$C^{1,1}$ and $C^{1,1}_{\text{loc}}$ $C^{1,1}$ convex extensions of jets, and some applications

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Two related problems: $C^{1,1}$ extension and $C^{1,1}$ convex extensions of 1-jets

Problem ($C^{1,1}$ convex extension of 1-jets)

Given E a subset of a Hilbert space X, and a 1-jet (f,G) on E (meaning a pair of functions $f : E \to \mathbb{R}$ and $G : E \to X$), how can we tell whether there is a $C^{1,1}$ *convex* function $F : X \to \mathbb{R}$ which extends this jet (meaning that $F(x) = f(x)$ and $\nabla F(x) = G(x)$ for all $x \in E$)?

Problem ($C^{1,1}$ extension of 1-jets)

Given E a subset of a Hilbert space X, and a 1-jet (f,G) on E, how can we tell whether there is a $C^{1,1}$ function $F : X \to \mathbb{R}$ which extends (f,G)?

Previous solutions to the $C^{1,1}$ extension problem for 1-jets

The $C^{1,1}$ version of the classical Whitney extension theorem theorem tells us that there exists a function $F \in C^{1,1}(\mathbb{R}^n)$ with $F = f$ on C and $\nabla F = G$ on E if and only there exists a constant $M > 0$ such that

$$|f(x) - f(y) - \langle G(y), x - y\rangle| \leq M|x-y|^2, \quad \text{and} \quad |G(x) - G(y)| \leq M|x-y|$$

for all $x, y \in E$.

We can trivially extend (f, G) to the closure $\overline{E}$ of E so that the inequalities hold on $\overline{C}$ with the same constant M. The function F can be explicitly defined by

$$F(x) = \begin{cases} f(x) & \text{if } x \in \overline{C} \\ \sum_{Q \in \mathcal{Q}} \left(f(x_Q) + \langle G(x_Q), x - x_Q\rangle\right) \varphi_Q(x) & \text{if } x \in \mathbb{R}^n \setminus \overline{C}, \end{cases}$$

where $\mathcal{Q}$ is a family of *Whitney cubes* that cover the complement of the closure $\overline{C}$ of C, $\{\varphi_Q\}_{Q\in\mathcal{Q}}$ is the usual Whitney partition of unity associated to $\mathcal{Q}$, and x_Q is a point of $\overline{C}$ which minimizes the distance of $\overline{C}$ to the cube Q.

Recall also that $\text{Lip}(\nabla F) \leq k(n)M$, where $k(n)$ is a constant depending only on n (but with $\lim_{n\to\infty} k(n) = \infty$).

In 1973 J.C. Wells improved this result and extended it to Hilbert spaces.

Theorem (Wells, 1973)

Let E be an arbitrary subset of a Hilbert space X, and $f : E \to \mathbb{R}$, $G : E \to X$. There exists $F \in C^{1,1}(X)$ such that $F_{|_E} = f$ and $(\nabla F)_{|_E} = G$ if and only if there exists $M > 0$ so that

$$f(y) \leq f(x) + \frac{1}{2}\langle G(x) + G(y), y - x\rangle + \frac{M}{4}\|x - y\|^2 - \frac{1}{4M}\|G(x) - G(y)\|^2 \qquad (W^{1,1})$$

for all $x, y \in E$.
In such case one can find F with $Lip(F) \leq M$.

We will say jet (f, G) on $E \subset X$ satisfies condition $(W^{1,1})$ if it satisfies the inequality of the theorem.

It can be checked that $(W^{1,1})$ is absolutely equivalent to the condition in the $C^{1,1}$ version of Whitney's extension theorem, which we will denote $(\widetilde{W}^{1,1})$.

Well's proof was quite complicated, and didn't provide any explicit formula for the extension when E is infinite.

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In 2009 Erwan Le Gruyer showed, by very different means, another version of Well's result.

Theorem (Erwan Le Gruyer, 2009)

Given a Hilbert space X, a subset E of X, and functions $f : E \to \mathbb{R}$, $G : E \to X$, a necessary and sufficient condition for the 1*-jet* (f, G) *to have a $C^{1,1}$ extension $(F, \nabla F)$ to the whole space X is that*

$$\Gamma(f, G, E) := \sup_{x,y \in E} \left(\sqrt{A_{x,y}^2 + B_{x,y}^2} + |A_{x,y}| \right) < \infty, \tag{2.1}$$

where

$$A_{x,y} = \frac{2(f(x) - f(y)) + \langle G(x) + G(y), y - x \rangle}{\|x - y\|^2} \quad \textit{and}$$

$$B_{x,y} = \frac{\|G(x) - G(y)\|}{\|x - y\|} \quad \textit{for all} \quad x, y \in E, x \neq y.$$

Moreover, $\Gamma(F, \nabla F, X) = \Gamma(f, G, E) = \|(f, G)\|_E$, *where*

$$\|(f, G)\|_E := \inf\{\mathrm{Lip}(\nabla H) \ : \ H \in C^{1,1}(X) \ \textit{and} \ (H, \nabla H) = (f, G) \ \textit{on} \ E\}$$

is the trace seminorm of the jet (f, G) *on* E.

The number $\Gamma(f, G, E)$ is the smallest $M > 0$ for which (f, G) satisfies Well's condition $(W^{1,1})$ with constant $M > 0$.

In particular Le Gruyer's condition is also absolutely equivalent to the condition in the $C^{1,1}$ version of Whitney's extension theorem.

Le Gruyer's theorem didn't provide any explicit formula for the extension either (it uses Zorn's lemma).

What about the convex case?

Theorem (Azagra-Mudarra, 2016)

Let E be a subset of $\mathbb{R}^n$, and $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$ be functions. There exists a convex *function $F \in C^{1,\omega}(\mathbb{R}^n)$ if and only if there exists $M > 0$ such that, for all $x, y \in E$,*

$$f(x) - f(y) - \langle G(y), x - y \rangle \geq \frac{1}{2M} |G(x) - G(y)|^2.$$

Moreover, $\sup_{x \neq y} \frac{|\nabla F(x) - \nabla F(y)|}{|x-y|} \leq k(n)M$.

Here, as in Whitney's theorem, $k(n)$ only depends on n, but goes to ∞ as $n \to \infty$ (not surprising, as Whitney's extension techniques were used in the proof, which was constructive).

We say that (f, G) satisfies condition $(CW^{1,1})$ if it satisfies the inequality of the theorem for some $M > 0$.

A constructive optimal solution to these $C^{1,1}$ extension problems.

Theorem (Azagra-Le Gruyer-Mudarra, 2017)

Let (f, G) be a 1-jet defined on an arbitrary subset E of a Hilbert space X. There exists $F \in C^{1,1}_{conv}(X)$ such that $(F, \nabla F)$ extends (f, G) if and only if

$$f(x) \geq f(y) + \langle G(y), x-y\rangle + \frac{1}{2M}|G(x) - G(y)|^2 \quad \textit{for all} \quad x, y \in E,$$

where

$$M = M(G, E) := \sup_{x,y\in E,\, x\neq y} \frac{|G(x) - G(y)|}{|x-y|}.$$

The function

$$F(x) = conv\left(\inf_{y\in E}\{f(y) + \langle G(y), x-y\rangle + \tfrac{M}{2}|x-y|^2\}\right)$$

defines such an extension, with the property that $Lip(\nabla F) \leq M$.

Recall that

$$\mathrm{conv}(g)(x) = \sup\{h(x) \, : \, h \text{ is convex and continuous}, \, h \leq g\}.$$

Other useful expressions for $\mathrm{conv}(g)$ are given by

$$\mathrm{conv}(g)(x) = \inf\left\{ \sum_{j=1}^{n+1} \lambda_j g(x_j) \, : \, \lambda_j \geq 0, \sum_{j=1}^{n+1} \lambda_j = 1, \, x = \sum_{j=1}^{n+1} \lambda_j x_j, \, n \in \mathbb{N} \right\}$$

(or with n fixed as the dimension of $\mathbb{R}^n$ in the case $X = \mathbb{R}^n$), and by the Fenchel biconjugate of g, that is,

$$\mathrm{conv}(g) = g^{**},$$

where

$$h^*(x) := \sup_{v \in \mathbb{R}^n} \{\langle v, x \rangle - h(v)\}.$$

Corollary (Wells 1973, Le Gruyer 2009, Azagra-Le Gruyer-Mudarra 2017)

Let E be an arbitrary subset of a Hilbert space X, and $f : E \to \mathbb{R}$, $G : E \to X$. There exists $F \in C^{1,1}(X)$ such that $F_{|_E} = f$ and $(\nabla F)_{|_E} = G$ if and only if there exists $M > 0$ so that

$$f(y) \leq f(x) + \frac{1}{2}\langle G(x)+G(y), y-x\rangle + \frac{M}{4}|x-y|^2 - \frac{1}{4M}|G(x)-G(y)|^2 \quad (W^{1,1})$$

for all $x, y \in E$. Moreover,

$$F = conv(g) - \tfrac{M}{2}|\cdot|^2, \text{ where}$$
$$g(x) = \inf_{y \in E}\{f(y) + \langle G(y), x-y\rangle + \tfrac{M}{2}|x-y|^2\} + \tfrac{M}{2}|x|^2, \quad x \in X,$$

defines such an extension, with the additional property that $Lip(\nabla F) \leq M$.

