COMMUTATIVE ALGEBRA 2008-09

Midterm exam, December 9, 2008

Time: 2 hours

1) Answer a) or b):

a) Let $(A, \mathfrak{m})$ be a local ring and let M be a finite A–module. Prove Nakayama's lemma, that is, prove that if $\mathfrak{m}M = M$, then $M = 0$.

b) Let A be a Noetherian integral domain and let $t \in A$, t not a unit. Prove $\bigcap_{n=1}^{\infty}(t^n) = (0)$.

2) Let $n \in N$ and let

$$B = \frac{\mathbf{k}[X, Y, Z]}{(X^n - Z, YZ - 1)}.$$

a) Prove that B is an integral domain.

b) The ring homomorphism

$$\begin{array}{ccc} \mathbf{k}[X] & \longrightarrow & B \\ c & \mapsto & \bar{c} \quad (c \in \mathbf{k}) \\ X & \mapsto & \overline{X} \end{array}$$

makes B into a $\mathbf{k}[X]$–algebra. Prove that it is not integral over $\mathbf{k}[X]$.

c) The ring homomorphism

$$\begin{array}{ccc} \mathbf{k}[Y] & \longrightarrow & B \\ c & \mapsto & \bar{c} \quad (c \in \mathbf{k}) \\ Y & \mapsto & \overline{Y - X} \end{array}$$

makes B into a $\mathbf{k}[Y]$–algebra. Prove that it is integral over $\mathbf{k}[Y]$.