COMMUTATIVE ALGEBRA 2008-09

Final exam, September 1, 2009

Time: 3 hours

You may answer this exam in English or in Spanish

1) (2 points) State and prove the Strong Hilbert's Nullstellensatz; in your proof say how the Weak Nullstellensatz is used.

2) (3 points) Answer the following questions. Explain your answers.

a) Let $A = \mathbf{R}[[X, Y]]$ and let M a module, finitely generated over A. Consider the ideal $\mathfrak{m} = (X, Y)$ of A. If $M/\mathfrak{m}M$ is an $\mathbf{R}$–vector space of dimension 3, what is the cardinal of a minimal number of generators of M as an A–module?

b) Give an exmaple of a ring that is Noetherian but not Artinian; give an example of an Artinian ring.

c) Describe $\mathrm{Spec}\mathbf{Z}_{(5)}$.

3) (2 points) Let $\mathbf{k}$ be a field. Consider the ring homomorphism

$$\begin{array}{ccc} \mathbf{k}[X,Y] & \xrightarrow{\Phi} & \mathbf{k}[T] \\ f(X,Y) & \mapsto & f(T^3, T^7) \end{array}$$

and let B be the image of Φ inside $\mathbf{k}[T]$.

a) Give a set of generators for the kernel of Φ.

b) Is B a normal ring? If not, find the normalization of B.

4) (3 points) Let $A = \mathbf{C}[X, Y, Z]$ and let $J = (X^2 - Y^2 - Z^2, X - Y, X - Z) \cap (X^2, Y, Z) \cap (X - 1, Y, Z)$ be an ideal of A.

a) Find a set of generators for J.

b) Find a shortest primary decomposition of J, giving a set of generators for each one of the primary components of the decomposition. Is the shortest primary decomposition found unique?

c) Compute the Krull dimension of A/J.