Key to the proof of the Corollary: it's well known that $F : X \to \mathbb{R}$ is of class $C^{1,1}$, with $\text{Lip}(\nabla F) \leq M$, if and only if $F + \frac{M}{2}|\cdot|^2$ is convex and $F - \frac{M}{2}|\cdot|^2$ is concave. This result generalizes to jets:

Lemma

Given an arbitrary subset E of a Hilbert space X and a 1-jet (f, G) defined on E, we have:

(f, G) satisfies $(W^{1,1})$ on E, with constant $M > 0$, if and only if the 1-jet $(\tilde{f}, \tilde{G})$ defined by $\tilde{f} = f + \frac{M}{2}|\cdot|^2$, $\tilde{G} = G + MI$, satisfies $(CW^{1,1})$ on E with constant $2M$.

(Proof: See "Further details" below.)

Necessity of $(W^{1,1})$

Proposition

(i) *If (f, G) satisfies $(W^{1,1})$ on E with constant M, then G is M-Lipschitz on E.*

(ii) *If F is a function of class $C^{1,1}(X)$ with $\mathrm{Lip}(\nabla F) \leq M$, then $(F, \nabla F)$ satisfies $(W^{1,1})$ on $E = X$ with constant M.*

Shown by Wells (or see “Further details” below).

Proof of the Corollary: (f, G) satisfies $(W^{1,1})$ with constant M if and only if $(\widetilde{f}, \widetilde{G}) := (f + \frac{M}{2}|\cdot|^2, g + MI)$ satisfies $(CW^{1,1})$ with constant $2M$. Then, by the $C^{1,1}$ convex extension theorem for jets,

$$\tilde{F} = \mathrm{conv}(\widetilde{g}), \quad \tilde{g}(x) = \inf_{y \in E}\{\tilde{f}(y) + \langle \tilde{G}(y), x - y\rangle + M|x-y|^2\}, \quad x \in X,$$

is convex and of class $C^{1,1}$ with $(\tilde{F}, \nabla\tilde{F}) = (\tilde{f}, \tilde{G})$ on E, and $\mathrm{Lip}(\nabla\tilde{F}) \leq 2M$. By an easy calculation,

$$\tilde{g}(x) = \inf_{y \in E}\{f(y) + \langle G(y), x - y\rangle + \tfrac{M}{2}|x-y|^2\} + \tfrac{M}{2}|x|^2, \quad x \in X.$$

Now, by the necessity of $(CW^{1,1})$, $(\tilde{F}, \nabla\tilde{F})$ satisfies condition $(CW^{1,1})$ with constant $2M$ on X. Thus, if

$$F(x) = \tilde{F}(x) - \frac{M}{2}|x|^2, \quad x \in X$$

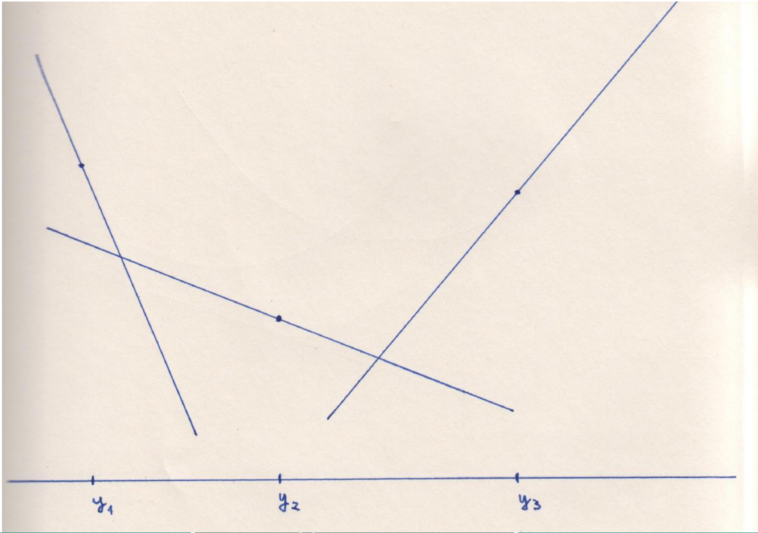
then (again by the preceding lemma) $(F, \nabla F)$ satisfies $(W^{1,1})$ with constant M on X. Hence, by the previous proposition, F is of class $C^{1,1}(X)$, with $\mathrm{Lip}(\nabla F) \leq M$. From the definition of $\tilde{f}, \tilde{G}, \tilde{F}$ and F it is immediate that $F = f$ and $\nabla F = G$ on E.

Sketch of the proof of the $C^{1,1}_{\mathbf{conv}}(X)$ extension result for 1-jets

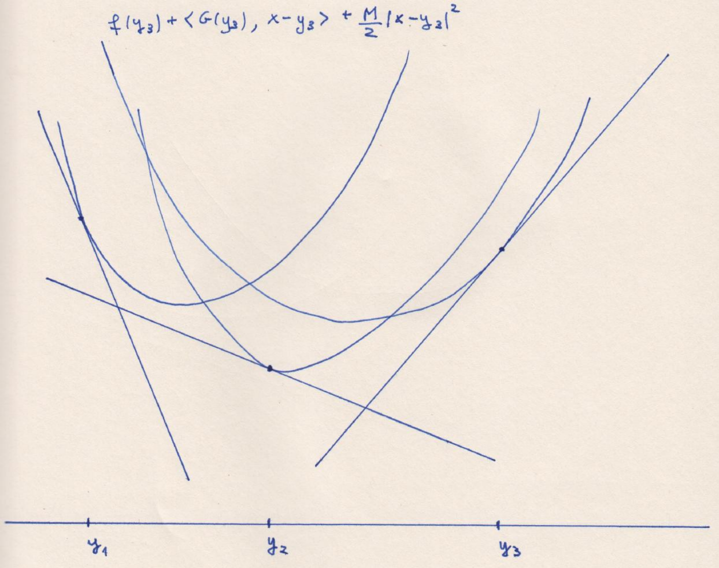
Necessity: we'll see later.

Sketch of the proof of the $C^{1,1}_{\mathrm{conv}}(X)$ extension result for 1-jets

Sufficiency: **1.** Define $m(x) = \sup_{y \in E}\{f(y) + \langle G(y), x - y\rangle\}$, the *minimal convex extension* of (f, G). This function is not necessarily differentiable.



2. Use condition $(CW^{1,1})$ to check that, for all $y, z \in E$, $x \in X$,
$f(z) + \langle G(z), x - z\rangle \leq f(y) + \langle G(y), x - y\rangle + \frac{M}{2}|x - y|^2$.



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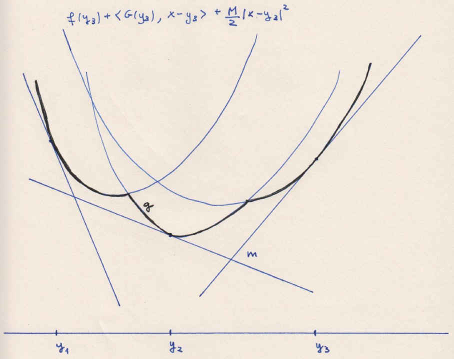
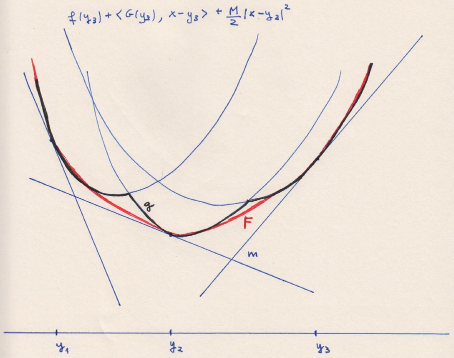
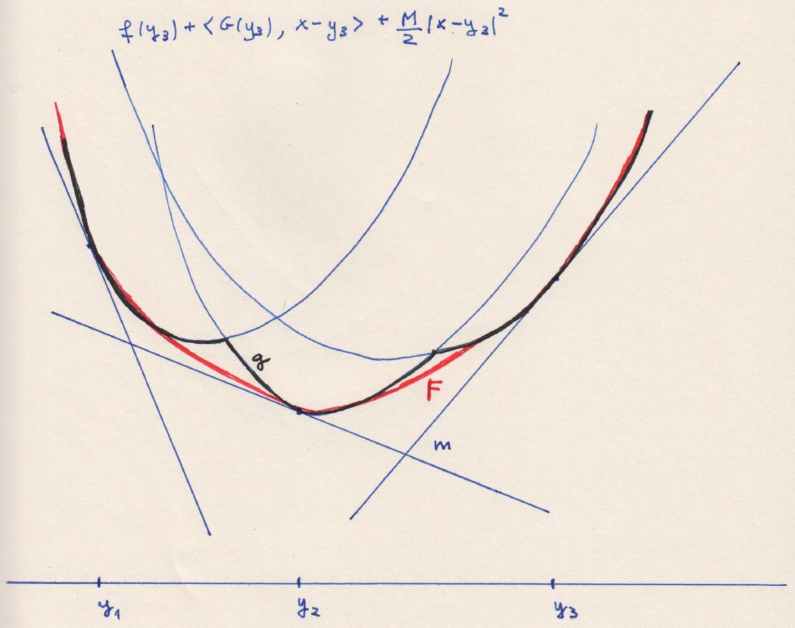
$f(z) + \langle G(z), x - z\rangle \leq f(y) + \langle G(y), x - y\rangle + \frac{M}{2}|x-y|^2$. Hence (taking $\inf_{z\in E}$ on the left and then $\inf_{y\in E}$ on the right)
$m(x) \leq \inf_{y\in E}\{f(y) + \langle G(y), x-y\rangle + \frac{M}{2}|x-y|^2\} =: g(x)$ for all $x \in X$.
Besides $f \leq m \leq g \leq f$ on E, and in particular $m = f = g$ on E.



