COMMUTATIVE ALGEBRA. 2008-09. *Exercise sheet* 1

Throughout this exercise sheet A will denote a ring and $\mathbf{k}$ will denote a field.

1) Compute a system of generators for the ideal $(X, Y^2) \cap (X^2, Y)$ of $\mathbf{k}[X, Y]$.

2) Compute a system of generators for the ideal consisting of the polynomials of $\mathbf{k}[X, Y]$ vanishing at the following subsets of $\mathbf{A}^2_k$:

a) $\{(0, 0), (1, 0)\}$; b) $\{(0, 0), (1, 0), (0, 1)\}$; c) $\{(0, 0), (1, 0), (0, 1), (1, 1)\}$.

3) We say that two ideals I and J of A are *strongly coprime* if $I + J = A$.

a) Prove that if I and J are strongly coprime then $IJ = I \cap J$ and $A/IJ \cong (A/I) \times (A/J)$.

b) Prove that if I and J are strongly coprime then so are I^n and J^n for all $n \geq 1$.

4) Let A and B be rings.

a) Let $f : A \longrightarrow B$ be a ring homomorphism. Prove that if $\mathfrak{p}$ is a prime ideal of B then $f^{-1}(\mathfrak{p})$ is a prime ideal of A. In particular, if A is a subring of B and $\mathfrak{p}$ is a prime ideal of B then $A \cap \mathfrak{p}$ is a prime ideal of A.

b) Let I be an ideal of A and let $\pi : A \longrightarrow A/I$ be the canonical projection onto the quotient of A by I. Prove that prime ideals $\mathfrak{q}$ of A/I are in one–to–one correspondence with the prime ideals $\mathfrak{p}$ of A containing I; this correspondence is given by $\mathfrak{q} = \pi(\mathfrak{p})$ and $\mathfrak{p} = \pi^{-1}(\mathfrak{q})$.

5) Prove or give a counterexample:

a) The intersection of two prime ideals is a prime ideal.

b) The sum of two prime ideals is a prime ideal.

c) The inverse image of a maximal ideal by a ring homomorphism is a maximal ideal.

d) Let I be an ideal of A; the inverse image of a maximal ideal of A/I by the canonical projection $\pi : A \longrightarrow A/I$ is a maximal ideal of A.

6) Compute the ideal I consisting of all the polynomials of $\mathbf{R}[X, Y]$ that vanish when evaluated at $(1+i, 1)$. Is I a maximal ideal of $\mathbf{R}[X, Y]$?

7) Let $\mathbf{k} \subset K$ be an algebraic field extension. Let $a = (a_1, \ldots, a_n) \in K^n$. Consider the evaluation homomorphism $e_a : \mathbf{k}[X_1, \ldots, X_n] \longrightarrow K$ defined by $f \mapsto f(a_1, \ldots, a_n)$. Determine the kernel and the image of e_a and deduce that the intersection $(X_1 - a_1, \ldots, X_n - a_n) \cap \mathbf{k}[X_1, \ldots, X_n]$ is a maximal ideal of $\mathbf{k}[X_1, \ldots, X_n]$.

8) Let I be the ideal of $\mathbf{k}[X, Y, Z]$ consisting of those polynomials f such that $f(T^3, T^4, T^5) = 0$ in $\mathbf{k}[T]$.

a) Give a family of generators of I.

b) Decide if I is a prime or a maximal ideal.

c) Show that I cannot be generated by two elements.

d) Prove that if $\mathbf{k}$ is an infinite field (for example, if $\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$), then I is the ideal of the curve C of $\mathbf{A}^3_{\mathbf{k}}$ parametrized by $x = t^3$, $y = t^4$ and $z = t^5$. Thus, this is an example of an affine algebraic subvariety whose ideal cannot be generated by as many polynomials as its codimension.

9) Let $I = (X - 4Z - 3, Y + 2Z + 1)$ be an ideal of $\mathbf{k}[X, Y, Z]$.

a) Prove that $I \cap \mathbf{k}[X, Y] = (X + 2Y - 1) \subset \mathbf{k}[X, Y]$.

b) Prove that I is a prime ideal of $\mathbf{k}[X, Y]$ but not a maximal ideal; find a maximal ideal containing I.

10) Let I be an ideal of A.

a) Prove that the set $\sqrt{I} = \{a \in A | \exists n \in \mathbf{N} \text{ such that } a^n \in I\}$ is an ideal of A (the *radical of* I).

b) Prove that $\sqrt{I}$ is the intersection of all prime ideals of A containing I.

c) Prove that $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

11) Let A be a UFD and let $f = f_1^{n_1} \ldots f_r^{n_r}$ be the unique (up to multiplication by units) factorization of $f \in A$ into irreducible elements.

a) Show that $\sqrt{(f)} = (f_1 \dots f_r)$.

b) This is for students that know the basics about plane affine algebraic curves: Let $\mathbf{k}$ be an algebraically closed field (for example, let $\mathbf{k} = \mathbf{C}$), let $A = \mathbf{C}[X,Y]$ and consider the curve $C = V(f) = \{(x,y) \in \mathbf{A}^2_{\mathbf{k}} | \ f(x,y) = 0\}$. Prove that the ideal $I(C) = \{g \in \mathbf{k}[X,Y] \mid g(x,y) = 0 \text{ for all } (x,y) \in C\}$ satisfies $I = \sqrt{(f)}$ (hint: use Study's lemma). This result is the *Nullstellensatz* (that we will see in Section 5 of this course) for the particular case of plane curves.

