CORRIGENDUM TO "LOCALLY DEFINABLE GROUPS IN O-MINIMAL STRUCTURES"

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In [3], the second author develops the theory of locally definable groups. In particular, using the notion of connectedness given by Definition 3.7 in [3], he studies the connected locally definable subgroups of a locally definable group obtaining similar results as in the definable case (see Section 3.1 in [3]). However, we show that this definition of connectedness is not correct for the category of locally definable groups. Moreover, we show that with this definition some results in [3] are false.

Counterexample (to Proposition 3.9 in [3]). Let R be an $\aleph_1$-saturated real closed field and consider the definable sets $Z_n = (-n, -\frac{1}{n}) \cup (\frac{1}{n}, n)$ for $n \in \mathbb{N}$, $n > 1$. Then $\mathcal{Z} = \bigcup_{n>1} Z_n$ is a connected locally definable group with the multiplicative operation of R. Intuitively, we see that $\mathcal{Z}$ is the disjoint union of $\bigcup_{n>1}(\frac{1}{n}, n)$ and $\bigcup_{n>1}(-n, -\frac{1}{n})$, but neither of these sets is definable. Consider also the locally definable subgroup $\mathcal{H} = \bigcup_{n>1}(\frac{1}{n}, n)$ of $\mathcal{Z}$. Both $\mathcal{Z}$ and $\mathcal{H}$ are compatible, connected, normal and $dim(\mathcal{Z}) = dim(\mathcal{H})$, which is a contradiction with Proposition 3.9 in [3].

The flaw in the proof of Proposition 3.9 in [3] comes from the following incorrect statement: given a locally definable group $\mathcal{Z}$, a connected compatible locally definable normal subgroup of $\mathcal{Z}$ with the same dimension contains all connected locally definable subgroups of $\mathcal{Z}$.

Inspired by the theory of locally semi-algebraic spaces from [1], Definition 3.7 must be replaced by the following.

Definition. *Let $\mathcal{Z}$ be a locally definable group. We say that a set $X \subset \mathcal{Z}$ is* ***connected*** *if there is no subset $U \subset \mathcal{Z}$ such that (i) the intersection of U with every definable subset of $\mathcal{Z}$ is definable and (ii) $U \cap X$ is a non-empty proper subset of X which is closed and open in the topology induced on X by $\mathcal{Z}$.*

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Observe that every definable subset of a locally definable group satisfies condition (i) above and that therefore the new notion of connectedness is stronger than Definition 3.7 in [3]. Also note that, using the terminology of [2], clearly a subset of a locally definable group satisfying (i) above is a compatible locally definable subset (see Definition 2.8 in [2]).

With this new definition, we can prove a correct version of Proposition 3.9 in [3] just following its proof.

Proposition 1. *Let $\mathcal{Z}$ be a locally definable group. Then there is a unique connected compatible locally definable subgroup $\mathcal{Z}^0$ of $\mathcal{Z}$ with dimension* $\dim\mathcal{Z}$*. Moreover, the following hold:*
(i) $\mathcal{Z}^0$ contains all connected locally definable subgroups of $\mathcal{Z}$;
(ii) $\mathcal{Z}^0$ is the smallest compatible locally definable subgroup of $\mathcal{Z}$ such that $(\mathcal{Z} : \mathcal{Z}^0) < \aleph_1$, and
(iii) $\mathcal{Z}^0$ is normal.

Proposition 3.9 in [3] (compare with Proposition 2.18 in [2]) is used extensively in both papers [3] and [2]. Therefore in both papers the definition of connectedness must be replace by the new one, in which case all the results remain true.

In spite of the fact that Definition 3.7 in [3] is not correct for the category of locally definable groups, it still makes sense and therefore we will call it "weakly connected". The next proposition gives us a relation between the notions of connectedness and weakly connectedness.

Proposition 2. *Let $\mathcal{Z}$ be a locally definable group which is not weakly connected. Then $\mathcal{Z}^0$ (that of Proposition 1) is definable and contains all weakly connected locally definable subgroups of $\mathcal{Z}$.*
Proof. Let U be a definable subset of $\mathcal{Z}$ such that U is both open and closed. By Corollary 3.6 in [3] there exist a locally definable subset $\{z_r : r \in S\}$ of $\mathcal{Z}$ such that $\mathcal{Z} = \bigcup_{r\in S} z_r\mathcal{Z}^0$. Since $\mathcal{Z}^0$ is connected it is easy to prove that if $z_r\mathcal{Z}^0 \cap U \neq \emptyset$ then $z_r\mathcal{Z}^0 \subset U$. Therefore $U = \bigcup_{r\in S'} z_r\mathcal{Z}^0$ for some $S' \subset S$. Since U is definable, by saturation we deduce that S' is finite and $z_r\mathcal{Z}^0$ is definable for each $r \in S'$. Hence $\mathcal{Z}^0$ is definable and it is closed and open. Suppose $\mathcal{H}$ is a weakly connected locally definable subgroup of $\mathcal{Z}$. Since $\mathcal{Z}^0$ is definable, open, closed and $\mathcal{H} \cap \mathcal{Z}^0 \neq \emptyset$ we deduce that $\mathcal{H} \cap \mathcal{Z}^0 = \mathcal{H}$, i.e, $\mathcal{H}$ is contained in $\mathcal{Z}^0$. □

Corollary. *Let $\mathcal{Z}$ be a locally definable group. Then there exists a unique weakly connected locally definable subgroup of $\mathcal{Z}$ which contains all weakly connected locally definable subgroups of $\mathcal{Z}$. Moreover, this weakly connected locally definable subgroup equals $\mathcal{Z}$ or $\mathcal{Z}^0$ (so in particular is compatible).*

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