2. Use condition $(CW^{1,1})$ to check that, for all $y, z \in E$, $x \in X$, $f(z) + \langle G(z), x - z\rangle \leq f(y) + \langle G(y), x - y\rangle + \frac{M}{2}|x-y|^2$. Hence (taking $\inf_{z\in E}$ on the left and then $\inf_{y \in E}$ on the right) $m(x) \leq \inf_{y\in E}\{f(y) + \langle G(y), x-y\rangle + \frac{M}{2}|x-y|^2\} =: g(x)$ for all $x \in X$. Besides $f \leq m \leq g \leq f$ on E, and in particular $m = f = g$ on E. Since m is convex, $m \leq F := \mathrm{conv}(g) \leq g$. Therefore $F = f$ on E.



A constructive optimal solution to these $C^{1,1}$ extension problems.



Sketch of the proof of the $C^{1,1}_{\text{conv}}(X)$ extension result for 1-jets

Sufficiency:

1. Define $m(x) = \sup_{y\in E}\{f(y) + \langle G(y), x-y\rangle\}$, the *minimal convex extension* of (f,G). This function is not necessarily differentiable.

2. Use condition $(CW^{1,1})$ to check that, for all $y, z \in E$, $x \in X$,

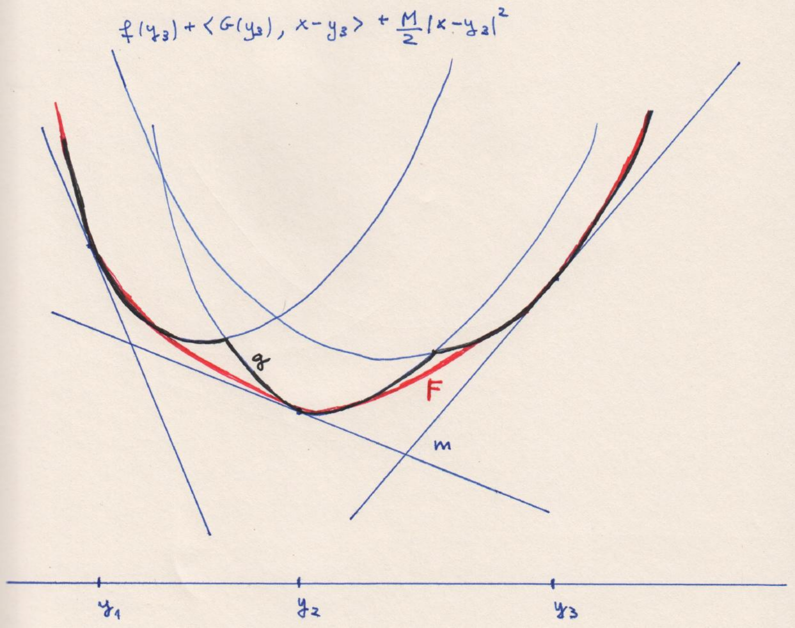
$$f(z) + \langle G(z), x-z\rangle \le f(y) + \langle G(y), x-y\rangle + \frac{M}{2}|x-y|^2.$$

Hence (taking $\inf_{z\in E}$ on the left and then $\inf_{y\in E}$ on the right)

$$m(x) \le \inf_{y\in E}\{f(y) + \langle G(y), x-y\rangle + \frac{M}{2}|x-y|^2\} =: g(x)$$

for all $x \in X$. Besides $f \le m \le g \le f$ on E, and in particular $m = f = g$ on E. Since m is convex, $m \le F := \text{conv}(g) \le g$. Therefore $F = f$ on E.

3. Check that $g(x+h) + g(x-h) - 2g(x) \le M|h|^2$. This inequality is preserved when we take $F = \text{conv}(g)$. Since F is convex, this implies $F \in C^{1,1}(X)$. **4.** Also check that $\nabla F(x) = G(x)$ for every $x \in E$.


f(y3) + <G(y3), x − y3> + M/2 |x − y3|²
g
F
m
y1
y2
y3

Some details: **Step 2: we'll see later. Step 3:**

Lemma

We have $g(x+h)+g(x-h)-2g(x)\leq M|h|^2$ *for all* $x,h\in X$.

Proof: Given $x,h\in X$ and $\varepsilon>0$, by definition of g, we can pick $y\in E$ with

$$g(x)\geq f(y)+\langle G(y),x-y\rangle+\frac{M}{2}|x-y|^2-\varepsilon.$$

We then have

$$\begin{aligned}
g(x+h)+g(x-h)-2g(x) &\leq f(y)+\langle G(y),x+h-y\rangle+\frac{M}{2}|x+h-y|^2\\
&\quad+f(y)+\langle G(y),x-h-y\rangle+\frac{M}{2}|x-h-y|^2\\
&\quad-2\left(f(y)+\langle G(y),x-y\rangle+\frac{M}{2}|x-y|^2\right)+2\varepsilon\\
&=\frac{M}{2}\left(|x+h-y|^2+|x-h-y|^2-2|x-y|^2\right)+2\varepsilon\\
&=M|h|^2+2\varepsilon.
\end{aligned}$$

Some details: Step 3.

$C^{1,1}$ smoothness of F: let's first recall some known facts.

Proposition

For a continuous convex function $f : X \to \mathbb{R}$, the following statements are equivalent.

(i) *There exists $M > 0$ such that*

$$f(x+h) + f(x-h) - 2f(x) \leq M|h|^2 \quad \textit{for all} \quad x, h \in X.$$

(ii) *f is differentiable on X with $\mathrm{Lip}(\nabla f) \leq M$.*

Some details: Step 3.

$C^{1,1}$ smoothness of F.

Proposition (Azagra-Le Gruyer-Mudarra)

Let X be a Banach space. Suppose that a function $g : X \to \mathbb{R}$ has a convex continuous minorant, and satisfies

$$g(x+h) + g(x-h) - 2g(x) \leq M|h|^2 \quad \textit{for all} \quad x, h \in X.$$

Then $H := conv(g)$ is a continuous convex function satisfying the same property. Hence H is of class $C^{1,1}(X)$, with $\text{Lip}(\nabla\psi) \leq M$.

In particular, for a function $\varphi \in C^{1,1}(X)$, we have that $conv(\varphi) \in C^{1,1}(X)$, with $\text{Lip}(\nabla conv(\varphi)) \leq \text{Lip}(\nabla\varphi)$.

Some details: Proof of the $C^{1,1}$ smoothness of this convex envelope:

Given $x, h \in X$ and $\varepsilon > 0$, we can find $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n > 0$ such that

$$H(x) \geq \sum_{i=1}^{n} \lambda_i g(x_i) - \varepsilon, \quad \sum_{i=1}^{n} \lambda_i = 1 \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i x_i = x.$$

Since $x \pm h = \sum_{i=1}^{n} \lambda_i (x_i \pm h)$, we have $H(x \pm h) \leq \sum_{i=1}^{n} \lambda_i g(x_i \pm h)$. Therefore

$$H(x+h)+H(x-h)-2H(x) \leq \sum_{i=1}^{n} \lambda_i \left(g(x_i + h) + g(x_i - h) - 2g(x_i)\right)+2\varepsilon,$$

and by the assumption on g we have

$$g(x_i + h) + g(x_i - h) - 2g(x_i) \leq M|h|^2 \quad i = 1, \ldots, n.$$

Thus

$$H(x + h) + H(x - h) - 2H(x) \leq M|h|^2 + 2\varepsilon, \tag{3.1}$$

and since ε is arbitrary we get the inequality of the statement.

Some details: **Step 4.**

Also note that $m \leq F$ on X and $F = m$ on E, where m is convex and F is differentiable on X. This implies that m is differentiable on E with $\nabla m(x) = \nabla F(x)$ for all $x \in E$.

It is clear, by definition of m, that $G(x) \in \partial m(x)$ (the subdifferential of m at x) for every $x \in E$, and these observations show that $\nabla F = G$ on E.

Application 1: Kirszbraun's extension theorem via an explicit formula.

Corollary (Kirszbraun's Theorem via an explicit formula)

Let X, Y be two Hilbert spaces, E a subset of X and $G : E \to Y$ a Lipschitz mapping. There exists $\widetilde{G} : X \to Y$ with $\widetilde{G} = G$ on E and $\mathrm{Lip}(\widetilde{G}) = \mathrm{Lip}(G)$. In fact, if $M = \mathrm{Lip}(G)$, then the function

$$\widetilde{G}(x) := \nabla_Y(conv(g))(x, 0), \quad x \in X, \quad where$$

$$g(x,y) = \inf_{z \in E} \left\{ \langle G(z), y \rangle_Y + \frac{M}{2} \|x - z\|_X^2 \right\} + \frac{M}{2} \|x\|_X^2 + M \|y\|_Y^2, \ \ (x,y) \in X \times Y,$$

and $\nabla_Y := P_Y \circ \nabla$, defines such an extension.

