This work is devoted to the approximation of differentiable semialgebraic functions by Nash functions. This problem is well-known in case that the functions are defined on an affine Nash manifold $M$, and here we extend it to functions defined on Nash sets with a special kind of singularities.

**Definition.** Let $M$ be an affine Nash manifold of $\mathbb{R}^n$. Let $X \subset M$ be a Nash subset and let $x \in X$. The germ $X_x$ is a monomial singularity if there is a neighborhood $U$ of $x$ in $M$ equipped with a Nash diffeomorphism $u: U \to \mathbb{R}^n$ with $u(x) = 0$ that maps $X \cap U$ to an union of coordinate linear varieties. That is, there is a (finite) family $\lambda$ of subsets of indices $\lambda = \{\lambda_1, \ldots, \lambda_s\}$ of possibly different cardinality $r \leq m$ such that

$$X \cap U = \bigcup_{\lambda} \{u = 0\}$$

where $u = (u_1, \ldots, u_m)$ and $\{u = 0\}$ denotes $\{u_1 = \cdots = u_m = 0\}$. For simplicity we assume that there are no immersed components, that is, if $X, X' \subset A$ are different then $X \not\subset X' \not\subset X$. This assures that the germs $\{u = 0\}$ are the irreducible components of the germ $X_x$. We say that $X$ has a monomial singularity of type $\Lambda$ at $x$. A Nash set $X \subset M$ has monomial singularities in $M$ if all germs $X_x$, $x \in X$, are monomial singularities.

Let us do some remarks concerning the notion of type:

- A different Nash diffeomorphism $\phi$ may provide a different type $\phi'$ and therefore a monomial singularity has several types. Thus, two types $\Lambda$ and $\Lambda'$ will be called equivalent if the union of the coordinate linear varieties given by $A$ in Nash diffeomorphims to the one given by $A'$ as germs in the origin (and as we see below, it is possible to show that this equivalence is in fact global via linear isomorphisms).

- For example, if we have a union of hyperplanes $X \cap U = \{u_i = 0\}$ over the obvious linear change of coordinates. That is, in the context of Nash normal crossings, the number of hyperplanes determines the type up to linear isomorphism. As we will see now, for monomial singularities the characterization of the type is far more involved.

**So how can we determine the equivalence class of a type?** We can do it even arithmetically. Let $\mathcal{L} = \{L_1, \ldots, L_s\}$ be a family of coordinate linear varieties of $\mathbb{R}^n$ without immersions. For each subset $I \subset \{1, \ldots, s\}$ and each $1 \leq p \leq s$ we denote $L_{I_p} = \bigcap_{i \in I_p} L_i$. Next, to each family of different nonempty subsets $I_1, \ldots, I_s \subset \{1, \ldots, s\}$, $r \geq 1$, we associate the number $\dim (L_{I_1} + \cdots + L_{I_r})$. The collection of all the previous dimensions will be called the load of $\mathcal{L}$.

**Proposition.** Let $\mathcal{L} = \{L_1, \ldots, L_s\}$ and $\mathcal{L}' = \{L'_1, \ldots, L'_s\}$ be coordinate linear varieties of $\mathbb{R}^n$. There is a linear isomorphism $f$ of $\mathbb{R}^n$ such that $f(L_i) = L_i'$ for all $i$ if and only if the loads of $\mathcal{L}$ and $\mathcal{L}'$ coincide.

**Properties of Nash sets with monomial singularities**

From the definition it is not clear a priori if the set of points of a Nash set where the germ is a monomial singularity is a semialgebraic set. The answer is positive:

**Theorem.** Let $X \subset M$ be a Nash set. Fix a type $\Lambda$ and put $T^\Lambda = \{x \in X \mid X_x \text{ has a monomial singularity of type } \Lambda \text{ at } x\}$. Then, the set $T^\Lambda$ is semialgebraic.

The proof of the above fact uses in a crucial way Artin’s approximation theorem with bounds. Another important fact is that following finiteness property:

**Theorem.** Let $X \subset M$ be a Nash set with monomial singularities. Then $X$ can be covered with finitely many open sets $U$ of $M$ each one equipped with a Nash diffeomorphism $u: U \to \mathbb{R}^n$ that maps $X \cap U$ onto an union of coordinate linear varieties.

To prove the above result it is important to study first Nash functions on Nash sets with monomial singularities. Recall that if $X$ is a Nash subset of a Nash manifold $M$ then we say that a function $f : X \to \mathbb{R}$ is a Nash function if there exists an open semialgebraic neighborhood $U$ of $X$ and a Nash extension $F : U \to \mathbb{R}$ of $f$. We denote by $N(X)$ the ring of Nash functions of $X$. We say that $f : X \to \mathbb{R}$ is $C^q$ Nash if its restriction to each irreducible component is a Nash function. We denote by $N^q(X)$ the ring of $C^q$ functions. Similarly, we define the ring $S^q(X)$ of $C^q$ semialgebraic functions on $X$ and the ring $S^q(X)$ of $C^q$ semialgebraic functions on $X$ for $q \geq 1$. By means of analytic coherence of Nash sets with monomial singularities we have the following weak normality property:

**Theorem.** If $X$ is a Nash set with monomial singularities then $N(X) = N^1(X)$.

**References**

