 *convex extension of the jet* (f_1^*, G_1^*) *does have a solution, and therefore the same is true of the jet* (f_1, G_1).

This example shows that in some cases the C^1 convex extension problem for a 1-jet (f, G) may be geometrically underdetermined in the sense that we may not have been given enough differential data so as to have condition (iii) of the second corollary satisfied with $Y = \text{span}\{G(x) - G(y) \ : \ x, y \in E\}$, and yet it may be possible to find a few more jets (β_j, w_j) associated to finitely many points $p_j \in \mathbb{R}^n \setminus \overline{E}$, $j = 1, \ldots, m$, so that, if we define $E^* = E \cup \{p_1, \ldots, p_m\}$ and extend the functions f and G from E to E^* by setting

$$f(x_j) := \beta_j, \quad G(p_j) := w_j \quad \text{for} \quad j = 1, \ldots, m, \tag{9.1}$$

then the new extension problem for (f, G) defined on E^* does satisfy condition (iii) of the second corollary.

Notice that, the larger Y grows, the weaker condition (iii) of the second corollary becomes.

Theorem (Azagra-Mudarra, 2017)

There exists a convex function $F : \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that $(F, \nabla F)$ extends (f, G) from E to $\mathbb{R}^n$ if and only if:

(i) *G is continuous and $f(x) \geq f(y) + \langle G(y), x-y \rangle$.*

(ii) *If $\lim_{k\to\infty} |G(x_k)| = +\infty$, then $\lim_{k\to\infty} \frac{\langle G(x_k), x_k \rangle - f(x_k)}{|G(x_k)|} = +\infty$.*

(iv) *Fix $Y := span\{G(x) - G(y) \ : \ x, y \in E\}$. There exists a subspace $X \supseteq Y$ such that, either $Y = X$, or else, if we denote $k = \dim Y$, $d = \dim X$, there exist $p_1, \ldots, p_{d-k} \in \mathbb{R}^n \setminus \overline{E}$, $\beta_1, \ldots, \beta_{d-k} \in \mathbb{R}$, and $w_1, \ldots, w_{d-k} \in \mathbb{R}^n$ such that:*

  (a) $X = span\left(\{u - v \ : \ u, v \in G(E) \cup \{w_1, \ldots, w_{d-k}\}\}\right)$.

  (b) $\beta_j > \max_{1\leq i \neq j \leq d-k}\{\beta_i + \langle w_i, p_j - p_i \rangle\}$ *for all* $1 \leq j \leq d-k$.

  (c) $\beta_j > \sup_{z\in E,\ |G(z)|\leq N}\{f(z) + \langle G(z), p_j - z \rangle\}$ *for all* $1 \leq j \leq d-k$ *and* $N \in \mathbb{N}$.

  (d) $\inf_{x\in E,\ |P_X(x)|\leq N}\{f(x) - \max_{1\leq j\leq d-k}\{\beta_j + \langle w_j, x - p_j \rangle\}\} > 0$ *for all* $N \in \mathbb{N}$.

(iii) *If $(x_k)_k, (z_k)_k \subset E$, with $(P_X(x_k))_k$ and $(G(z_k))_k$ bounded, then $f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k \rangle \to 0 \implies |G(x_k) - G(z_k)| \to 0$.*

The complicated condition, with subconditions $(a)-(d)$, are the analytical expression of the following geometrical condition:

- Either there are data which force all possible C^1 convex extensions to be essentially coercive in the directions of the corners at infinity that (f,G) may have;
- or else there is *enough room in E to add finitely many new data* to the given extension problem, in such a way that the new data are compatible with the old ones (meaning that the new problem still satisfies (i)), and they force all possible extensions to be coercive in the direction of the possible corners, so that condition (iii) holds with $Y=$ the span of the new putative derivatives.

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Let's see a more general version of this result which characterizes the C^1 convex extendibility of 1-jets with prescribed subspaces of essential coerciveness.