Proof of this version of Kirszbraun's theorem:

Define the 1-jet (f^*, G^*) on $E \times \{0\} \subset X \times Y$ by $f^*(x,0) = 0$ and $G^*(x,0) = (0, G(x))$. It's easy to see that (f^*, G^*) satisfies condition $(W^{1,1})$ on $E \times \{0\}$ with constant M. Therefore the function F

$$F = \text{conv}(g) - \frac{M}{2}\|\cdot\|^2, \quad \text{where}$$

$$g(x,y) = \inf_{z \in E}\{f^*(z,0) + \langle G^*(z,0), (x-z,y)\rangle + \frac{M}{2}\|(x-z,y)\|^2\} + \frac{M}{2}\|(x,y)\|^2$$

is of class $C^{1,1}(X \times Y)$ with $(F, \nabla F) = (f^*, G^*)$ on $E \times \{0\}$ and $\text{Lip}(\nabla F) \leq M$. In particular, the mapping $X \ni x \mapsto \widetilde{G}(x) := \nabla_Y F(x,0) \in Y$ is M-Lipschitz and extends G. Finally, the expressions defining $\widetilde{G}$ and g can be simplified as

$$\widetilde{G}(x) = \nabla_Y\left(\text{conv}(g) - \frac{M}{2}\|\cdot\|^2\right)(x,0) = \nabla_Y(\text{conv}(g))(x,0)$$

and

$$g(x,y) = \inf_{z \in E}\left\{\langle G(z), y\rangle_Y + \frac{M}{2}\|x-z\|_X^2\right\} + \frac{M}{2}\|x\|_X^2 + M\|y\|_Y^2.$$

New results: $C^{1,1}_{\text{loc}}(\mathbb{R}^n)$ convex extensions of 1-jets.

Theorem (Reformulation of Azagra-LeGruyer-Mudarra's 2017 theorem)

Let E be an arbitrary nonempty subset of $\mathbb{R}^n$. Let $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$ be given functions. Assume that

$$f(z) + \langle G(z), x - z\rangle \leq f(y) + \langle G(y), x - y\rangle + \frac{M}{2}|x - y|^2$$

for every $y, z \in E$ and every $x \in \mathbb{R}^n$. Then the formula

$$F = conv\left(x \mapsto \inf_{y\in E}\left\{f(y) + \langle G(y), x - y\rangle + \frac{M}{2}|x - y|^2\right\}\right)$$

defines a $C^{1,1}$ convex extension of f to $\mathbb{R}^n$ which satisfies $\nabla F = G$ on E and $Lip(\nabla F) \leq M$.

Geometrically speaking, the epigraph of F is the closed convex envelope in $\mathbb{R}^{n+1}$ of the union of the family of paraboloids $\{\mathcal{P}_y : y \in E\}$, where $\mathcal{P}_y = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : t = f(y) + \langle G(y), x - y\rangle + \frac{M}{2}|x - y|^2, x \in \mathbb{R}^n\}$, which must lie above the putative tangent hyperplanes.

Lemma

The following conditions are equivalent:

1. $f(y) - f(z) - \langle G(z), y - z\rangle \geq \frac{1}{2M}\,|G(y) - G(z)|^2$ *for every* $y, z \in E$;
2. $f(z) + \langle G(z), x - z\rangle \leq f(y) + \langle G(y), x - y\rangle + \frac{M}{2}|x - y|^2$ *for every* $y, z \in E,\ x \in X$.

Proof of $(1) \implies (2)$: Given $y, z \in E,\ x \in X$, condition $(CW^{1,1})$ implies

$$
\begin{aligned}
&f(y) + \langle G(y), x - y\rangle + \tfrac{M}{2}|x - y|^2 \\
&\geq f(z) + \langle G(z), y - z\rangle + \tfrac{1}{2M}|G(y) - G(z)|^2 + \langle G(y), x - y\rangle + \tfrac{M}{2}|x - y|^2 = \\
&f(z) + \langle G(z), x - z\rangle + \tfrac{1}{2M}|G(y) - G(z)|^2 + \langle G(z) - G(y), y - x\rangle + \tfrac{M}{2}|x - y|^2 \\
&= f(z) + \langle G(z), x - z\rangle + \tfrac{1}{2M}|G(y) - G(z) + 2M(y - x)|^2 \\
&\geq f(z) + \langle G(z), x - z\rangle.
\end{aligned}
$$

Lemma

The following conditions are equivalent:

1. $f(y) - f(z) - \langle G(z), y - z\rangle \geq \frac{1}{2M}\,|G(y) - G(z)|^2$ *for every* $y, z \in E$*;*
2. $f(z) + \langle G(z), x - z\rangle \leq f(y) + \langle G(y), x - y\rangle + \frac{M}{2}|x - y|^2$ *for every* $y, z \in E,\ x \in X$.

Proof of $(2) \implies (1)$: Minimize the function $\psi(x) := f(y) + \langle G(y), x - y\rangle + \frac{M}{2}|x - y|^2 - (f(z) + \langle G(z), x - z\rangle)$ in order to find $x := y + \frac{1}{M}\,(G(z) - G(y))$. Use inequality (2) for this x, and simplify the expression to obtain (1).

Now we will be looking for analogues of this result for the much more complicated case of $C^{1,1}_{\text{loc}}$ convex extensions of 1-jets.

If the given jet (f, G) has the property that

$$\text{span}\{G(y) - G(z) : y, z \in E\} = \mathbb{R}^n$$

(which is rather generic), then our result is easier to understand and use.

In this talk we will focus on this case and ignore the general situation where $\text{span}\{G(y) - G(z) : y, z \in E\}$ does not necessarily coincide with all of $\mathbb{R}^n$, which is more difficult to handle.

Theorem (2019)

Assume that $\text{span}\{G(x) - G(y) : x, y \in E\} = \mathbb{R}^n$. *Then there exists a convex function* $F \in C^{1,1}_{loc}(\mathbb{R}^n)$ *such that* $(F, \nabla F) = (f, G)$ *on* E *if and only if for each* $y \in E$ *there exists a (not necessarily convex)* $C^{1,1}_{loc}$ *function* $\varphi_y : \mathbb{R}^n \to [0, \infty)$ *such that:*

$$\varphi_y(y) = 0, \nabla\varphi_y(y) = 0;$$

$$\sup\left\{ \frac{|\nabla\varphi_y(x) - \nabla\varphi_y(z)|}{|x - z|} : x, z \in B(0,R), x \neq z, \ y \in E \cap B(0,R) \right\} < \infty$$

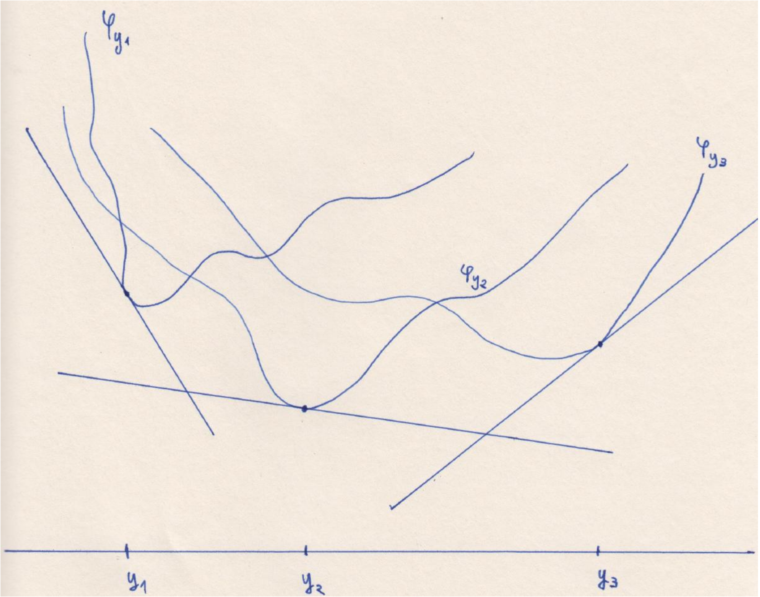
for every $R > 0$*, and*

$$f(z) + \langle G(z), x - z\rangle \leq f(y) + \langle G(y), x - y\rangle + \varphi_y(x) \quad \forall y, z \in E \ \ \forall x \in \mathbb{R}^n.$$

