COMMUTATIVE ALGEBRA 2008-09

Final exam, February 9, 2009

Time: 3 hours

You may answer this exam in English or in Spanish

1) (2 points) Let $A \subset B$ be a ring extension and let $y \in B$. Prove that y is integral over A if and only if the subring $A[y]$ of B is a finite module over A.

2) (3 points) Decide if the following statements are true or false. Explain your answers.

a) If $\mathfrak{q}$ is a $\mathfrak{p}$–primary ideal of a commutative ring A, then the homomorphism

$$\begin{array}{ccc} A/\mathfrak{q} & \longrightarrow & (A/\mathfrak{q})_{\mathfrak{p}} \\ \overline{a} & \mapsto & \overline{a}/\overline{1} \end{array}$$

is injective.

b) The ring $\mathbf{k}[X,Y]/(X^n, Y^n)$ is a local ring ($\mathbf{k}$ is a field).

c) The ring $\mathbf{Z}[X]/(X^2-2)$ has Krull dimension 2.

3) (3 points) Let $\mathbf{k}$ be a field, let $A = \mathbf{k}[X,Y,Z]$ and let $\mathfrak{m} = (X,Y,Z)$. Consider the ring homomorphism

$$\begin{array}{ccc} \mathbf{k}[X,Y,Z] & \stackrel{\varphi}{\longrightarrow} & \mathbf{k}[T] \\ f(X,Y,Z) & \mapsto & f(T^2,T^3,T^5). \end{array}$$

Let I be the kernel of φ and let $B = A/I$.

a) Prove that I is generated by $Y^2 - X^3$ and $XY - Z$.

b) Find a finite free resolution of A/I (that is, give an exact sequence

$$0 \longrightarrow F_k \stackrel{\Psi_k}{\longrightarrow} F_{k-1} \stackrel{\Psi_{k_1}}{\longrightarrow} \cdots \stackrel{\Psi_2}{\longrightarrow} F_1 \stackrel{\Psi_1}{\longrightarrow} F_0 \stackrel{\Psi_0}{\longrightarrow} B \longrightarrow 0,$$

where the F_i are free A–modules of finite rank, giving explicitely the homomorphisms Ψ_i).

c) Find out the minimal number of elements needed to generate $\mathfrak{m}B$.

4) (2 points) Let $A = \mathbf{C}[X,Y]$ and let $J = (Y^2 - X^3, X^2 + Y^2 - 2X)$ be an ideal of A.

a) Find the irreducible components of $V(I)$ and find $\sqrt{I}$.

b) Prove that I has a unique shortest primary decomposition and find it.