Theorem (Azagra-Mudarra 2017)

Let $E \subset \mathbb{R}^n$*,* X *a subspace,* $P = P_X : \mathbb{R}^n \to X$ *o.p.,*
$f : E \to \mathbb{R}$*,* $G : E \to \mathbb{R}^n$*. There exists* $F \in C^1_{conv}(\mathbb{R}^n)$ *so that* $F_{|_E} = f$*,*
$(\nabla F)_{|_E} = G$*, and* $X_F = X$*, if and only if:*

- (i) G *is continuous and* $f(x) \geq f(y) + \langle G(y), x - y \rangle$.
- (ii) $(x_k)_k \subset E$, $\lim_k |G(x_k)| = \infty \implies \lim_k \frac{\langle G(x_k), x_k \rangle - f(x_k)}{|G(x_k)|} = \infty$.
- (iii) *If* $(x_k)_k, (z_k)_k \subset E$*, with* $(P(x_k))_k$ *and* $(G(z_k))_k$ *bounded,*
  $f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k \rangle \to 0 \implies |G(x_k) - G(z_k)| \to 0$.
- (iv) $Y := span\left(\{G(x) - G(y) : x, y \in E\}\right) \subseteq X$.
- (v) *Si* $Y \neq X$*,* $k = \dim Y$*,* $d = \dim X$*, there exist* $w_1, \ldots, w_{d-k} \in \mathbb{R}^n$*,*
  $p_1, \ldots, p_{d-k} \in \mathbb{R}^n \setminus \overline{E}$*, y* $\beta_1, \ldots, \beta_{d-k} \in \mathbb{R}$ *such that*
  - (a) $X = span\left(\{u - v \,:\, u, v \in G(E) \cup \{w_1, \ldots, w_{d-k}\}\}\right)$.
  - (b) $\beta_j > \max_{1 \leq i \neq j \leq d-k} \{\beta_i + \langle w_i, p_j - p_i \rangle\}$
  - (c) $\beta_j > \sup_{z \in E,\, |G(z)| \leq N} \{f(z) + \langle G(z), p_j - z \rangle\}$ *for every* $N \in \mathbb{N}$*,* j*.*
  - (d) $\inf_{x \in E,\, |P(x)| \leq N} f(x) > \max_{1 \leq j \leq d-k} \{\beta_j + \langle w_j, x - p_j \rangle\}$ *for every* $N \in \mathbb{N}$*,* j*.*

Idea of the proof:

1. Use the *complicated condition* to add finitely many data to the problem and find the right projection $P_Y : \mathbb{R}^n \to Y$ so that
$Y = \text{span}\{G(x) - G(y) : x, y \in E\} = Y$ and (f, G) satisfies that:
if $(x_k)_k, (z_k)_k \subset E$, with $(P(x_k))_k$ y $(G(z_k))_k$ bounded, and
$f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k \rangle$, then $|G(x_k) - G(z_k)| \to 0$.

2. *Use P_Y to project the new problem (f, G) onto Y*, obtaining a jet $(\hat{f}, \hat{G})$ defined on $\overline{P_Y(E)} \subset Y$. Corners won't appear at finite points, so on Y we will have:
if $(x_k)_k, (z_k)_k \subset E$, with $(x_k)_k$ and $(\hat{G}(z_k))_k$ bounded, and
$\hat{f}(x_k) - \hat{f}(z_k) - \langle \hat{G}(z_k), x_k - z_k \rangle$, then $|\hat{G}(x_k) - \hat{G}(z_k)| \to 0$.

3. Use the ideas of the solution to the compact case, replacing points with sequences where it's appropriate, to solve the new extension problem in Y. That is: find an $F_Y \in C^1_{\text{conv}}(Y)$ such that $(F_Y, \nabla F_Y)$ extends the *projected jet* $(\hat{f}, \hat{G})$ from $\overline{P_Y(E)}$ to Y.

4. Define $F = F_Y \circ P : \mathbb{R}^n \to \mathbb{R}$. Then $(F, \nabla F)$ extends (f, G) to $\mathbb{R}^n$.

The Lipschitz case

If G is bounded it's natural to expect to find $F \in C^1_{\mathrm{conv}}(\mathbb{R}^n)$ which is Lipschitz as well, and so that $\mathrm{Lip}\, F$ is controlled by $\|G\|_\infty$.

This is indeed possible, and in this case the conditions are simpler.