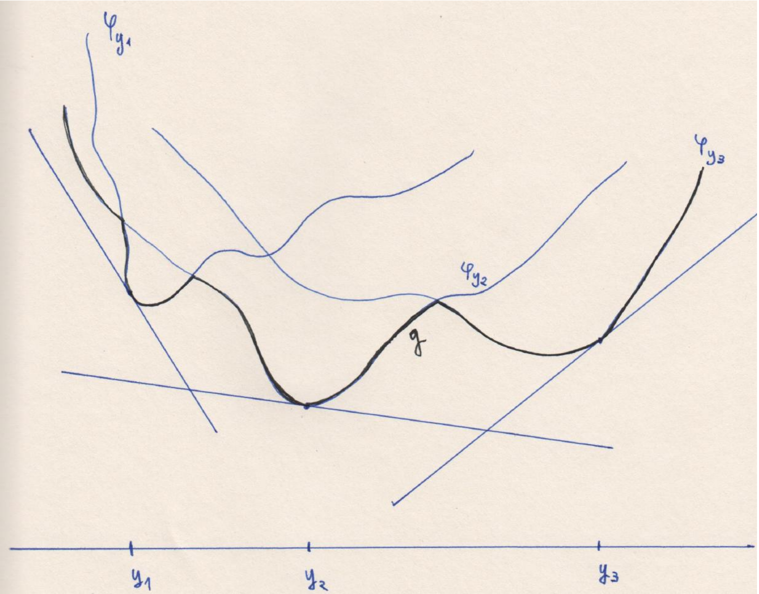
Whenever these conditions are satisfied, we can take (for any $a > 0$*)*

$$F = conv\left(x \mapsto \inf_{y \in E}\left\{ f(y) + \langle G(y), x - y\rangle + \varphi_y(x) + a|x - y|^2 \right\} \right).$$

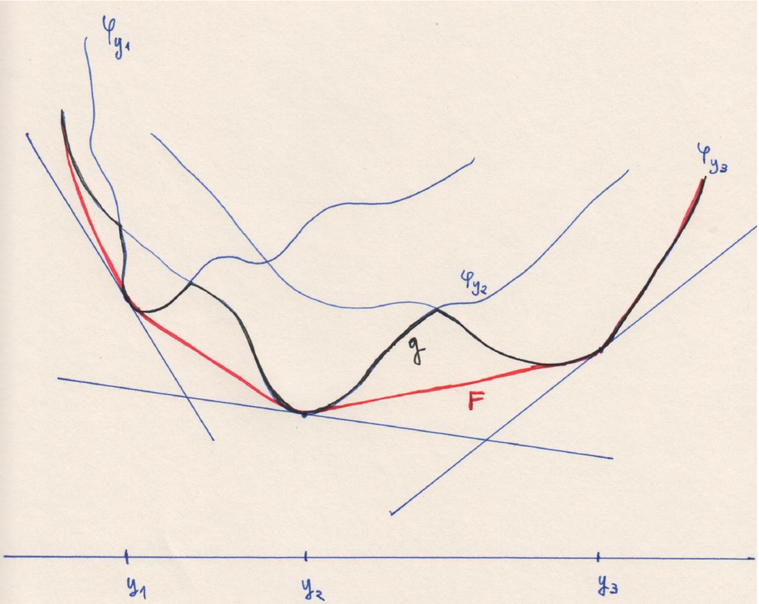
A sketch of the proof of the Theorem.



A sketch of the proof of the Theorem.



A sketch of the proof of the Theorem.



But we must be careful: in dimensions $n \geq 2$, the convex envelope of a $C^{1,1}_{\text{loc}}$ function may not be differentiable!

For instance, $\text{conv}\left((x,y) \mapsto \sqrt{x^2 + \exp(-y^2)}\right) = |x|$.

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In order for these ideas to work we need that g, or at least a linear perturbation of g, be coercive, and overcome some technical difficulties (which we cannot explain in this talk).

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In order for these ideas to work we need that g, or at least a linear perturbation of g, be coercive, and overcome some technical difficulties (which we cannot explain in this talk).

Back to the theorem's statement:

Theorem (2019)

Assume that $span\{G(x)-G(y) : x, y \in E\} = \mathbb{R}^n$*. Then there exists a convex function* $F \in C^{1,1}_{loc}(\mathbb{R}^n)$ *such that* $(F, \nabla F) = (f, G)$ *on* E *if and only if for each* $y \in E$ *there exists a (not necessarily convex)* $C^{1,1}_{loc}$ *function* $\varphi_y : \mathbb{R}^n \to [0, \infty)$ *such that:*

$$\varphi_y(y) = 0, \nabla\varphi_y(y) = 0;$$

$$\sup\left\{\frac{|\nabla\varphi_y(x) - \nabla\varphi_y(z)|}{|x-z|} : x, z \in B(0,R), x \neq z, \ y \in E \cap B(0,R)\right\} < \infty$$

for every $R > 0$*, and*

$$f(z) + \langle G(z), x-z\rangle \leq f(y) + \langle G(y), x-y\rangle + \varphi_y(x) \quad \forall y, z \in E \ \ \forall x \in \mathbb{R}^n.$$

Whenever these conditions are satisfied, we can take (for any $a > 0$*)*

$$F = conv\left(x \mapsto \inf_{y \in E}\left\{f(y) + \langle G(y), x-y\rangle + \varphi_y(x) + a|x-y|^2\right\}\right).$$

The above result is quite general and may be very useful in some situations, as it gives us a lot of freedom in choosing a suitable family of functions $\{\varphi_y\}_{y\in E}$, but of course they do not tell us how to find such a family, which may be inconvenient in other situations. In order to decide whether or not such functions exist and, if they do, how to build them, we need to know something about the global behavior of at least one convex extension ψ of f satisfying $\psi(x) \geq f(y) + \langle G(y), x - y\rangle$ for all $x \in \mathbb{R}^n$ and $y \in E$. The most natural (and minimal) of such extensions is given by

$$m(x) := \sup_{y\in E}\{f(y) + \langle G(y), x - y\rangle\}.$$

The following Corollary gives us a practical condition for the existence of convex extensions F of the jet (f, G).

Corollary (2019)

Let E be an arbitrary nonempty subset of $\mathbb{R}^n$. Let $f : E \to \mathbb{R}$, $G : E \to \mathbb{R}^n$ be functions such that

$$span\{G(x) - G(y) : x, y \in E\} = \mathbb{R}^n.$$

Then there exists a convex function $F \in C^{1,1}_{loc}(\mathbb{R}^n)$ such that $F_{|_E} = f$ and $(\nabla F)_{|_E} = G$ if and only if for each $k \in \mathbb{N}$ there exists a number $A_k \geq 2$ such that

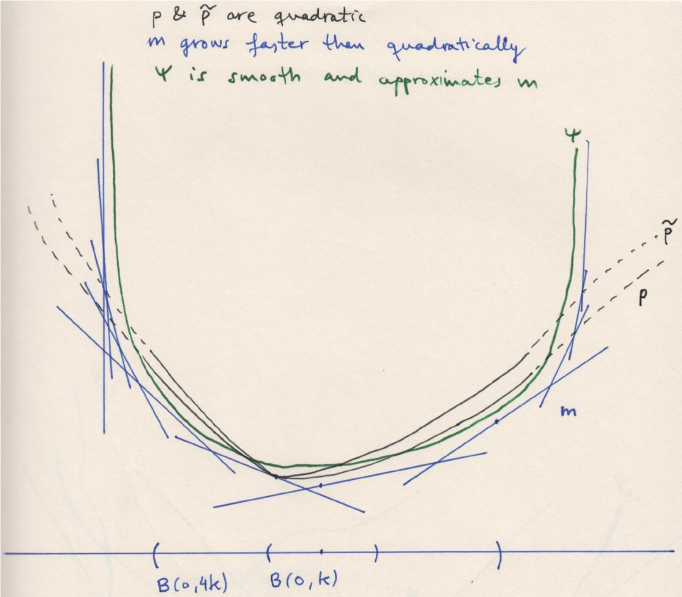
$$m(x) \leq f(y) + \langle G(y), x-y \rangle + \frac{A_k}{2}|x-y|^2 \quad \forall y \in E \cap B(0,k) \;\; \forall x \in B(0,4k).$$

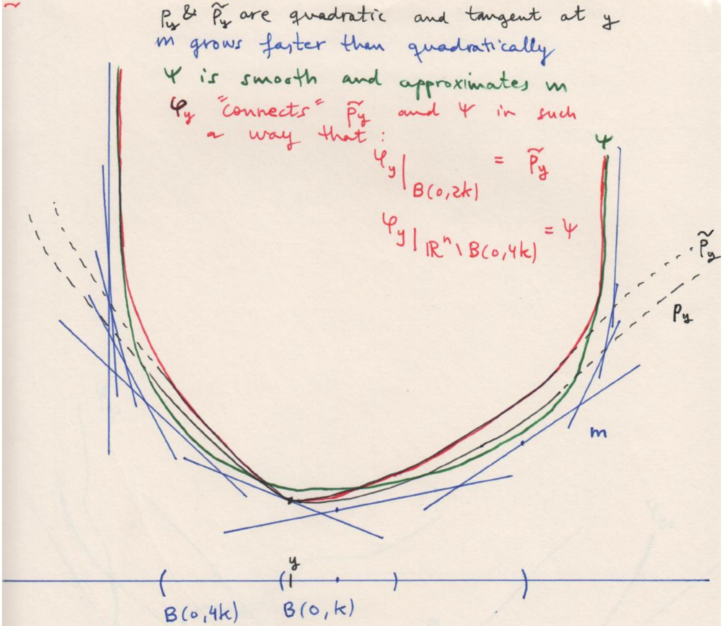
Equivalently,

$$f(z) + \langle G(z), x - z \rangle \leq f(y) + \langle G(y), x - y \rangle + \frac{A_k}{2}|x-y|^2$$

for every $z \in E$, every $y \in E \cap B(0,k)$, and every $x \in B(0,4k)$.