12) Use Zorn's lemma to prove that any prime ideal $\mathfrak{p}$ contains a minimal prime ideal.

13) Let $S \subset A$ be a multiplicative set and let I be an ideal of A such that S and I are disjoint. Prove that there is a prime ideal $\mathfrak{p}$ containing I such that S and $\mathfrak{p}$ are disjoint.

14) Let Σ be the set consisting of all the ideals I of A such that each element of I is either 0 or a zero divisor. Prove that Σ has maximal elements and that each maximal element of Σ is a prime ideal. Conclude that the subset of A consisting of 0 and the zero divisors of A is a union of prime ideals.

15) Let Σ be the set of multiplicative sets of A not containing 0. Prove that S is a maximal element of Σ if and only if $A \setminus S$ is a minimal prime ideal of A.

16) Let $X \subset \mathbf{A}^n_{\mathbf{k}}$ be an algebraic variety such that $I(X)$ is generated by $f_1, \dots, f_r$ and let $f \in \mathbf{k}[X_1, \dots, X_n]$. Prove that $f_1, \dots, f_r, X_{n+1}f - 1$ generate the ideal of the set
$Y = \{(a_1, \dots, a_n, a_{n+1}) \in \mathbf{A}^{n+1}_{\mathbf{k}} | (a_1, \dots, a_n) \in X, \ a_{n+1}f(a_1, \dots, a_n) = 1\}$.

17) Find the nilpotent elements and the minimal prime ideals of the following rings:

a) $\mathbf{k}[X,Y]/(XY, Y^2)$; b) $\mathbf{Z}_{300}$;

c) $\mathbf{k}[X,Y]/(f)$, where $f = cf_1^{n_1} \cdots f_s^{n_s}$, ($c \in \mathbf{k}, f_1, \dots, f_s$ irreducible elements of $\mathbf{k}[X,Y]$) is the unique factorization of f into irreducible elements.

18) Let A be a reduced ring having finitely many minimal prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$. Prove that there is an injective homomorphism $\varphi : A \longrightarrow A/\mathfrak{p}_1 \times \cdots \times A/\mathfrak{p}_n$ whose image has non zero intersection which each $A/\mathfrak{p}_i$. As a corollary, describe the injective homomorphism $\mathbf{k}[X,Y]/(XY) \hookrightarrow \mathbf{k}[X] \times \mathbf{k}[Y]$.

19) Prove that the ideal $J = (X + Y - X^2 + XY - Y^2, X(X + Y - 1)) \subset \mathbf{k}[X,Y]$ is not equal to the ideal I consisting of those polynomials vanishing at the subset $\{(0,0),(1,0),(0,1)\}$ of $\mathbf{A}^2_{\mathbf{k}}$, despite of the fact that $V(J) = \{(0,0),(1,0),(0,1)\}$.

20) Let $A[[X]] = \{\Sigma_{n=0}^{\infty} a_n X^n | a_n \in A\}$ (the set of *formal power series* with coefficients in A).

a) Prove that $A[[X]]$ is a ring with respect to the obvious addition and multiplication.

b) Prove that a formal power series $\Sigma_{n=0}^{\infty} a_n X^n$ is a unit of $A[[X]]$ if and only if a_0 is a unit of A. Compute the inverse of $1 + X$ in $\mathbf{k}[[X]]$.

c) Find all the ideals of $\mathbf{k}[[X]]$.

21) Let $A[[X_1, \dots, X_n]]$ be the set of expressions of the form $\Sigma_{n=0}^{\infty} f_n$, where f_n is a degree n homogeneous polynomial in the variables $X_1, \dots, X_n$ with coefficients in A.

a) Prove that $A[[X_1, \dots, X_n]]$ is a ring with respect to the obvious addition and multiplication.

b) Prove that $\Sigma_{n=0}^{\infty} f_n$ is a unit of $A[[X_1, \dots, X_n]]$ if and only if f_0 is a unit of A.

c) Conclude that $\mathbf{k}[[X_1, \dots, X_n]]$ only posseses one maximal ideal, that is, conclude that $\mathbf{k}[[X_1, \dots, X_n]]$ is a local ring.

22) Prove that the following rings are local and find their maximal ideal:

a) $\{f/g | f, g \in \mathbf{k}[X] \text{ and } g(1) \neq 0\}$.

b) $\{f/g | f, g \in \mathbf{k}[X,Y] \text{ and } g(0,0) \neq 0\}$.

c) $\mathbf{k}\{X\}$, the ring of power series over $\mathbf{k} = \mathbf{R}$ or $\mathbf{C}$ with radius of convergence > 0.

d) The ring of continuous function germs at the origin $0 \in \mathbf{R}^n$. To define this ring we consider the quotient of the set $\{(U,h) | \ U \text{ is an open neighborhood of } 0 \text{ and } h : U \longrightarrow \mathbf{R} \text{ is a continuous function}\}$ by the following equivalence relation: (U,h) is equivalent to (U',h') if and only if there exists an open neighborhood W of 0 contained in $U \cap U'$ such that $h|_W = h'|_W$. Then the ring of continuous function germs at 0 is the quotient set by this equivalent relation with addition and multiplication defined in the obvious way.