Theorem (Azagra-Mudarra, 2017)

Let $E \subset \mathbb{R}^n$*, a subspace* $X \subset \mathbb{R}^n$*, and* $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$*. Then there exists a Lipschitz* $F \in C^1_{conv}(\mathbb{R}^n)$ *such that* $F_{|_E} = f$, $(\nabla F)_{|_E} = G$*, and* $X_F = X$*, if and only if:*

(i) *G is continuous and bounded, and* $f(x) \geq f(y) + \langle G(y), x - y \rangle$.

(ii) $Y := span\left(\{G(x) - G(y) : x, y \in E\}\right) \subseteq X$.

(iii) *If* $Y \neq X$ *and* $k = \dim Y$, $d = \dim X$*, there exist* $p_1, \ldots, p_{d-k} \in \mathbb{R}^n \setminus \overline{E}$, $\varepsilon \in (0,1)$*, and l.i.n. vectors* $w_1, \ldots, w_{d-k} \in X \cap Y^{\perp}$ *such that, for every* $j = 1, \ldots, d-k$*, the cone* $V_j := \{x \in \mathbb{R}^n \ : \ \varepsilon \langle w_j, x - p_j \rangle \geq |P_Y(x - p_j)|\}$ *does not contain any point of* $\overline{E}$.

(iv) *If* $(x_k)_k, (z_k)_k \subset E$*, with* $(P_X(x_k))_k$ *bounded, then* $\lim_{k\to\infty} \left(f(x_k) - f(z_k) - \langle G(z_k), x_k - z_k \rangle\right) = 0 \implies \lim_{k\to\infty} |G(x_k) - G(z_k)| = 0.$

Moreover, $\mathrm{Lip}(F) = \sup_{x \in \mathbb{R}^n} |\nabla F(x)| \leq C(n) \sup_{y \in E} |G(y)|$.

Two applications

Theorem (Azagra-Mudarra, 2017)

Let $E \subset \mathbb{R}^n$, $N : E \to \mathbb{S}^{n-1}$ continuous, X subspace. Then there exists a (possibly unbounded) convex body W of class C^1 such that $E \subset \partial W$, $0 \in int(W)$, $N(x) = n_W(x)$ for all $x \in E$, and $X = span\,(n_W(\partial W))$, if and only if:

1. $\langle N(y), x - y \rangle \leq 0$ *for all* $x, y \in E$.
2. *If* $(x_k)_k, (z_k)_k \subset E$*, with* $(P(x_k))_k$ *bounded,*
$$\lim_{k\to\infty} \langle N(z_k), x_k - z_k \rangle = 0 \implies \lim_{k\to\infty} |N(z_k) - N(x_k)| = 0.$$
3. $0 < \inf_{y \in E} \langle N(y), y \rangle$.
4. *Denoting* $d = dim(X)$, $Y = span(N(E))$, $\ell = dim(Y)$, *we have that* $Y \subseteq X$, *and if* $Y \neq X$ *then there exist l.i.n. vectors* $w_1, \ldots, w_{d-k} \in X \cap Y^{\perp}$, *points* $p_1, \ldots, p_{d-\ell} \in \mathbb{R}^n$, $\varepsilon \in (0,1)$ *such that* $\left(\overline{E} \cup \{0\}\right) \cap \left(\bigcup_{j=1}^{d-\ell} V_j\right) = \emptyset$, *where* $V_j := \{x \in \mathbb{R}^n \ : \ \varepsilon \langle w_j, x - p_j \rangle \geq |P_Y(x - p_j)|\}$.

Theorem (Azagra-Hajłasz 2017)

Let $f:\mathbb{R}^n\to\mathbb{R}$ be a convex function. Then f has the Lusin property of class $C^1_{conv}(\mathbb{R}^n)$ (meaning that for every $\varepsilon>0$ there exists $g\in C^1_{conv}(\mathbb{R}^n)$ so that $\mathcal{L}^n\left(\{x\in\mathbb{R}^n : f(x)\neq g(x)\}\right)<\varepsilon$) if and only if:

- *either f is essentially coercive,*
- *or else f is already C^1, in which case taking $g=f$ is the unique choice.*

Thank you very much for your attention!