A sketch of the proof of the Corollary.




P_y & $\tilde{P}_y$ are quadratic and tangent at y
m grows faster than quadratically
ψ is smooth and approximates m
φ_y "connects" $\tilde{P}_y$ and ψ in such a way that:
$\varphi_y|_{B(0,2k)} = \tilde{P}_y$
$\varphi_y|_{\mathbb{R}^n \setminus B(0,4k)} = \psi$
ψ
$\tilde{P}_y$
P_y
m
y
$B(0,4k)$
$B(0,k)$

Application 2: Lusin properties of convex functions

Let $\mathcal{A}$ and $\mathcal{C}$ be two classes of functions contained in the class of all functions from $\mathbb{R}^n$ (or an open subset of $\mathbb{R}^n$) into $\mathbb{R}$. If for a given $f \in \mathcal{A}$ and every $\varepsilon > 0$ we can find a function $g \in \mathcal{C}$ such that

$$\mathcal{L}^n\left(\{x : f(x) \neq g(x)\}\right) < \varepsilon, \tag{6.1}$$

we will say that f *has the Lusin property of class* $\mathcal{C}$. Here $\mathcal{L}^n$ denotes Lebesgue's outer measure in $\mathbb{R}^n$. If every function $f \in \mathcal{A}$ satisfies this property, we will also say that the $\mathcal{A}$ has the Lusin property of class $\mathcal{C}$.

This terminology comes from the well known Lusin's theorem (1912): for every Lebesgue-measurable function $f : \mathbb{R}^n \to \mathbb{R}$ and every $\varepsilon > 0$ there exists a continuous function $g : \mathbb{R}^n \to \mathbb{R}$ such that $\mathcal{L}^n\left(\{x : f(x) \neq g(x)\}\right) < \varepsilon$. That is, measurable functions have the Lusin property of class $C(\mathbb{R}^n)$.

Several authors have shown that one can take g of class C^k if f has some weaker regularity properties of order k:

- H. Federer (1944): almost everywhere differentiable functions (and in particular locally Lipschitz functions) have the Lusin property of class C^1.
- H. Whitney (1951) improved this result by showing that a function $f : \mathbb{R}^n \to \mathbb{R}$ has approximate partial derivatives of first order a.e. if and only if f has the Lusin property of class C^1.
- Calderon and Zygmund (1961) proved analogous results for $\mathcal{A} = W^{k,p}(\mathbb{R}^n)$ (the class of Sobolev functions) and $\mathcal{C} = C^k(\mathbb{R}^n)$.
- Other authors, including Liu, Bagby, Michael-Ziemer, Bojarski-Hajłasz-Strzelecki, and Bourgain-Korobkov-Kristensen have improved Calderon and Zygmund's result in several directions, by obtaining additional estimates for $f - g$ in the Sobolev norms, as well as the Bessel capacities or the Hausdorff contents of the exceptional sets where $f \neq g$.
- Generalizing Whitney's result, and Liu and Tai independently established that a function $f : \mathbb{R}^n \to \mathbb{R}$ has the Lusin property of class C^k if and only if f is approximately differentiable of order k almost everywhere.

The Whitney extension technique, or other techniques also related to Whitney cubes and associated partitions of unity, play a key role in the proofs of all of these results.

The Whitney extension technique, or other techniques also related to Whitney cubes and associated partitions of unity, play a key role in the proofs of all of these results.

For the special class of *convex* functions $f : \mathbb{R}^n \to \mathbb{R}$:
G. Alberti (1994) and S.A. Imonkulov (1992) independently showed that every convex function has the Lusin property of class C^2. However, given a convex function f and $\varepsilon > 0$, the function $g \in C^2(\mathbb{R}^n)$ satisfying $\mathcal{L}^n(\{x : f(x) \neq g(x)\}) < \varepsilon$ that they obtained is not convex. This fact is rather disappointing and may thwart the applicability of their result.

Our $C^{1,1}_{\mathrm{loc}}$ convex extension results allow us to show:

Theorem (Azagra-Hajłasz 2019)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then f has the Lusin property of class $C^{1,1\,loc}_{conv}(\mathbb{R}^n)$ (meaning that for every $\varepsilon > 0$ there exists $g : \mathbb{R}^n \to \mathbb{R}$ convex and of class $C^{1,1}_{loc}$ with $\mathcal{L}^n\left(\{x \in \mathbb{R}^n : f(x) \neq g(x)\}\right) < \varepsilon$) if and only if:

- *either f is essentially coercive (in the sense that $\lim_{|x|\to\infty} f(x) - \ell(x) = \infty$ for some linear function ℓ),*
- *or else f is already of class $C^{1,1}_{loc}$, in which case taking $g = f$ is the only possible choice.*

Sketch of the proof of the Theorem.

Consider a convex function $f : \mathbb{R}^n \to \mathbb{R}$, and assume w.l.o.g. that f is coercive. By Alexandroff's theorem we know that, for almost every $x \in \mathrm{Diff}(f)$ there exists an $n \times n$ matrix $\nabla^2 f(x)$ such that

$$\lim_{y \to x} \frac{f(y) - f(x) - \langle \nabla f(x), y - x \rangle - \frac{1}{2}\langle \nabla^2 f(x)(y-x), y - x \rangle}{|y-x|^2} = 0 \quad (\mathcal{A}_2).$$

Let $\varepsilon \in (0,1)$ be given. Since $(\mathcal{A}_2)$ holds for almost every $x \in \mathbb{R}^n$, there exists a closed subset set $A = A_\varepsilon$ of $\mathbb{R}^n$ such that

$$A \subseteq \{x \in \mathbb{R}^n \, : \, (\mathcal{A}_2) \text{ holds at } x\},$$

and

$$\mathcal{L}^n \left(\mathbb{R}^n \setminus A \right) \leq \varepsilon / 8.$$

In particular $\nabla f(x)$ exists for every $x \in A$, and by convexity the restriction of ∇f to A is continuous.

We set $B_0 = \emptyset$, and for each $k \in \mathbb{N}$, we define

$$B_k := B(0,k), \text{ and } A_k := A \cap (B_k \setminus B_{k-1}).$$

We have that

$$A = \bigcup_{k=1}^{\infty} A_k.$$

We also consider the sets

$$E_j := \{y \in A \, : f(x) - f(y) - \langle \nabla f(y), x-y \rangle \leq j|x-y|^2 \; \forall x \in \mathbb{R}^n \text{ s.t. } |x-y| \leq \frac{1}{j}\},$$

for which we have

$$A = \bigcup_{j=1}^{\infty} E_j,$$

and

$$E_j \subset E_{j+1} \text{ for all } j \in \mathbb{N}.$$

Lemma

For each $j \in \mathbb{N}$, the set E_j is closed, and in particular it is measurable.

Proof. Since A is closed, it is enough to see that E_j is closed in A. Let $(y_k)_{k\in\mathbb{N}} \subset E_j$ be such that $\lim_{k\to\infty} y_k = y \in A$, and let us check that $y \in E_j$. Given $x \in \mathbb{R}^n$ with $|x-y| < 1/j$, since $\lim_{k\to\infty} y_k = y$ there exists k_0 large enough so that $|x-y_k| < 1/j$ for all $k \geq k_0$. As $y_k \in E_j$, this implies that

$$f(x) - f(y) - \langle \nabla f(y_k), x - y_k \rangle \leq j|x - y_k|^2$$

for all $k \geq k_0$. Since the restriction of ∇f to A is continuous (f being convex and differentiable on A), by taking limits as $k \to \infty$ we obtain

$$f(x) - f(y) - \langle \nabla f(y), x - y \rangle \leq j|x - y|^2.$$

We have shown that this inequality holds for every y in the open ball of center x and radius $1/j$. By continuity, $y \in E_j$. □

Now, for each $k \in \mathbb{N}$, since the sequence $\{E_j\}_{j\in\mathbb{N}}$ is increasing and $A_k = \bigcup_{j=1}^{\infty} (E_j \cap A_k)$, we can find $j_k \in \mathbb{N}$ such that

$$\mathcal{L}^n(A_k \setminus E_{j_k}) \leq \frac{\varepsilon}{2^{k+3}},$$

and define, for each $k \in \mathbb{N}$,

$$C_k := E_{j_k} \cap A_k,$$

and

$$C := \bigcup_{k=1}^{\infty} C_k.$$

We may obviously assume that

$$j_k \leq j_{k+1} \text{ for all } k \in \mathbb{N}. \tag{6.2}$$

We then have that

$$\mathcal{L}^n(A \setminus C) \leq \sum_{k=1}^{\infty} \mathcal{L}^n(A_k \setminus C_k) \leq \frac{\varepsilon}{8}. \tag{6.3}$$

Lemma

For each $k \in \mathbb{N}$ there exists a number $\beta_k \geq 2$ such that:

$$f(x) - f(y) - \langle \nabla f(y), x - y \rangle \leq \beta_k |x-y|^2 \text{ for all } y \in C \cap B_k \text{ and all } x \in B_{4k}.$$

Proof. Take $y \in C_k$, and note that since (j_k) is increasing we have $C \cap B_k \subseteq E_{j_k} \cap B_k \subset B_{4k}$. If $x \in \mathbb{R}^n$ is such that $|x-y| \leq 1/j$, the inequality we seek obviously holds with $\beta_k = j_k$, because of the definition of E_{j_k}. On the other hand, if $|x-y| > 1/j_k$ and $x \in B_{4k}$, then, since f is Lipschitz on the ball B_{4k}, we have

$$f(x) - f(y) - \langle \nabla f(y), x - y \rangle \leq 2 \operatorname{Lip}\left(f_{|_{B_{4k}}}\right) |x-y| \leq 2 \operatorname{Lip}\left(f_{|_{B_{4k}}}\right) j_k |x-y|^2.$$

In any case the Lemma is satisfied with
$\beta_k = \max\left\{2, j_k, \ 2 j_k \operatorname{Lip}\left(f_{|_{B_{4k}}}\right)\right\}$. □

Since f is convex we have, for all $z \in C$, $x \in \mathbb{R}^n$, that

$$f(z) + \langle \nabla f(z), x - z \rangle \leq f(x),$$

hence $m(x) \leq f(x)$ for all $x \in \mathbb{R}^n$, which combined with the preceding lemma shows that the jet $(f(y), \nabla f(y))$, $y \in C$, satisfies the condition of our result for $C^{1,1}_{\mathrm{loc}}$ convex extension:

Theorem (2019)

Let C be an arbitrary nonempty subset of $\mathbb{R}^n$. Let $f : C \to \mathbb{R}$, $G : C \to \mathbb{R}^n$ be functions such that $span\{G(x) - G(y) : x, y \in E\} = \mathbb{R}^n$. Then there exists a convex function $F \in C^{1,1}_{loc}(\mathbb{R}^n)$ such that $F_{|C} = f$ and $(\nabla F)_{|C} = G$ if and only if for each $k \in \mathbb{N}$ there exists a number $A_k \geq 2$ such that

$$m(x) \leq f(y) + \langle G(y), x - y \rangle + \frac{A_k}{2}|x - y|^2 \quad \forall y \in E \cap B_k \;\; \forall x \in B_{4k}.$$

(We omit the proof that $\text{span}\{\nabla f(y) - \nabla f(z) : y, z \in C\} = \mathbb{R}^n$.)

End of the proof: We have thus checked that the 1-jet $(f(y), \nabla f(y))$, $y \in C$, satisfies all the conditions of the preceding Theorem, and therefore there exists a locally $C^{1,1}$ convex function $F : \mathbb{R}^n \to \mathbb{R}$ such that $F = f$ on C, and also $\nabla F = \nabla f$ on C. In particular we have that

$$\mathcal{L}^n \left(\{ x \in \mathbb{R}^n \, ; f(x) \neq F(x) \text{ or } \nabla f(x) \neq \nabla F(x) \} \right) \leq \mathcal{L}^n \left(\mathbb{R}^n \setminus C \right) \leq \varepsilon.$$

For the other part of the theorem, see "Further details" below.

Thank you for your attention!

Further details

Details: **Necessity of** $(W^{1,1})$**.**
Proof: (i) Given $x, y \in E$, we have

$$f(y) \leq f(x) + \frac{1}{2}\langle G(x) + G(y), y - x\rangle + \frac{M}{4}\|x - y\|^2 - \frac{1}{4M}\|G(x) - G(y)\|^2$$
$$f(x) \leq f(y) + \frac{1}{2}\langle G(y) + G(x), x - y\rangle + \frac{M}{4}\|x - y\|^2 - \frac{1}{4M}\|G(x) - G(y)\|^2.$$

By combining both inequalities we easily get $\|G(x) - G(y)\| \leq M\|x - y\|$.

(ii) Fix $x, y \in X$ and $z = \frac{1}{2}(x + y) + \frac{1}{2M}(\nabla F(y) - \nabla F(x))$.

Details: **Necessity of** $(W^{1,1})$**.**
Using Taylor's theorem we obtain

$$F(z) \leq F(x) + \langle \nabla F(x), \tfrac{1}{2}(y-x) \rangle + \tfrac{1}{2M}(\nabla F(y) - \nabla F(x)) \rangle$$

$$+\tfrac{M}{2}\left\| \tfrac{1}{2}(y-x) + \tfrac{1}{2M}(\nabla F(y) - \nabla F(x)) \right\|^2$$

and

$$F(z) \geq F(y) + \langle \nabla F(y), \tfrac{1}{2}(x-y) \rangle + \tfrac{1}{2M}(\nabla F(y) - \nabla F(x)) \rangle$$

$$-\tfrac{M}{2}\left\| \tfrac{1}{2}(x-y) + \tfrac{1}{2M}(\nabla F(y) - \nabla F(x)) \right\|^2.$$

Details: **Necessity of** $(W^{1,1})$**.**
Then we get

$$\begin{aligned}
F(y) \le\ & F(x) + \langle \nabla F(x), \tfrac{1}{2}(y-x) \rangle + \tfrac{1}{2M}(\nabla F(y) - \nabla F(x)) \rangle \\
& + \tfrac{M}{2} \left\| \tfrac{1}{2}(y-x) \rangle + \tfrac{1}{2M}(\nabla F(y) - \nabla F(x)) \right\|^2 \\
& - \langle \nabla F(y), \tfrac{1}{2}(x-y) \rangle - \tfrac{1}{2M}(\nabla F(y) - \nabla F(x)) \rangle \\
& + \tfrac{M}{2} \left\| \tfrac{1}{2}(x-y) + \tfrac{1}{2M}(\nabla F(y) - \nabla F(x)) \right\|^2 \\
=\ & F(x) + \frac{1}{2} \langle \nabla F(x) + \nabla F(y), y-x \rangle + \\
& + \tfrac{M}{4} \|x-y\|^2 - \tfrac{1}{4M} \|\nabla F(x) - \nabla F(y)\|^2.
\end{aligned}$$

Details: **Relationship between** $(CW^{1,1})$ **and** $(W^{1,1})$**.**

Lemma

(f, G) satisfies $(W^{1,1})$ on E, with constant $M > 0$, if and only if $(\tilde{f}, \tilde{G})$, defined by $\tilde{f}(x) = f(x) + \frac{M}{2}\|x\|^2$, $\tilde{G}(x) = G(x) + Mx$, $x \in E$, satisfies property $(CW^{1,1})$ on E, with constant $2M$.

Details: **Relationship between** $(CW^{1,1})$ **and** $(W^{1,1})$**.**
Proof: Suppose first that (f,G) satisfies $(W^{1,1})$ on E with constant $M>0$. We have, for all $x,y\in E$,

$$\begin{aligned}
&\tilde{f}(x)-\tilde{f}(y)-\langle \tilde{G}(y),x-y\rangle-\frac{1}{4M}\|\tilde{G}(x)-\tilde{G}(y)\|^2\\
&=f(x)-f(y)+\frac{M}{2}\|x\|^2-\frac{M}{2}\|y\|^2-\langle G(y)+My,x-y\rangle\\
&\quad-\frac{1}{4M}\|G(x)-G(y)+M(x-y)\|^2\\
&\geq \frac{1}{2}\langle G(x)+G(y),x-y\rangle-\frac{M}{4}\|x-y\|^2+\frac{1}{4M}\|G(x)-G(y)\|^2\\
&\quad+f(x)-f(y)+\frac{M}{2}\|x\|^2-\frac{M}{2}\|y\|^2-\langle G(y)+My,x-y\rangle\\
&\quad-\frac{1}{4M}\|G(x)-G(y)+M(x-y)\|^2\\
&=\frac{M}{2}\|x\|^2+\frac{M}{2}\|y\|^2-M\langle x,y\rangle-\frac{M}{2}\|x-y\|^2=0.
\end{aligned}$$

Details: **Relationship between** $(CW^{1,1})$ **and** $(W^{1,1})$**.**
Conversely, if $(\tilde{f}, \tilde{G})$ satisfies $(CW^{1,1})$ on E with constant $2M$, we have

$$
\begin{aligned}
& f(x) + \frac{1}{2}\langle G(x) + G(y), y - x\rangle + \frac{M}{4}\|x-y\|^2 - \frac{1}{4M}\|G(x) - G(y)\|^2 - f(y) \\
&= \tilde{f}(x) - \frac{M}{2}\|x\|^2 + \frac{1}{2}\langle \tilde{G}(x) + \tilde{G}(y) - M(x+y), y - x\rangle + \frac{M}{4}\|x-y\|^2 \\
&\quad - \frac{1}{4M}\|\tilde{G}(x) - \tilde{G}(y) - M(x-y)\|^2 - \tilde{f}(y) + \frac{M}{2}\|y\|^2 \\
&= \tilde{f}(x) - \tilde{f}(y) + \frac{1}{2}\langle \tilde{G}(x) + \tilde{G}(y), y - x\rangle + \frac{M}{4}\|x-y\|^2 \\
&\quad - \frac{1}{4M}\|\tilde{G}(x) - \tilde{G}(y) - M(x-y)\|^2 \\
&\geq \langle \tilde{G}(y), x - y\rangle + \frac{1}{4M}\|\tilde{G}(x) - \tilde{G}(y)\|^2 + \frac{1}{2}\langle \tilde{G}(x) + \tilde{G(y)}, y - x\rangle \\
&\quad + \frac{M}{4}\|x-y\|^2 - \frac{1}{4M}\|\tilde{G}(x) - \tilde{G}(y) - M(x-y)\|^2 = 0.
\end{aligned}
$$

Details: **Absolute equivalence of $(W^{1,1})$ and the classical Whitney $C^{1,1}$ extension condition.**

Let E be a subset of a Hilbert space X and $(f,G): E \to \mathbb{R} \times X$ be a 1-jet. Given $M > 0$, we will say that (f,G) satisfies the condition $(W^{1,1}_M)$ on E if the inequality

$$f(y) \leq f(x) + \frac{1}{2}\langle G(x) + G(y), y - x\rangle + \frac{M}{4}\|x-y\|^2 - \frac{1}{4M}\|G(x) - G(y)\|^2,$$

holds for every $x, y \in E$. Also, given $M_1, M_2 > 0$, we will say that (f,G) satisfies the condition $(W^{1,1}_{M_1,M_2})$ on E provided that the inequalities

$$|f(y) - f(x) - \langle G(x), y-x\rangle| \leq M_1\|x-y\|^2, \quad \|G(x) - G(y)\| \leq M_2\|x-y\|,$$

are satisfied for every $x, y \in E$.

Details: **Absolute equivalence of** $(W^{1,1})$ **and the classical Whitney** $C^{1,1}$ **extension condition.**

Claim

$(W^{1,1}_M) \implies (\widetilde{W^{1,1}_{\frac{M}{2},M}})$

First, we have that, for all $x, y \in E$,

$$f(y) \leq f(x) + \frac{1}{2}\langle G(x) + G(y), y - x\rangle + \frac{M}{4}\|x - y\|^2 - \frac{1}{4M}\|G(x) - G(y)\|^2$$

and

$$f(x) \leq f(y) + \frac{1}{2}\langle G(y) + G(x), x - y\rangle + \frac{M}{4}\|y - x\|^2 - \frac{1}{4M}\|G(y) - G(x)\|^2.$$

Details: **Absolute equivalence of $(W^{1,1})$ and the classical Whitney $C^{1,1}$ extension condition.**
By summing both inequalities we get $\|G(x)-G(y)\| \leq M\|x-y\|$. On the other hand, by using $(W^{1,1}_M)$, we can write

$$f(y)-f(x)-\langle G(x), y-x\rangle \leq \frac{1}{2}\langle G(x)+G(y), y-x\rangle - \langle G(x), y-x\rangle$$
$$+\frac{M}{4}\|x-y\|^2 - \frac{1}{4M}\|G(x)-G(y)\|^2$$
$$=\frac{1}{2}\langle G(y)-G(x), y-x\rangle + \frac{M}{4}\|x-y\|^2 - \frac{1}{4M}\|G(x)-G(y)\|^2$$
$$=\frac{M}{2}\|x-y\|^2 - \frac{M}{4}\left(\|x-y\|^2 + \frac{1}{M^2}\|G(x)-G(y)\|^2 - 2\langle\frac{1}{M}(G(y)-G(x)), y$$
$$=\frac{M}{2}\|x-y\|^2 - \frac{M}{4}\left\|\frac{1}{M}(G(x)-G(y))-(y-x)\right\|^2 \leq \frac{M}{2}\|x-y\|^2.$$

Details: **Absolute equivalence of $(W^{1,1})$ and the classical Whitney $C^{1,1}$ extension condition.**

Also, we have

$$f(x)-f(y)-\langle G(x),x-y\rangle \leq \frac{1}{2}\langle G(x)+G(y),x-y\rangle - \langle G(x),x-y\rangle$$
$$+\frac{M}{4}\|x-y\|^2-\frac{1}{4M}\|G(x)-G(y)\|^2$$
$$=\frac{1}{2}\langle G(y)-G(x),x-y\rangle+\frac{M}{4}\|x-y\|^2-\frac{1}{4M}\|G(x)-G(y)\|^2$$
$$=\frac{M}{2}\|x-y\|^2-\frac{M}{4}\left(\|x-y\|^2+\frac{1}{M^2}\|G(x)-G(y)\|^2-2\langle\frac{1}{M}(G(y)-G(x)),x$$
$$=\frac{M}{2}\|x-y\|^2-\frac{M}{4}\left\|\frac{1}{M}(G(x)-G(y))-(x-y)\right\|^2\leq\frac{M}{2}\|x-y\|^2.$$

This leads us to

$$|f(y)-f(x)-\langle G(x),y-x\rangle|\leq\frac{M}{2}\|x-y\|^2,$$

which proves the first claim.

Details: **Absolute equivalence of $(W^{1,1})$ and the classical Whitney $C^{1,1}$ extension condition.**

Claim

$(\widetilde{W^{1,1}_{M_1,M_2}}) \implies (W^{1,1}_M)$, *where* $M = (3+\sqrt{10})\max\{M_1, M_2\}$.

Using that $f(y) - f(x) - \langle G(x), y-x\rangle \leq M_1\|x-y\|^2$, we can write

$$
\begin{aligned}
& f(y) - f(x) - \frac{1}{2}\langle G(x)+G(y), y-x\rangle - \frac{M}{4}\|x-y\|^2 + \frac{1}{4M}\|G(x)-G(y)\|^2 \\
& \leq \langle G(x), y-x\rangle + M_1\|x-y\|^2 - \frac{1}{2}\langle G(x)+G(y), y-x\rangle - \frac{M}{4}\|x-y\|^2 + \frac{1}{4M} \\
& = \frac{1}{2}\langle G(x)-G(y), y-x\rangle + \left(M_1 - \frac{M}{4}\right)\|x-y\|^2 + \frac{1}{4M}\|G(x)-G(y)\|^2 \\
& \leq \frac{1}{2}ab + \left(M_1 - \frac{M}{4}\right)a^2 + \frac{1}{4M}b^2,
\end{aligned}
$$

where $a = \|x-y\|$ and $b = \|G(x)-G(y)\|$. Since G is M_2-Lipschitz, we have the inequality $b \leq M_2 a$.

Details: **Absolute equivalence of $(W^{1,1})$ and the classical Whitney $C^{1,1}$ extension condition.**

So, the last term in the above chain of inequalities is smaller than or equal to

$$(\tfrac{1}{2}M_2 + (M_1 - \tfrac{M}{4}) + \tfrac{1}{4M}M_2^2)a^2 \leq (\tfrac{1}{2}K + (K - \tfrac{M}{4}) + \tfrac{1}{4M}K^2)a^2,$$

where $K = \max\{M_1, M_2\}$. Now, the last term is smaller than or equal to 0 if and only if $-M^2 + 6MK + K^2 \leq 0$. But, in fact, for $M = (3+\sqrt{10})K$ the term $-M^2 + 6MK + K^2$ is equal to 0. This proves the second Claim.

Details: The other part of the Azagra-Hajłasz theorem on Lusin properties of convex functions follows from:

Theorem (2013)

For every convex function $f : \mathbb{R}^n \to \mathbb{R}$, there exist a unique linear subspace X_f of $\mathbb{R}^n$, a unique vector $v_f \in X_f^{\perp}$, and a unique essentially coercive function $c_f : X_f \to \mathbb{R}$ such that f can be written in the form

$$f(x) = c_f(P_{X_f}(x)) + \langle v_f, x \rangle \text{ for all } x \in \mathbb{R}^n.$$

Proposition (Azagra-Hajłasz 2019)

Let $P : \mathbb{R}^n \to X$ be the orthogonal projection onto a linear subspace X of $\mathbb{R}^n$ of dimension k, with $1 \leq k \leq n-1$, let $c : X \to \mathbb{R}$ be a convex function, and define $f(x) = c(P(x))$. Then f is the only convex function $g : \mathbb{R}^n \to \mathbb{R}$ such that $\mathcal{L}^n(\{x \in \mathbb{R}^n : f(x) \neq g(x)\}) < \infty$.

Proof of the Proposition. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that $\mathcal{L}^n(A) < \infty$, where $A := \{x \in \mathbb{R}^n : f(x) \neq g(x)\}$. By Fubini's theorem, for $\mathcal{H}^k$-almost every point $x \in X$, we have that for $\mathcal{H}^{n-k-1}$-almost every direction $v \in X^\perp$, $|v| = 1$, the line $L(x, v) := \{x + tv : t \in \mathbb{R}\}$ must intersect A in a set of finite 1-dimensional measure. This implies that for all such $x \in X, v \in X^\perp$, the set $L(x, v) \cap (\mathbb{R}^n \setminus A)$ contains sequences

$$x_j^{\pm} := x + t_{x,j}^{\pm} v \in \mathbb{R}^n \setminus A, j \in \mathbb{N}$$

with $\lim_{j \to \pm\infty} t_{x,j}^{\pm} = \pm\infty$. Since $f = f \circ P$, this means that

$$f(x) = f(x + t_{x,j}^{\pm} v) = g(x + t_{x,j}^{\pm} v),$$

and because $t \mapsto g(x + tv)$ is convex we see that

$$f(x + tv) = f(x) = g(x + tv)$$

for all $t \in \mathbb{R}$ and every such x, v. By continuity of f and g this implies that

$$f(x + tv) = g(x + tv)$$

for all $x \in X$, $v \in X^\perp$, and this shows that $f = g$ on $\mathbb{R}^n$. □